

RECURSIVE ENUMERABILITY AND THE JUMP OPERATOR

BY

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By degree we mean degree of recursive unsolvability as defined by Kleene and Post in [4]. Following Shoenfield [7], we say a degree c is recursively enumerable in a degree b if there is a set of degree c which is the range of a function of degree less than or equal to b , and we call a degree recursively enumerable if it is recursively enumerable in 0 (i.e., if it is the degree of a recursively enumerable set). The jump operator, which takes the degree d to the degree d' (the completion of d), was defined in [4] and has the following properties: if h is recursively enumerable in d , then $h \leq d'$; $d' > d$; and d' is recursively enumerable in d . In [4] a degree c is said to be complete if there exists a degree d such that $d' = c$. Friedberg [1] showed that a degree c is complete if and only if $c \geq 0'$.

For any degree b , if $b \leq d \leq b'$, then $b' \leq d' \leq b$ and d' is recursively enumerable in b' . Shoenfield [7] proved that if $b' \leq c \leq b''$ and c is recursively enumerable in b' , then there is a degree d such that $b \leq d \leq b'$ and $d' = c$. Thus the degrees which lie between b' and b'' and are recursively enumerable in b' can be viewed as the completions of the degrees which lie between b and b' . He also showed there is a degree greater than b and less than b' which is not recursively enumerable in b .

Our main result below is that the degrees which lie between b' and b'' and are recursively enumerable in b' can be viewed as the completions of the degrees which lie between b and b' and are recursively enumerable in b . Our notation is that of [3].

THEOREM 1. *Let a , b and c be degrees such that $a \not\leq b$, $a \leq b' \leq c$ and c is recursively enumerable in b' . Then there exists a degree d such that $a \not\leq d$, $b \leq d$, $d' = c$ and d is recursively enumerable in b .*

Proof. We first prove the theorem when $b = 0$, and then indicate the changes needed when $b > 0$. Thus we have degrees a and c such that $a > 0$, $a \leq 0' \leq c$ and c is recursively enumerable in $0'$, and we wish find a recursively enumerable degree d such that $a \not\leq d$ and $d' = c$.

Let f be a function of degree less than or equal to $0'$ whose range is a set C of

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degree $\mathbf{c}$. Let $\mathbf{c}$ be the representing function of C . Let g be a recursive function whose range is a set J of degree $\mathbf{0}'$. Let j be the representing function of J . We define

$$j(s, n) = \begin{cases} 0 & \text{if } (Ek)_{k < s}(g(k) = n), \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $j(s, n)$ is a recursive function, and that for each n , $\lim_s j(s, n)$ exists and is equal to $j(n)$. Since f is recursive in j , there is a Gödel number z_1 such that

$$f(n) = \{z_1\}^j(n) = U(\mu y T_1^1(j(y), z_1, n, y))$$

for all n . We define a recursive function $f(s, n)$ of supreme importance to our argument;

$$f(s, n) = \begin{cases} U\left(\mu y T_1^1\left(\prod_{i < y} p_i^{j(s, n)}, z_1, n, y\right)\right) & \text{if } (Ey)_{y \leq s} T_1^1\left(\prod_{i < y} p_i^{j(s, n)}, z_1, n, y\right), \\ s + 1 & \text{otherwise.} \end{cases}$$

We claim that $\lim_s f(s, n)$ exists and is equal to $f(n)$ for all n . Our claim is a consequence of the fact that $f(n) = \{z_1\}^j(n)$ and $\lim_s j(s, n) = j(n)$ for all n .

Let a be an everywhere positive function of degree $\mathbf{a}$, and let z_2 be a Gödel number such that $\{z_2\}^j(n) = a(n)$ for all n . We define

$$a(s, n) = \begin{cases} U\left(\mu y T_1^1\left(\prod_{i < y} p_i^{j(s, n)}, z_2, n, y\right)\right) & \text{if } (Ey)_{y \leq s} \left[T_1^1\left(\prod_{i < y} p_i^{j(s, n)}, z_2, n, y\right) \& U(y) \geq 1 \right], \\ 1 & \text{otherwise.} \end{cases}$$

The function $a(s, n)$ is recursive; for each n , $\lim_s a(s, n)$ exists and is equal to $a(n)$.

A useful property of the Gödel numbering devised by Kleene in [3] to arithmetize his formalism for recursive functions is: the Gödel number of a deduction is greater than the intuitive counterpart of any formal numeral occurring in the deduction. We will denote this fact by GND. It follows from GND that $a(s, n) = 1$ whenever $n \geq s$.

We define two recursive functions, $t(s, n)$ and $h(s, n)$, by means of an induction on s :

$$t(s, n) = \mu m_{m < s}(f(s, m) = n);$$

$$h(0, n) = 0;$$

$$h(s + 1, n) = h(s, n) + \text{sg}(|t(s + 1, n) - t(s, n)|).$$

Recall that the bounded least number operator is defined in such a way that $t(s, n) = s$ if and only if there is no $m < s$ such that $f(s, m) = n$.

We now proceed to define four recursive functions, $y(s, n, e)$, $m(s, e)$, $r(s, n, e)$ and $d(s, n)$, simultaneously by induction on s . The function $d(s, n)$ will be such that

$$0 \leq d(s+1, n) \leq d(s, n) \leq 1$$

for all s and n . Thus for each n , $\lim_s d(s, n)$ will exist; furthermore, $\lim_s d(s, n)$ will be the representing function of a recursively enumerable set D . The degree of D will be the desired degree $\mathbf{d}$. At stage s of the construction we put finitely or infinitely many natural numbers in D ; our main objective is to see that $\mathbf{c} \leq \mathbf{d}'$; however, with the aid of a system of priorities, we exercise restraint when we add members to D in order to insure that $\mathbf{a} \not\leq \mathbf{d}$ and $\mathbf{d}' \leq \mathbf{c}$.

Stage $s = 0$. We set $y(0, n, e) = r(0, n, e) = 0$, $m(0, e) = e + 1$ and $d(0, n) = 1$ for all n and e .

Stage $s > 0$. We define $y(s, n, e)$ for all n and e :

$$y(s, n, e) = \begin{cases} \mu y T_1^1 \left(\prod_{i < y} p_i^{d(s-1, i)}, e, n, y \right) \\ \text{if } n \geq e \ \& (Ey)_{y \leq s} T_1^1 \left(\prod_{i < y} p_i^{d(s-1, i)}, e, n, y \right), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from GND that $y(s, n, e) = 0$ whenever $n \geq s$.

We define $m(s, e)$ for all e ; there are three mutually exclusive cases.

Case 1. $y(s, e, e) = 0$. We set $m(s, e) = e + 1$.

