SOME SUBGROUP THEOREMS FOR FREE $\mathfrak{v}$-GROUPS (1)

BY

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1. **Introduction.**

1.1. This paper is concerned with some analogues of the theorem which asserts that every subgroup of a free group is free (Nielsen [1], Schreier [2], Levi [3]). These analogues arise naturally if one considers certain nonempty classes $\mathfrak{v}$ of groups which are closed with respect to the formation of factor groups, subgroups and cartesian products. Such classes of groups are called *varieties.* Now a group G is said to be a free $\mathfrak{v}$-group if

(i) G is a $\mathfrak{v}$-group, i.e., G is in $\mathfrak{v}$, and

(ii) there is a set X of generators of G such that for every $\mathfrak{v}$-group H and every mapping μ of X into H there is a homomorphism η of G into H which coincides with μ on X. It is with the subgroups of certain free $\mathfrak{v}$-groups that our interests lie.

1.2. Following P. Hall [4] let us call a set X of generators of a free $\mathfrak{v}$-group which satisfies (ii) (above) a *canonical* set of generators. The main theorem of this paper deals with a free $\mathfrak{v}$-group G which is residually a finite p-group for an infinite set of primes p (2). It is easy to prove that G/G' is free abelian, where here G' denotes the commutator subgroup of G. If one considers a subset Y of G which freely generates, modulo G', a free abelian group, then it turns out that the group H generated by Y is a free $\mathfrak{v}$-group; moreover Y is a canonical set of generators of H. This is the main theorem cited above — it will be proved in §2 as Theorem 1.

1.3. Finitely generated torsion-free nilpotent groups are residually finite p-groups for every prime p (Gruenberg [5]). It follows that a torsion-free nilpotent free $\mathfrak{v}$-group G is residually a finite p-group for every prime p and so Theorem 1 applies to such groups G. This fact constitutes a far reaching generalisation of the corresponding theorem for free nilpotent groups (cf. e.g. Gruenberg [5] for the relevant definition) proved by Malcev [14] and generalised by Gol'dina [6]. It is perhaps worth noting that Malcev's theorem follows easily from the representation of a free nilpotent group in a free nilpotent ring (cf. e.g. M. Hall [7]) about which much is known.

Another source of free $\mathfrak{v}$-groups which satisfy the requirements of Theorem 1

Received by the editors August 2, 1962.

(1) Supported by Grant G 19674 from the National Science Foundation.

(2) If $\mathscr{P}$ is a property pertaining to groups, then P. Hall [4] defines a group G to be residually $\mathscr{P}$ if to each x in G ($x \neq 1$) there corresponds a normal subgroup N_x of G, which does not contain x, such that G/N_x is $\mathscr{P}$.

are the *free polynilpotent* groups of Gruenberg [5] (cf. [5] for an explanation of the terms used here). In particular, Theorem 1 applies to free soluble groups; indeed we shall completely determine all the free soluble subgroups of a free soluble group by employing a theorem of Malcev [8] and Theorem 1. To be precise, we show that H is a free soluble subgroup of the free soluble group G if and only if we can find a set Y of generators of H which generate freely, modulo some term of the derived series of G, a free abelian group. This will be proved in §3 as Theorem 2. We hope to return to some generalisations of this theorem at a later date.

It is perhaps worth explaining a way of getting a new supply of free $\mathfrak{v}$-groups which are residually finite p-groups for infinitely many primes p. This is connected with the product variety $\mathfrak{u}\mathfrak{w}$ of two varieties $\mathfrak{u}$ and $\mathfrak{w}$, introduced by Hanna Neumann [9]. We recall that if G denotes a group, then

$$\mathfrak{u}\mathfrak{w} = \{G \mid \text{there exists } N \lhd G \text{ with } G/N \in \mathfrak{w} \text{ and } N \in \mathfrak{u}\}.$$

In a forthcoming work (Baumslag [15]) we shall show, by refining some important new work of B. H. Neumann, Hanna Neumann and Peter M. Neumann [10], that if the free groups in $\mathfrak{u}$ and the free groups in $\mathfrak{v}$ are residually finite p-groups for the same prime p, then so also are the free groups in $\mathfrak{u}\mathfrak{v}$ (notice that this theorem includes the theorem of Gruenberg on free polynilpotent groups). So Theorem 1 applies also here.