Case 2. $y(s, e, e) > 0$ and there is an n such that

$$e < n < m(s-1, e) \ \& \ y(s, n, e) \neq y(s-1, n, e) \ \& \ a(s, n) \neq U(y(s, n, e)).$$

We set

$$m(s, e) = \mu n_{e < n} [y(s, n, e) \neq y(s-1, n, e) \ \& \ a(s, n) \neq U(y(s, n, e))].$$

Case 3. Otherwise.

We set

$$m(s, e) = \mu n [m(s-1, e) \leq n < 2m(s-1, e) + s \\ \ \& (Et) (e < t \leq n \ \& \ a(s, t) \neq U(y(s, t, e)))].$$

Note that Case 3 of the definition of $m(s, e)$, the least number operator is bounded.

We define $r(s, n, e)$ and $d(s, p_e^n)$ for all n and e by means of a simultaneous induction on e . Let $e \geq 0$ and suppose $r(s, n, i)$ and $d(s, p_i^n)$ have been defined for all $i < e$ and all n ; we define $r(s, n, e)$ and $d(s, p_e^n)$ for all n as follows:

$$r(s, n, e) = \begin{cases} 0 & \text{if } (Ei)(Em)(Et) [i < e \leq t \leq n \ \& \ p_i^m < y(s, t, e) \\ & \ \& \ d(s, p_i^m) \neq d(s-1, p_i^m)], \\ 1 & \text{otherwise;} \end{cases}$$

$$d(s, p_e^n) = \begin{cases} d(s-1, p_e^n) & \text{if } n \geq h(s, e), \\ d(s-1, p_e^n) & \text{if } (Ei)(Em)[i \leq e \& i \leq m < m(s, i) \\ & \& r(s, m, i) = 1 \& p_e^n < y(s, m, i)], \\ 0 & \text{otherwise.} \end{cases}$$

We conclude the construction by setting $d(s, n) = d(s-1, n)$ for all n not a power of a prime. It is readily verified by the method of [4] that each of the four functions just defined is recursive. Such a verification is possible for two reasons: each of the functions $a(s, n)$ and $h(s, n)$ is recursive; at stage $s > 0$, all quantifiers, as well as all applications of the least number operator, are bounded. For each n , let

$$d(n) = \lim_s d(s, n);$$

it is clear that $d(n) = 0$ if and only if there is an s such that $d(s, n) = 0$. Thus d is the representing function of a recursively enumerable set. Let $\mathbf{d}$ be the degree of d .

We list some remarks which will be needed in vital parts of the body of our argument:

(R1) $(s)(e)[m(s, e) > e]$;

(R2) $(s)(n)(e)[r(s, n, e) = 0 \rightarrow r(s, n+1, e) = 0]$;

(R3) $(s)(n)(e)[(y(s, n, e) = 0 \& n > e) \rightarrow m(s, e) \leq n]$.

Remark (R1) is easily proved by induction on s if the definition of the bounded least number operator is kept in mind.

We prove remark (R3) by induction on s . We have

$$(n)(e)[(y(0, n, e) = 0 \& n > e) \rightarrow m(0, e) \leq n].$$

Let s be such that $s > 0$ and

$$(n)(e)[(y(s-1, n, e) = 0 \& n > e) \rightarrow m(s-1, e) \leq n].$$

Let e and n be such that

$$y(s, n, e) = 0 \quad \text{and } n > e.$$

Then $a(s, n) \neq U(y(s, n, e))$, since $a(s, n) \geq 1$ and $U(0) = 0$. First we suppose $n < m(s-1, e)$. Then $y(s-1, n, e) > 0$ as a consequence of the induction hypothesis. But then either Case 1 or Case 2 of the definition of $m(s, e)$ holds, and so $m(s, e) \leq n$. Now we suppose $m(s-1, e) \leq n$. If either Case 1 or Case 2 of the definition of $m(s, e)$ holds, then $m(s, e) \leq m(s-1, e) \leq n$ by remark (R1). If Case 3 holds and $n < 2m(s-1, e) + s$, then $m(s, e) \leq n$. If Case 3 holds and $n \geq 2m(s-1, e) + s$, then $m(s, e) \leq 2m(s-1, e) + s \leq n$. (Note that if Case 3 holds and

$$(t)[e < t < 2m(s-1, e) + s \rightarrow a(s, t) = U(y(s, t, e))],$$

then $m(s, e) = m(s-1, e) + s$; this last is a consequence of the definition of the bounded least number operator.)

We introduce two predicates:

$A(e)$: if the set $\{m(s, e) \mid s \geq 0\}$ is infinite, then there is an $n \geq e$ such that $\lim_s y(s, n, e)$ either does not exist or is equal to 0.

$B(e)$: $\lim_n d(p_e^n)$ exists and is equal to $1 - c(e)$.

We will prove $(e)A(e)$ and $(e)B(e)$ by means of a simultaneous induction on e . From $(e)A(e)$ it will follow that $\mathbf{a} \not\leq \mathbf{d}$. From $(e)B(e)$ it will follow that $\mathbf{c} \leq \mathbf{d}'$. Fix $e^* \geq 0$ and suppose $A(e)$ and $B(e)$ are true for all $e < e^*$. We proceed to prove $A(e^*)$ and $B(e^*)$.

LEMMA 1. *Let $y(s, n, e^*) > 0$ and $m(s, e^*) > n \geq e^*$. Let $d(s, p_i^m) = d(s - 1, p_i^m)$ for all i, m and t such that $i < e^* \leq t \leq n$ and $p_i^m < y(s, t, e^*)$. Then $y(s, n, e^*) = y(s + 1, n, e^*)$.*

Proof. Since $y(s, n, e^*) > 0$, we have

$$y(s, n, e^*) = \mu y T_1^1 \left(\prod_{i < y} p_i^{d(s-1, i)}, e^*, n, y \right).$$

We suppose $y(s + 1, n, e^*) \neq y(s, n, e^*)$ and then show there is an i , an m and a t such that

$$i < e^* \leq t \leq n \text{ \& } p_i^m < y(s, t, e^*) \text{ \& } d(s, p_i^m) \neq d(s - 1, p_i^m).$$

Since $y(s + 1, n, e^*) \neq y(s, n, e^*)$, there must be a $j < y(s, n, e^*)$ such that $d(s, j) \neq d(s - 1, j)$. Recall that $d(s, w) = d(s - 1, w)$ for all w not a power of a prime. Thus there is an i' and an m' such that

$$d(s, p_{i'}^{m'}) \neq d(s - 1, p_{i'}^{m'})$$

and $p_{i'}^{m'} < y(s, n, e^*)$. But then by the hypothesis of the lemma, $e^* \leq i'$. Thus we have

$$e^* \leq i' \text{ \& } e^* \leq n < m(s, e^*) \text{ \& } p_{i'}^{m'} < y(s, n, e^*) \text{ \& } d(s, p_{i'}^{m'}) \neq d(s - 1, p_{i'}^{m'}).$$

It follows from the definition of $d(s, p_{i'}^{m'})$ that $r(s, n, e^*) = 0$. But this last means the desired i, m and t exist.

LEMMA 2. *Let $y(s, n, e^*) > 0$ and $m(s, e^*) > n > e^*$. Let $d(s, p_i^m) = d(s - 1, p_i^m)$ for all i, m and t such that $i < e^* \leq t \leq n$ and $p_i^m < y(s, t, e^*)$. Then $m(s + 1, e^*) > n$.*

Proof. Since $m(s, e^*) > n > e^*$, it follows from remark (R3) and Case 1 of the definition of $m(s, e)$ that

$$y(s, t, e^*) > 0$$

for all t such that $e^* \leq t \leq n$. But then by Lemma 1,

$$y(s, t, e^*) = y(s + 1, t, e^*)$$

for all t such that $e^* \leq t \leq n$. Suppose $m(s + 1, e^*) \leq n$. Then $m(s + 1, e^*) < m(s, e^*)$,

and consequently, Case 2 of the definition of $m(s+1, e^*)$ holds. This means there is a t (namely, $m(s+1, e^*)$) such that

$$e^* < t \leq n \text{ \& } y(s, t, e^*) \neq y(s+1, t, e^*).$$

LEMMA 3. $A(e^*)$.

Proof. By the hypothesis of our theorem the function a is nonrecursive. We suppose $A(e^*)$ is false and show a is recursive. Thus the set $\{m(s, e^*) \mid s \geq 0\}$ is infinite, and for each $n \geq e^*$, $\lim_s y(s, n, e^*)$ exists and is positive. Let $R(n, s)$ denote the predicate

$$\begin{aligned} m(s, e^*) > n \text{ \& } (e)(m)(t) [(p_e^m < y(s, t, e^*) \text{ \& } e < e^* \leq t \leq n) \\ \rightarrow d(s-1, p_e^m) = d(p_e^m)]. \end{aligned}$$

We know $B(e)$ is true for all $e < e^*$. This means $\lim_m d(p_e^m)$ exists for all $e < e^*$. For each $e < e^*$, let $g(e)$ be such that

$$(m) [m \geq g(e) \rightarrow d(p_e^m) = d(p_e^{g(e)})].$$

We define a recursive function $z(n)$ as follows: first we require that $z(n) = 1$ for all n not a power of a prime; then we specify

$$z(p_e^m) = \begin{cases} d(p_e^{g(e)}) & \text{if } e > e^* \text{ \& } m \geq g(e), \\ d(p_e^m) & \text{if } e > e^* \text{ \& } m < g(e), \\ 1 & \text{otherwise.} \end{cases}$$

The predicate $R(n, s)$ can now be rewritten as

$$\begin{aligned} m(s, e^*) > n \text{ \& } (e)(m)(t) [(p_e^m < y(s, t, e^*) \text{ \& } e < e^* \leq t \leq n) \\ \rightarrow d(s-1, p_e^m) = z(p_e^m)]. \end{aligned}$$

It is clear that $R(n, s)$ is recursive, since the functions m, y and z are recursive.

Now we show $(n)(Es)R(n, s)$. Fix n . Since $\lim_s y(s, n, e^*)$ exists for all $n \geq e^*$ there is a y such that

$$y \geq y(s, t, e^*)$$

for all t and s such that $e^* \leq t \leq n$. Let s' be so large that

$$d(s-1, w) = d(w)$$

for all s and w such that $s \geq s'$ and $w < y$. Since the set $\{m(s, e^*) \mid s \geq 0\}$ is infinite, there is an $s \geq s'$ such that $m(s, e^*) > n$. But then $R(n, s)$.

Let $w(n)$ denote the recursive function $\mu s R(n, s)$. Note that $w(n+1) \geq w(n)$ for all n .

Next we prove $y(w(n), n, e^*) = \lim_s y(s, n, e^*)$ for all $n > e^*$. Fix $n > e^*$. We show by induction on s that $y(w(n), n, e^*) = y(s, n, e^*)$ for all $s \geq w(n)$. Let s be such that $s \geq w(n)$ and

$$y(w(n), e^*) = y(s, n, e^*) \& R(n, s).$$

Since $m(s, e^*) > n > e^*$, it follows from remark (R3) and Case 1 of the definition of $m(s, e^*)$ that $y(s, t, e^*) > 0$ for all t such that $e^* \leq t \leq n$. By the definition of $R(n, s)$, we have

$$d(s-1, p_e^m) = d(p_e^m)$$

for all e, m and t such that $e < e^* \leq t \leq n$ and $p_e^m < y(s, t, e^*)$. Recall that if $d(s-1, w) = d(w)$ then $d(s', w) = d(w)$ for all $s' \geq s$. It follows from Lemma 1 that

$$y(s, t, e^*) = y(s+1, t, e^*)$$

for all t such that $e^* \leq t \leq n$. It follows from Lemma 2 that

$$m(s+1, e^*) > n.$$

But then

$$y(w(n), n, e^*) = y(s+1, n, e^*) \& R(n, s+1).$$

Thus $y(w(n), n, e^*) = y(s, n, e^*)$ for all $s \geq w(n)$, and $\lim_s y(s, n, e^*) = y(w(n), n, e^*)$.

Finally, we show by means of a reductio ad absurdum that

$$a(n) = U(y(w(n), n, e^*))$$

for all $n > e^*$. It will then follow that a is recursive, since w is recursive. Fix $n > e^*$ and suppose $a(n) \neq U(y(w(n), n, e^*))$. Since $y(w(n), n, e^*) = \lim_s y(s, n, e^*)$, and since $a(n) = \lim_s a(s, n)$, there is an s^* such that for all $s \geq s^*$,

$$a(n) = a(s, n) \neq U(y(s, n, e^*)) = U(y(w(n), n, e^*)).$$

Let $s > s^*$ and suppose $m(s-1, e^*) \leq m(s^*, e^*) + n + e^* + 1$. If either Case 1 or Case 2 of the definition of $m(s, e^*)$ holds, then

$$m(s, e^*) \leq \max(e^* + 1, m(s-1, e^*)) \leq m(s^*, e^*) + n + e^* + 1.$$

If Case 3 holds and $n < 2m(s-1, e^*) + s$, then $m(s, e^*) \leq n$. If Case 3 holds and $2m(s-1, e^*) + s \leq n$, then $m(s, e^*) \leq n$. Thus we have shown by induction on s that

$$m(s, e^*) \leq m(s^*, e^*) + n + e^* + 1$$

for all $s \geq s^*$. But this last is absurd, since the set $\{m(s, e^*) \mid s \geq 0\}$ is infinite.