Finally we turn our attention to varieties $\mathfrak{w}\mathfrak{x}$, where $\mathfrak{w}$ is any variety and $\mathfrak{x}$ is such that the free $\mathfrak{x}$-groups are residually finite p-groups for an infinite set of primes p. We shall prove, Theorem 3 in §4, that if G is a free $\mathfrak{w}\mathfrak{x}$-group with a set X of canonical generators, then the subgroup H generated by

$$Y = \{x^{r_x} \mid x \in X,\ r_x \text{ a nonzero integer}\}$$

is a free $\mathfrak{w}\mathfrak{x}$-group; moreover Y is a canonical set of generators of H. It is interesting to notice that here Y is a special set of elements which freely generates, modulo G', a free abelian group. The natural extension is presumably true; however the method adopted here is of no use in this more general situation.

It is a pleasure to acknowledge that this work has benefited from a correspondence with K. W. Gruenberg.

2. **The proof of Theorem 1.**

2.1. We begin by recalling the following fact.

LEMMA 1. *Let $\mathfrak{v}$ be a variety of groups and let G be a free $\mathfrak{v}$-group. Further, let Y be a subset of G and let H be the subgroup generated by Y. Then H is a free $\mathfrak{v}$-group and Y is a canonical set of generators of H if and only if every finite subset of Y generates a free $\mathfrak{v}$-group and is a canonical set of generators of the group it generates.*

The proof of Lemma 1 is straightforward and is omitted.

2.2. Next we prove the useful

LEMMA 2. *Let $\mathfrak{v}$ be a variety of groups and let G be a free $\mathfrak{v}$-group. If G is residually a p-group for infinitely many primes p, then G/G' is a free abelian group. Moreover if X is any canonical set of generators of G, then X freely generates, modulo G', the free abelian group G/G'.*

Proof. Let $a \in X$. If the prime q divides the order of a, then every normal subgroup N of G with a p-group G/N as factor group $(p \neq q)$ will contain $a^{n/q}$, where n is the order of a, contradicting the hypothesis. So every element of X is of infinite order. Hence $\mathfrak{v}$ contains $\mathfrak{a}$, the variety of all abelian groups. Therefore G/G' is free abelian and X has the required property.

2.3. Before proceeding to the proof of Theorem 1, it is perhaps worth while to state it explicitly here.

THEOREM 1. *Let $\mathfrak{v}$ be a variety of groups. Let G be a free $\mathfrak{v}$-group and let Y be a subset of G which freely generates, modulo G', a free abelian group. Further, let H be the subgroup generated by Y. If G is residually a finite p-group for an infinite set of primes p, then H is a free $\mathfrak{v}$-group; in addition, Y is a canonical set of generators of H.*

Proof. By Lemma 1 we may suppose G is finitely generated. Thus suppose

$$a_1, a_2, \cdots, a_m$$

is a canonical set of generators of G. Then, by Lemma 2, $a_1, a_2, \cdots, a_m$ freely generate, modulo G', the free abelian group G/G'. Consequently, on invoking the basis theorem for free abelian groups (cf. e.g. Kurosh [11, Vol. 1, p. 149]), we find that we can choose m elements

$$b_1, b_2, \cdots, b_m$$

of G such that, first,

$$b_1, b_2, \cdots, b_m \text{ generate } G \text{ modulo } G' \tag{1}$$

and, second,

$$HG'/G' \text{ is a subgroup of finite index, say } j, \text{ in } gp(b_1, b_2, \cdots, b_k)G'/G'. \tag{2}$$

Notice that (2) implies $|Y| = k$; thus $Y = \{c_1, c_2, \cdots, c_k\}$, say.

We now enlarge Y slightly to

$$\tilde{Y} = \{c_1, c_2, \cdots, c_k, c_{k+1} = b_{k+1}, \cdots, c_m = b_m\}. \tag{3}$$

Now let η be the endomorphism of G defined by

$$a_1\eta = c_1,\ a_2\eta = c_2,\ \cdots,\ a_m\eta = c_m. \tag{4}$$

We shall show that η is a monomorphism.