For each $e \geq 0$, we say e is stable if for all $n \geq e$, $\lim_s y(s, n, e)$ exists and is positive. Note that if e is not the Gödel number of a system of equations, then $y(s, n, e) = 0$ for all s and n , and consequently, e is not stable. It follows that there are infinitely many e which are not stable, since there are infinitely many e which are not Gödel numbers of systems of equations. We define

$$e_0 = \mu e \text{ (} e \text{ is not stable);}$$

$$e_{j+1} = \mu e \text{ (} e > e_j \text{ and } e \text{ is not stable).}$$

Thus $e_0 < e_1 < e_2 < \dots$ is a listing of all the e which are not stable. For each $j \geq 0$, let n_j be the least $n \geq e_j$ such that $\lim_s y(s, n, e_j)$ either does not exist or is equal to 0.

The most important part of our argument is contained in Lemma 4. If the proof of our theorem is a heavy meal, then the proof of Lemma 4 is the main course; furthermore, it is there that the combinatorial flavor of our reasoning is strongest.

LEMMA 4. *For each k and v , there is an $s \geq v$ such that*

$$(j)_{j < k} [m(s, e_j) \leq n_j \vee r(s, n_j, e_j) = 0 \vee y(s, n_j, e_j) = 0].$$

Proof. Fix k and v . We suppose there is no s with the properties required by the lemma, and then show it is possible to define an infinite, descending sequence of natural numbers.

We propose the following system of equations as a means of defining two functions, $S(t)$ and $M(t)$, simultaneously by induction:

$$S(0) = \mu s \text{ (} s \geq v \text{);}$$

$$M(t) = \mu j [j < k \ \& \ n_j < m(S(t), e_j)$$

$$\ \& \ r(S(t), n_j, e_j) = 1 \ \& \ y(S(t), n_j, e_j) > 0];$$

$$S(t+1) = \mu s (Em) [s \geq S(t) \ \& \ m < y(S(t), n_{M(t)}, e_{M(t)})$$

$$\ \& \ d(s, m) \neq d(S(t) - 1, m)].$$

Clearly $S(0)$ is well defined and greater than or equal to v . Suppose $t \geq 0$ and $S(t)$ is well defined and greater than or equal to v . Then $M(t) < k$, since we have supposed the lemma to be false. Thus

$$y(S(t), n_{M(t)}, e_{M(t)}) > 0$$

and $\lim_s y(s, n_{M(t)}, e_{M(t)})$ does not exist or is equal to 0. Then there must be an $s > S(t)$ such that

$$y(s, n_{M(t)}, e_{M(t)}) \neq y(S(t), n_{M(t)}, e_{M(t)});$$

note that $S(t) > 0$, since $y(0, n, e) = 0$ for all n and e ; this means there is an $s > S(t)$ and an m such that

$$m < y(S(t), n_{M(t)}, e_{M(t)}) \ \& \ d(s - 1, m) \neq d(S(t) - 1, m).$$

Then $S(t+1)$ is well defined and greater than or equal to v .

For each $t \geq 0$, let

$$u(t) = \mu m [d(S(t+1), m) \neq d(S(t) - 1, m)].$$

Now we show $u(t) < u(t-1)$ for all $t > 0$. Fix $t > 0$. Since we have

$$u(t) < y(S(t), n_{M(t)}, e_{M(t)})$$

by definition of u , it will be sufficient to show

$$y(S(t), n_{M(t)}, e_{M(t)}) \leq u(t-1).$$

Since $d(w) = 1$ for all w not a power of a prime, there must exist i and m such that $u(t-1) = p_i^m$. Note that $d(S(t), u(t-1)) \neq d(S(t)-1, u(t-1))$; this last follows from the definitions of $S(t)$ and $u(t-1)$. Let

$$e = e_{M(t)}, s = S(t) \text{ and } n = n_{M(t)}.$$

First we suppose $i < e$. This means

$$i < e \leq n \text{ \& } d(s, p_i^m) \neq d(s-1, p_i^m) \text{ \& } r(s, n, e) = 1,$$

since $M(t) < k$. But then it follows from the definition of $r(s, n, e)$ that $y(s, n, e) \leq p_i^m$. Now we suppose $i \geq e$. This means

$$e \leq i \text{ \& } e \leq n < m(s, e) \text{ \& } r(s, n, e) = 1 \text{ \& } d(s, p_i^m) \neq d(s-1, p_i^m),$$

since $M(t) < k$. But then it follows from the definition of $d(s, p_i^m)$ that $y(s, n, e) \leq p_i^m = u(t-1)$.

LEMMA 5. *If $c(e^*) = 0$, then $\lim_n d(p_{e^*}^n)$ exists and is equal to 1.*

Proof. Let t be the least m such that $f(m) = e^*$. Let s' be so large that $s' > t$ and $f(s, m) = f(m)$ for all s and m such that $s \geq s'$ and $m \leq t$. Then $t(s, e^*) = t$ for all $s \geq s'$, and consequently, $h(s, e^*) = h(s', e^*)$ for all $s \geq s'$. But then

$$d(s, p_{e^*}^n) = d(s-1, p_{e^*}^n)$$

for all s and n such that $s > 0$ and $n \geq h(s', e^*)$, since $h(s, e^*) \leq h(s', e^*)$ for all $s \leq s'$. It follows that $\lim_n d(s, p_{e^*}^n) = 1$ for all $n \geq h(s', e^*)$, since $d(0, w) = 1$ for all w . Then $d(p_{e^*}^n) = 1$ for all $n \geq h(s', e^*)$, and $\lim_n d(p_{e^*}^n) = 1$.

LEMMA 6. *If $c(e^*) = 1$, then $\lim_n d(p_{e^*}^n)$ exists and is equal to 0.*

Proof. First we show that the set $\{t(s, e^*) \mid s \geq 0\}$ is infinite. Suppose $t(s, e^*) \leq t$ for all s . Let s' be so large that $s' > t$ and $f(s, m) = f(m)$ for all s and m such that $s \geq s'$ and $m \leq t$. Then $f(s', t(s', e^*)) = e^*$, since $t(s', e^*) < s'$. But

$$f(s', t(s', e^*)) = f(t(s', e^*)),$$

since $t(s', e^*) \leq t$. But then $f(t(s', e^*)) = e^*$; this last is impossible because C is the range of f and $c(e^*) = 1$.

Since the set $\{t(s, e^*) \mid s \geq 0\}$ is infinite, it is clear that the set $\{h(s, e^*) \mid s \geq 0\}$ is infinite.

By Lemma 3, we know $A(e)$ holds for all $e \leq e^*$. This means that if $e \leq e^*$ and e

is stable, then the set $\{m(s, e^*) \mid s \geq 0\}$ is finite. If $e \leq e^*$ and e is stable, let $m(e)$ be the greatest member of $\{m(s, e^*) \mid s \geq 0\}$; if $e \leq e^*$ and e is not stable, let $m(e) = n_j$, where j is such that $e = e_j$. If $e \leq e^*$ and $e \leq m < m(e)$, then $\lim_s y(s, m, e)$ exists. Let y be so large that

$$y \geq y(s, m, e)$$

for all s, m and e such that $e \leq e^*$ and $e \leq m < m(e)$.

Fix $n > y$. We show $d(p_{e^*}^n) = 0$. It will suffice to find an s such that $d(p_{e^*}^n, s) = 0$. Let v be such that $h(v, e^*) > n$. Let k be such that if $e \leq e^*$ and e is not stable, then $e = e_j$ for some $j < k$. By Lemma 4 there is an $s \geq v$ such that

$$(j)_{j < k} [m(s, e_j) \leq n_j \vee r(s, n_j, e_j) = 0 \vee y(s, n_j, e_j) = 0].$$

We will show:

$$h(s, e^*) > n;$$

$$(e)(m) [(e \leq e^* \ \& \ e \leq m < m(s, e)) \rightarrow (r(s, m, e) = 0 \vee p_{e^*}^n \geq y(s, m, e))].$$

It will then follow from the definition of $d(s, p_{e^*}^n)$ that $d(s, p_{e^*}^n) = 0$. We have $h(s, e^*) > n$, since $s \geq v$ and $h(s, e^*)$ is a nondecreasing function of s . Fix e and m so that $e \leq e^*$ and $e \leq m < m(s, e)$. Suppose e is stable. Then $m < m(e)$, since $m(s, e) \leq m(e)$. But then $y \geq y(s, m, e)$, and consequently $p_{e^*}^n \geq y(s, m, e)$, since $n > y$.

Now suppose e is not stable. Then $e = e_j$, where $j < k$, and $m(e) = n_j$. If $m < n_j$, then $m < m(e)$ and $p_{e^*}^n \geq y(s, m, e)$. Suppose $m \geq n_j$. Then $m(s, e_j) > n_j$. This last means that either $r(s, n_j, e) = 0$ or $y(s, n_j, e) = 0$. If $r(s, n_j, e) = 0$, then by remark (R2), $r(s, m, e) = 0$, since $m \geq n_j$. Suppose $y(s, n_j, e) = 0$. Since $n_j \leq m < m(s, e)$, it follows from remark (R3) that $n_j = e$. But then $y(s, e, e) = 0$, and Case 1 of the definition of $m(s, e)$ holds. It follows that $m(s, e) = e + 1$, $m = e$ and $y(s, m, e) = 0$.

Thus $d(p_{e^*}^n) = 0$ for all $n > y$, and $\lim_n d(p_{e^*}^n)$ exists and is equal to 0.

Lemmas 5 and 6 constitute a proof of $B(e^*)$. That concludes our proof by induction of $(e)A(e)$ and $(e)B(e)$. It is now easily seen that $c \leq d'$. Observe that

$$(e)(Et) [(m)_{m \geq t} (d(p_e^m) = 1) \vee (m)_{m \geq t} (d(p_e^m) = 0)]$$

is an immediate consequence of $(e)B(e)$. We define

$$k(e) = \mu t [(m)_{m \geq t} (d(p_e^m) = 1) \vee (m)_{m \geq t} (d(p_e^m) = 0)].$$

The function k has degree less than or equal to d' , and by $(e)B(e)$,

$$c(e) = 1 - d(p_e^{k(e)})$$

for all e .

LEMMA 7. $a \leq d$.

Proof. We suppose there is a Gödel number e such that

$$a(n) = \{e\} (n)$$

for all n , and then show $A(e)$ is false. First we show that $\lim_s y(s, n, e)$ exists and is positive for all $n \geq e$. Fix $n \geq e$; let

$$w = \mu y T_1^1(\tilde{d}(y), e, n, y).$$

Let s' be so large that $d(s, m) = d(m)$ whenever $s \geq s'$ and $m < w$. Then

$$y(s, n, e) = w \ \& \ U(w) = a(n)$$

for all $s \geq s' + w$; $w > 0$, since 0 is not the Gödel number of a deduction.

Now we show the set $\{m(s, e) \mid s \geq 0\}$ is infinite. We fix $m > e$ and obtain an s' such that $m(s', e) > m$. Let s be so large that $s > m$ and

$$a(s, t) = a(t) = \{e\}^d(t) = U(y(s, t, e))$$

for all t such that $e \leq t \leq m$. If $m(s-1, e) > m$, then $s-1$ is the desired s' . Suppose $m(s-1, e) \leq m$. This means

$$a(s, t) = U(y(s, t, e))$$

for all t such that $e \leq t \leq m(s-1, e)$; in addition, $y(s, e, e) > 0$, since $a(s, e) \geq 1$ and $U(0) = 0$. But then Case 3 of the definition of $m(s, e)$ holds. It follows that $m(s, e) > m$, since $s > m$.

LEMMA 8. $\mathbf{d}' \leq \mathbf{c}$.

Proof. We will define two functions, $E(e, n)$ and $L(e)$, simultaneously by induction on e so that each is recursive in the function $c(n)$. We will combine the definition of E and L with a proof by induction on e of the following:

$$\begin{aligned} (e)(n) \{ E(e, n) = 0 \leftrightarrow [n \geq e \ \& \ (w)(Es)(s > w \ \& \ m(s, e) > n) \\ \ \& \ (m)(n \geq m \geq e \rightarrow (Ey) T_1^1(\tilde{d}(y), e, m, y))] \}; \\ (e)(m) [m \geq L(e) \rightarrow d(p_e^m) = d(p_e^{L(e)})]. \end{aligned}$$

It follows immediately from the above and remark (R1) that

$$(e)[E(e, e) = 0 \leftrightarrow (Ey) T_1^1(\tilde{d}(y), e, e, y)];$$

but then if E is recursive in c , we have $\mathbf{d}' \leq \mathbf{c}$.

Fix $e \geq 0$. Our induction hypothesis has two parts:

(1, e) for each $i < e$ and each n , $E(i, n)$ has been defined and

$$\begin{aligned} E(i, n) = 0 \leftrightarrow [n \geq i \ \& \ (w)(Es)(s > w \ \& \ m(s, i) > n) \\ \ \& \ (m)(n \geq m \geq i \rightarrow (Ey) T_1^1(\tilde{d}(y), i, m, y))]; \end{aligned}$$

(2, e) for each $i < e$, $L(i)$ has been defined and

$$(m) [m \geq L(i) \rightarrow d(p_i^m) = d(p_i^{L(i)})].$$

We proceed to define $E(e, n)$ for all n , verify $(1, e + 1)$, define $L(e)$ and verify $(2, e + 1)$.