For suppose the contrary. Then there is an element $u \in G$ such that

(5) $$u\eta = 1 \qquad (u \neq 1).$$

Now G is residually a finite p-group for infinitely many primes p. Hence we can choose a prime q and a normal subgroup N of G such that

(6) $$u \notin N, |G/N| = q^l \text{ and } (q, j) = 1.$$

By virtue of the fact that finite q-groups are nilpotent, G/N is nilpotent, of class c, say.

We now put

$$L = G_c \cdot G^{q^l},$$

where here G^{q^l} is the subgroup generated by q^lth powers of elements of G and G_c is the cth term of the lower central series of G. It follows (cf. (6)) that

$$L \leqq N.$$

Therefore, by (6),

(7) $$u \notin L.$$

We consider now

$$G^* = G/L;$$

we denote the natural homomorphism of G onto G^* by v. Then we see by (7) that

(8) $$uv \neq 1.$$

However since $u\eta = 1$ we have

(9) $$u\eta v = 1.$$

We notice, in addition, that G^* is a finite q-group and it is nilpotent of class c. Clearly

$$a_1 v,\ a_2 v,\ \cdots,\ a_m v$$

generate G^*. Moreover $c_1 v, c_2 v, \cdots, c_m v$ generate, modulo $(G^*)'$, a subgroup of G^*, which is simultaneously of index dividing j and q^l (cf. (1), (2), (3), (6)). Since $(q, j) = 1$ (by (6)), it follows that $c_1 v, c_2 v, \cdots, c_m v$ generate G^* modulo $(G^*)'$. So $c_1 v, c_2 v, \cdots, c_m v$ generate G^* itself. But G^* may be thought of as a free $\mathfrak{u}$-group, where $\mathfrak{u}$ is the variety of nilpotent groups in $\mathfrak{v}$ of exponent dividing q^l and class at most c. Clearly, then, $a_1 v, a_2 v, \cdots, a_m v$ are a canonical set of generators of G^*. Hence the mapping

$$\eta^* : a_i v \to c_i v \qquad (i = 1, 2, \cdots, m)$$

defines an endomorphism of G^*. But G^* is finite and the $c_i v$ generate G^*; so η^* is in fact an *automorphism* of G^* and hence η^* is certainly one-to-one. In particular, then (cf.(8)),

(10) $$uv\eta^* \neq 1.$$

However it is easy to see that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta} & G \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ G^* & \xrightarrow[\eta^*]{} & G^* \end{array}$$

is commutative. Therefore

$$u\eta\nu = u\nu\eta^*.$$

But (9) and (10) now stand in direct contradiction. Thus our initial assumption concerning η is invalid, i.e., η is a monomorphism. This ensures that $c_1, c_2, \cdots, c_m$ generate a free $\mathfrak{v}$-group (cf.(4)) and that $c_1, c_2, \cdots, c_m$ are a canonical set of generators of this group. But this yields immediately that H is a free $\mathfrak{v}$-group and that Y is a canonical set of generators of H (cf. (3)).

3. Some applications of Theorem 1.

3.1. The proof of the following lemma is straightforward — it depends only on the fact that a subgroup of a free group is free, and on the relation

$$(G/N)' = G'N/N,$$

for a group G with normal subgroup N. Consequently the details are left to the reader.

LEMMA 3. *Let G be a free soluble group. Then every member of the derived series of G is again a free soluble group.*

Besides Lemma 3 we need another preparatory lemma.

LEMMA 4. *Let G be a free soluble group. Then every noncyclic abelian subgroup of G is contained in the last nontrivial term of the derived series of G,*

Lemma 4 is a direct consequence of a theorem of Malcev [8] (cf. Auslander and Lyndon [12]).