Let $(1, e + 1, n)$ denote the following predicate: for each $t < n$, $E(e, t)$ has been defined and

$$E(e, t) = 0 \leftrightarrow [t \geq e \ \& \ (w)(Es)(s > w \ \& \ m(s, e) > t) \\ \& \ (m)(t \geq m \geq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y))].$$

To verify $(1, e + 1)$, it suffices to prove $(1, e + 1, n)$ for all n . We define $E(e, n)$ and prove $(1, e + 1, n)$ for all n by means of an induction on n . First we set $E(e, t) = 1$ for all $t < e$. Then it is clear that $(1, e + 1, t)$ holds for all $t \leq e$. Now we fix $n \geq e$ and suppose $(1, e + 1, n)$ holds. We define $E(e, n)$ and then prove $(1, e + 1, n + 1)$. The definition of $E(e, n)$ has two cases.

Case 1. $(Et)(e \leq t < n \ \& \ E(e, t) \neq 0)$. We set $E(e, n) = 1$.

Case 2. Otherwise. It follows from $(1, e + 1, n)$ that

$$(m)(n > m \geq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y)).$$

For each m such that $n > m \geq e$, let

$$y(m) = \mu y T_1^1(\tilde{d}(y), e, m, y).$$

Let y^* be the largest member of $\{y(m) \mid n > m \geq e\} \cup \{0\}$. Let

$$s^* = \mu s [(i)(i < y^* \rightarrow d(s - 1, i) = d(i)) \ \& \ s > y^*].$$

Recall that for each $s > 0$ and i , if $d(s - 1, i) = d(i)$, then $d(s', i) = d(i)$ for all $s' \geq s$. It follows from the definition of $y(s, m, e)$ and the fact that 0 is not the Gödel number of a deduction that

$$(s)(m)[(s \geq s^* \ \& \ n > m \geq e) \rightarrow y(s, m, e) = y(m) > 0].$$

We define

$$E(e, n) = \begin{cases} 0 & \text{if } (Es) \{y(s, n, e) > 0 \ \& \ n < m(s, e) \ \& \ s > s^* \ \& \\ & (i)_{i < e} [(m)(m < L(i) \rightarrow d(s - 1, p_i^m) = d(p_i^m)) \\ & \ \& \ (m)(y(s, n, e) > m \geq L(i) \rightarrow d(s - 1, p_i^m) \\ & \quad = d(p_i^{L(i)})]\}, \\ 1 & \text{otherwise.} \end{cases}$$

To verify $(1, e + 1, n + 1)$, it suffices to prove

$$E(e, n) = 0 \leftrightarrow [n \geq e \ \& \ (w)(Es)(s > w \ \& \ m(s, e) > n) \\ \& \ (m)(n \geq m \geq e \rightarrow (Ey)T_1^1(\tilde{d}(y), e, m, y))].$$

Suppose $E(e, n) = 0$. Then Case 2 of the definition of $E(e, n)$ must hold. Let s be the natural number whose existence is required by the fact $E(e, n) = 0$; thus

$$y(s, n, e) > 0 \ \& \ n < m(s, e) \ \& \ s > s^*.$$

It is our aim now to prove

$$y(s, n, e) = y(s', n, e) \ \& \ n < m(s', e)$$

for all $s' \geq s$ by means of an induction on s' . Fix $s' > s$ and suppose

$$y(s, n, e) = y(s' - 1, n, e) \ \& \ n < m(s' - 1, e).$$

Suppose for the sake of a *reductio ad absurdum* that there is a w such that $d(s' - 1, w) \neq d(s' - 2, w)$ and $w < y(s' - 1, n, e)$. Then there must be an i and an m such that (0) and (1) are true:

$$(0) \quad d(s' - 1, p_i^m) \neq d(s' - 2, p_i^m) \ \& \ p_i^m < y(s' - 1, n, e) = y(s, n, e);$$

$$(1) \quad (i')_{i' < i(m')} [d(s' - 1, p_{i'}^{m'}) = d(s' - 2, p_{i'}^{m'}) \vee p_{i'}^{m'} \geq y(s' - 1, n, e)].$$

If $i < e$, then it follows from the second half of (0), the definition of s and (2, e) that $d(s - 1, p_i^m) = d(p_i^m)$; but this last contradicts the first half of (0), since $s' > s$. Thus $i \geq e$. Since $e \leq n < m(s' - 1, e)$, it follows from (0) and the definition of $d(s' - 1, p_i^m)$ that $r(s' - 1, n, e) = 0$. This last means there is an i' , an m' and a t such that

$$i' < e \leq t \leq n \ \& \ p_{i'}^{m'} < y(s' - 1, t, e) \ \& \ d(s' - 1, p_{i'}^{m'}) \neq d(s' - 2, p_{i'}^{m'}).$$

If $t = n$, this last contradicts (1), since $i' < e \leq i$. Thus $t < n$, and

$$p_{i'}^{m'} < y(s' - 1, t, e) = y(s^*, t, e) \leq y^*,$$

since $s' > s > s^*$ and $e \leq t < n$. But this is absurd, since

$$d(s' - 1, p_{i'}^{m'}) = d(s' - 2, p_{i'}^{m'}) = d(s - 1, p_{i'}^{m'}) = d(p_{i'}^{m'})$$

is a consequence of the fact that $s' > s > s^*$ and $p_{i'}^{m'} < y^*$.

Since $d(s' - 1, w) = d(s' - 2, w)$ for all $w < y(s' - 1, n, e)$, and since $y(s' - 1, n, e) = y(s, n, e) > 0$, it must be that

$$y(s', n, e) = y(s' - 1, n, e) = y(s, n, e).$$

Then we have

$$(m) [n \geq m \geq e \rightarrow y(s', n, e) = y(s' - 1, n, e) > 0],$$

since $s' > s > s^*$. It follows that either Case 2 or Case 3 of the definition of $m(s', e)$ holds. If Case 2 of the definition of $m(s', e)$ holds, then it is clear $n < m(s', e)$. If Case 3 holds, then $n < m(s', e)$ because $n < m(s' - 1, e)$.

Thus we have shown that

$$y(s', n, e) = y(s, n, e) > 0 \ \& \ n < m(s', e)$$

for all $s' \geq s$. It follows immediately that $(Ey)T_1^1(\tilde{d}(y), e, n, y)$ and

$$(w)(Es)(s > w \ \& \ m(s, e) > n).$$

Note that

$$(m)(n > m \geq e \rightarrow (Ey)T_1^1(\check{d}(y), e, m, y))$$

is a consequence of the fact that Case 2 of the definition of $E(e, n)$ holds. That completes the first half of the verification of $(1, e + 1, n + 1)$; in order to verify the second half, we suppose

$$(w)(Es)(s > w \ \& \ m(s, e) > n) \ \& \ (m)(n \geq m \geq e \rightarrow (Ey)T_1^1(\check{d}(y), e, m, y)),$$

and then show $E(e, n) = 0$. It follows from $(1, e + 1, n)$ that Case 2 of the definition of $E(e, n)$ holds. Let v be so large that $v > L(i)$ for all $i < e$. Let $z = \mu y T_1^1(\check{d}(y), e, n, y)$. Let w be so large that $w > z$ and

$$d(w - 1, t) = d(t)$$

for all $t < p_e^{z+v}$. Let s be such that $s > w + s^*$ and $m(s, e) > n$. It follows easily from $(2, e)$ that s has the properties required to conclude $E(e, n) = 0$; note that $y(s, n, e) = z > 0$.

The definition of $L(e)$ has two cases:

Case 1. $c(e) = 0$. Then by $B(e)$, $\lim_m d(p_e^m) = 1$. We set

$$L(e) = \mu t(s)(m)[m \geq t \rightarrow d(s, p_e^m) = 1].$$

Case 2. $c(e) = 1$. It is a consequence of $(1, e + 1)$ and of the definition of $y(s, n, i)$ that for each $i \leq e$ and each n ,

$$E(i, n) = 0 \rightarrow [(Es)(m(s, i) > n) \ \& \ \lim_s y(s, n, i) \text{ exists and is positive}].$$

For each $i \leq e$, it follows from $A(i)$ that there is a t such that $t \geq i$ and $E(i, t) = 1$. For each $i \leq e$, let

$$t_i = \mu t(E(i, t) = 1 \ \& \ t \geq i).$$

It follows from $(1, e + 1)$ that for each $i \leq e$, t_i satisfies either (2) or (3):

(2) $(Ew)(s)(s > w \rightarrow m(s, i) \leq t_i)$;

(3) $\lim_s y(s, t_i, i)$ does not exist or is equal to 0.

Note that since $E(i, t) = 0$ whenever $t_i > t \geq i$, it follows from $(1, e + 1)$ that

$$(m)(t_i > m \geq i \rightarrow \lim_s y(s, m, i) \text{ exists and is positive}).$$

But then

$$(Ey)(i)(m)(s)[(i \leq e \ \& \ t_i > m \geq i \ \& \ s \geq 0) \rightarrow y(s, m, i) \leq y];$$

let $L(e)$ be the least such y . We now verify $(2, e + 1)$. What follows is similar to the proof of Lemma 6. Fix $n \geq L(e)$. We must show $d(p_e^n) = \lim_m d(p_e^m)$. If Case 1 of the definition of $L(e)$ holds, there is nothing to prove. Suppose Case 2 of the definition of $L(e)$ holds. Then $c(e) = 1$, and by $B(e)$, $\lim_m d(p_e^m) = 0$. In order to

show $d(p_e^n) = 0$, it suffices to find an s such that $d(s, p_e^n) = 0$. Let k be such that if $i \leq e$ and i is not stable, then $i = e_j$ for some $j < k$. Let w be so large that for all $i \leq e$, if (2) holds, then

$$(s)(s > w \rightarrow m(s, i) \leq t_i).$$

By the same argument as in Lemma 6, there is a $v > w$ such that

$$(s)(s \geq v \rightarrow h(s, e) > n).$$

By Lemma 4, there is an $s \geq v$ such that

$$(j)_{j < k} [m(s, e_j) \leq n_j \vee r(s, n_j, e_j) = 0 \vee y(s, n_j, e_j) = 0].$$

If we can show for each i and m that

$$(i \leq e \& i \leq m < m(s, i)) \rightarrow (r(s, m, i) = 0 \vee p_e^n \geq y(s, m, i)),$$

then it will be clear that $d(s, p_e^n) = 0$. Fix i and m so that $i \leq e$ and $i \leq m < m(s, i)$. Suppose (2) holds. Then $m(s, i) \leq t_i$, since $s \geq v > w$. But then $t_i > m \geq i$, and consequently,

$$p_e^n > n \geq L(e) \geq y(s, m, i).$$

Now suppose (3) holds. Then i is not stable, and there is a $j < k$ such that $e_j = i$; in addition, $n_j = t_i$. If $m < n_j = t_i$, then $p_e^n \geq y(s, m, i)$, since $n \geq L(e)$. Suppose $m \geq n_j$. Then $m(s, i) > n_j$. This last means either

$$r(s, n_j, e_j) = 0 \quad \text{or} \quad y(s, n_j, e_j) = 0.$$

If $r(s, n_j, i) = 0$, then by remark (R2), $r(s, m, i) = 0$, since $m \geq n_j$. Suppose $y(s, n_j, i) = 0$. Since $n_j \leq m < m(s, i)$, it follows from remark (R3) that $n_j = i$. But then $y(s, i, i) = 0$, $m(s, i) = i + 1$, $m = i$ and $y(s, m, i) = 0 < p_e^n$.

Inspection of the definitions of E and L readily reveals they are recursive in c . We make some informal remarks to indicate how to write equations defining E and L recursively in c . Fix $e \geq 0$ and $n \geq e$, and consider the definition of $E(e, n)$. The choice between Case 1 and Case 2 can be made effectively once the values of $E(e, t) (t < n)$ are known. Suppose Case 2 holds. The values of y^* and s^* are defined by means of a predicate recursive in d . Then the value of $E(e, n)$ is found from the values of $d(p_i^m) (m \leq L(i) \text{ and } i < e)$ by means of a predicate of degree less than or equal to $0'$. Thus the value of $E(e, n)$ can be expressed in terms of the values of $E(e, t) (t < n)$ and $L(i) (i < e)$ with the aid of a predicate of degree less than or equal to $\mathbf{d} \cup 0'$. Similarly, the value of $L(e)$ can be expressed in terms of the values of $E(i, n) (i \leq e \text{ and } n \geq 0)$ with the aid of a predicate of degree $\mathbf{c} \cup 0'$. Then $\mathbf{d}' \leq \mathbf{c}$, since $\mathbf{d}'$ is the degree of the function $E(e, e)$, and since $\mathbf{d} \cup 0' \cup \mathbf{c} = \mathbf{c}$.

When $\mathbf{b} > 0$, the changes needed in the above argument are largely notational. The notion of recursiveness is replaced throughout by the notion of recursiveness in a function of degree $\mathbf{b}$. The arguments contained in Lemmas 1–8 are retained

unaltered save for relativization to a function of degree $\mathbf{b}$. The functions $a(s, n)$, $f(s, n)$ and $d(s, n)$ are now recursive in a function of degree $\mathbf{b}$. We set $d(s, 2 \cdot 3^n) = b(n)$ for all s and all $n > 0$, where b has degree $\mathbf{b}$, in order to insure $\mathbf{b} \leq \mathbf{d}$. The value of $E(e, n)$ is now obtained from the values of $E(e, t)$ ($t < n$) and $L(i)$ ($i < e$) with the aid of a predicate of degree less than or equal to $\mathbf{b}' \cup \mathbf{d}$.