3.2. The following proposition is a straightforward application of Lemma 4.

PROPOSITION 1. *Let G be a free soluble group of derived length d and let H be a free soluble subgroup of G of derived length e. If $e > 1$ and $G^{(s)}$ denotes the sth term of the derived series of G, then*

(1) $$H \leqq G^{(s)}, \quad s = d - e$$

and

(2) $$H^{\,i} = H \cap G^{(s+i)} \qquad (i = 1, 2, \cdots, e).$$

Proof. We are assured by Lemma 4, of the inequality

(3) $$H \cap G^{(d-1)} \leqq H^{(e-1)}.$$

On the other hand, by Lemma 4,

$$H^{(e-1)} \leqq G^{(d-1)}.$$

Therefore,

(4) $$H \cap G^{(d-1)} \geqq H \cap H^{(e-1)} = H^{(e-1)}.$$

Putting (3) and (4) together yields

(5) $$H \cap G^{(d-1)} = H^{(e-1)}.$$

After utilising Lemma 3 and induction, we find that

$$HG^{(d-1)} \leqq G^{(s)}.$$

Therefore,

$$H \leqq G^{(s)}$$

and we have proved (1).

Again, by Lemma 3, we may apply induction, yielding

$$HG^{(d-1)} \cap G^{(s+i)} \leqq H^{(i)}\, G^{(d-1)} \qquad (i = 1, 2, \cdots, e-1).$$

Hence

(6) $$H \cap G^{(s+i)} \leqq H \cap H^{(i)}\, G^{(d-1)} \qquad (i = 1, 2, \cdots, e-1).$$

But, for $i = 1, 2, \cdots, e-1$,

(7) $$H \cap H^{(i)}\, G^{(d-1)} = H^{(i)}.$$

To see this suppose $u \in H \cap H^{(i)}G^{(d-1)}$. Then

$$u = hg \quad (h \in H^{(i)}, g \in G^{(d-1)}).$$

Since $u \in H$, we find (cf. (5))

$$g \in G^{(d-1)} \cap H = H^{(e-1)}.$$

So $u \in H^{(i)}$ which establishes (7). But this yields, via (6),

(8) $$H \cap G^{(s+i)} \leqq H^{(i)} \qquad (i = 1, 2, \cdots, e-1).$$

On the other hand, $H \leqq G^{(s)}$ (by (1)): therefore $H^{(i)} \leqq G^{(s+i)}$. Hence

(9) $$H \cap G^{(s+i)} \geqq H \cap H^{(i)} = H^{(i)} \qquad (i = 1, 2, \cdots, e-1).$$

Putting (8) and (9) together we have the remaining part required to complete the proof of Proposition 1.

3.3. We come now to the proof of

THEOREM 2. *A subgroup H of a free soluble group G is itself a free soluble* *group if and only if there exists a set Y of generators of H which freely generates, modulo some term of the derived series of G, a free abelian group.*

Proof. This has been made easy by the preceding lemmas. For on the one hand if H is a subgroup of a free soluble group G with a set Y of generators such that Y freely generates a free abelian group modulo some term, say $G^{(s+1)}$, of the derived series of G, then it follows from Lemma 4 and Lemma 3 that Y is contained in $G^{(s)}$. On applying Lemma 3 and Theorem 1 we can deduce that H is a free soluble group and Y is a canonical set of generators of H. On the other hand if H is a free soluble subgroup of G, then the existence of a set Y of canonical generators for H with the required property follows immediately from Proposition 1. This then completes the proof of Theorem 2.

4. **The proof of Theorem 3.**

4.1. We shall need some of the usual commutator-calculus notation. Thus we put

$$[x, x_2] = x_1^{-1} \ x_2^{-1} \ x_1 x_2,$$

and define inductively

$$[x_1, x_2, \cdots, x_n] = [[x_1, x_2, \cdots, x_{n-1}], x_n] \qquad (n > 2);$$

here $x_1, x_2, \cdots, x_n$ are, of course, elements from some group.

The following lemma is the key to the proof of Theorem 3.

LEMMA 5. *Let R be a free group of finite rank k which is freely generated by $x_1, x_2, \cdots, x_k$ and let S be the subgroup of R generated by*

$$x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k},$$

where the n_i $(i = 1, 2, \cdots, k)$ are nonzero integers. Then S' is a free factor of R', i.e., R' is a free product of the form

$$R' = S' * T,$$

for some subgroup T of R'.