COROLLARY 1. *If $\mathbf{b}$ and $\mathbf{c}$ are degrees, then the following conditions are equivalent:*

- (i) $\mathbf{b}' \leq \mathbf{c} \leq \mathbf{b}''$ and $\mathbf{c}$ is recursively enumerable in $\mathbf{b}'$;
- (ii) there is a $\mathbf{d}$ such that $\mathbf{b} \leq \mathbf{d} \leq \mathbf{b}'$ and $\mathbf{d}' = \mathbf{c}$;
- (iii) there is a $\mathbf{d}$ such that $\mathbf{b} \leq \mathbf{d} \leq \mathbf{b}'$, $\mathbf{d}$ is recursively enumerable in $\mathbf{b}$ and $\mathbf{d}' = \mathbf{c}$.

For each degree $\mathbf{b}$, let $R_{\mathbf{b}}$ denote the set of all degrees greater than or equal to $\mathbf{b}$, recursively enumerable in $\mathbf{b}$ and less than or equal to $\mathbf{b}'$. Let j denote the jump operator. Then Corollary 1 tells us that the order-preserving map

$$j: R_{\mathbf{b}} \rightarrow R_{\mathbf{b}'},$$

is onto. It also follows from Theorem 1 that any element of $R_{\mathbf{b}'}$ greater than $\mathbf{b}'$ is the image of more than one element of $R_{\mathbf{b}}$; Friedberg (result unpublished) has shown that $\mathbf{b}'$ does not have a unique pre-image in $R_{\mathbf{b}}$. We do not know if $R_{\mathbf{b}}$ and $R_{\mathbf{b}'}$ are order-isomorphic, but we conjecture that they are. We can show (announced in [6] for $\mathbf{b} = \mathbf{0}$): for any degree $\mathbf{b}$, $R_{\mathbf{b}}$ is a universal, countable partial ordering.

COROLLARY 2. *There exists a recursively enumerable degree $\mathbf{d}$ such that $\mathbf{d} < \mathbf{0}' < \mathbf{0}'' = \mathbf{d}'$.*

Proof. Let $\mathbf{b} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}''$ and $\mathbf{a} = \mathbf{0}'$, and apply Theorem 1 to obtain $\mathbf{d}$. Then $\mathbf{d}$ is recursively enumerable, $\mathbf{0}' \not\leq \mathbf{d}$ and $\mathbf{d}' = \mathbf{0}''$.

Note that Corollary 2 provides still another solution to Post's problem.

COROLLARY 3. *For each degree $\mathbf{b}$ and each natural number n , there is a degree $\mathbf{d}$ recursively enumerable in $\mathbf{b}$ such that*

$$\mathbf{b} < \mathbf{d} < \mathbf{b}' < \mathbf{d}' < \mathbf{b}'' < \dots < \mathbf{b}^{(n)} < \mathbf{d}^{(n)} < \mathbf{b}^{(n+1)}.$$

Proof. We know from [2] that there exists a degree $\mathbf{g}$ such that $\mathbf{g}$ is recursively enumerable in $\mathbf{b}^{(n)}$ and $\mathbf{b}^{(n)} < \mathbf{g} < \mathbf{b}^{(n+1)}$. By Theorem 1, there is a degree $\mathbf{h}_1$ such that $\mathbf{h}_1$ is recursively enumerable in $\mathbf{b}^{(n-1)}$, $\mathbf{b}^{(n-1)} < \mathbf{h}_1 < \mathbf{b}^{(n)}$ and $\mathbf{h}_1' = \mathbf{g}$. By making $n - 1$ further applications of Theorem 1, we obtain degrees $\mathbf{h}_2, \mathbf{h}_3, \dots, \mathbf{h}_n$ such that for $2 \leq i \leq n$, $\mathbf{h}_i$ is recursively enumerable in $\mathbf{b}^{(n-i)}$,

$$\mathbf{b}^{(n-i)} < \mathbf{h}_i < \mathbf{b}^{(n-i+1)} \quad \text{and} \quad \mathbf{h}_i' = \mathbf{h}_{i-1}.$$

Let $\mathbf{d} = \mathbf{h}_n$. Then $\mathbf{d}$ is recursively enumerable in $\mathbf{b}$, and for all $i \leq n$, $\mathbf{b}^{(i)} < \mathbf{d}^{(i)} < \mathbf{b}^{(i+1)}$.

Corollary 3 improves a result of Shoenfield [7]; he showed that for each degree $\mathbf{b}$, there is a degree $\mathbf{d}$ such that $\mathbf{b} < \mathbf{d} < \mathbf{b}' < \mathbf{d} < \mathbf{b}''$. We do not know if for any degree $\mathbf{b}$ there exists a degree $\mathbf{d}$ such that for all $n \geq 0$,

$$\mathbf{b}^{(n)} < \mathbf{d}^{(n)} < \mathbf{b}^{(n+1)};$$

if for some $\mathbf{b}$, such a $\mathbf{d}$ exists, then by Theorem 1, $\mathbf{d}$ can be given the additional property of recursive enumerability in $\mathbf{b}$.

Theorem 1 can be extended without any radical alteration of its proof. For example, we can show: if $\mathbf{g}$ is a recursively enumerable degree such that $\mathbf{g}' < \mathbf{0}''$, then there is a recursively enumerable degree $\mathbf{d}$ such that

$$\mathbf{g} < \mathbf{d} < \mathbf{0}' < \mathbf{0}'' = \mathbf{d}'.$$

We say a sequence $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \dots$ of degrees is simultaneously recursively enumerable if there is a sequence $A_0, A_1, A_2, \dots$ of simultaneously recursively enumerable sets such that $\mathbf{a}_i$ is the degree of A_i for all i . Using the method underlying the proof of Theorem 1, we can show: if $\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \dots$ is an infinite, ascending sequence of simultaneously recursively enumerable degrees, then there exists a recursively enumerable degree $\mathbf{d}$ such that

$$\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \dots < \mathbf{d} < \mathbf{0}'.$$

We end with a conjecture: the upper semi-lattice of recursively enumerable degrees is dense (i.e., if $\mathbf{b}$ and $\mathbf{c}$ are recursively enumerable degrees such that $\mathbf{b} < \mathbf{c}$, then there exists a recursively enumerable degree $\mathbf{d}$ such that $\mathbf{b} < \mathbf{d} < \mathbf{c}$). The only evidence we have to offer in favor of this conjecture is contained in the results we announced above and the result of Muchnik [5] that there is no minimal, nonzero, recursively enumerable degree⁽²⁾.

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(2) *Added in proof.* Our conjecture is true; we will give a proof elsewhere.