Proof. We observe that if Y is a free group freely generated by $y_1, y_2, \cdots, y_s$ then Y' is freely generated by the commutators

$$[y_{i_1}^{u_1}, y_{i_2}^{u_2}, \cdots, y_{i_t}^{u_t}]$$

where (i) $u_1, u_2, \cdots, u_t$ are nonzero integers, (ii) $i_1, i_2, \cdots, i_t$ are all distinct and (iii) $i_1 > i_2$, $i_2 < i_3 < \cdots < i_t$ and $i_1, i_2, \cdots, i_t \leqq s$ (Gruenberg [5]).

It follows immediately that we can find a set of free generators of R' and a set of free generators of S' so that the second is a subset of the first. Lemma 5 follows immediately from this observation.

4.2. Let R be a group and let $\mathfrak{u}$ be a variety of groups. Then we denote by $\mathfrak{u}(R)$ the intersection of all normal subgroups N of R whose quotient R/N lies in $\mathfrak{u}$. Clearly then $R/\mathfrak{u}(R)$ is in $\mathfrak{u}$ and indeed $R/\mathfrak{u}(R)$ is a free $\mathfrak{u}$-group whenever R is itself an ordinary free group.

We remind the reader that a finitely generated residually finite group is hopfian (Malcev [13]), i.e., it has no proper isomorphic quotient groups. This fact will be useful in the proof of

LEMMA 6. *Let $\mathfrak{x}$ be a variety of groups. Let R be a free group freely generated by $x_1, x_2, \cdots, x_k$ and let*

$$S = gp(x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k}) \qquad (n_i \neq 0,\ i = 1, 2, \cdots, k).$$

If $R/\mathfrak{x}(R)$ is residually a finite p-group for infinitely many primes p, then

$$S \cap \mathfrak{x}(R) = \mathfrak{x}(S).$$

Proof. The group $R/\mathfrak{x}(R)$ is a free $\mathfrak{x}$-group; moreover Theorem 1 applies. Thus

$$S\mathfrak{x}(R)/\mathfrak{x}(R) \cong R/\mathfrak{x}(R). \tag{1}$$

We have also

$$S\mathfrak{x}(R)/\mathfrak{x}(R) \cong S/\mathfrak{x}(R) \cap S. \tag{2}$$

Now S is a free group freely generated by $x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k}$; therefore (cf. (1),(2))

$$S/\mathfrak{x}(S)(\cong R/\mathfrak{x}(R)) \cong S/\mathfrak{x}(R) \cap S. \tag{3}$$

It is clear that

$$\mathfrak{x}(R) \cap S \geqq \mathfrak{x}(S).$$

In view of (3) and the hopficity of $S/\mathfrak{x}(S)$ it follows that this inequality is of necessity an equality. This completes the proof of Lemma 6.

The following consequence of Lemma 6 seems worth stating as a proposition; here we assume the notation and assumptions of Lemma 6.

PROPOSITION 2. *If $\mathfrak{x}$ is nontrivial, then $\mathfrak{x}(S)$ is a free factor of $\mathfrak{x}(R)$.*

Proof. Suppose F is a free group. Then $F/\mathfrak{x}(F)$ is a free $\mathfrak{x}$-group. By Lemma 2 it follows that

$$F/F' \cong (F/\mathfrak{x}(F))/(F/\mathfrak{x}(F))' \cong F/F'\mathfrak{x}(F).$$

Therefore,

$$\mathfrak{x}(F) \leqq F'. \tag{1}$$

It follows from (1) that $\mathfrak{x}(S)$ is a subgroup of S'. Now S' is a free factor of R' (Lemma 5). Hence, by the Kurosh subgroup theorem for free products (cf. Kurosh

[11, Vol. 2, p. 17]), $S' \cap \mathfrak{x}(R)$ is a free factor of $\mathfrak{x}(R)$, since $\mathfrak{x}(R)$ is a subgroup of of R' by (1). But

$$S' \cap \mathfrak{x}(R) = S' \cap (S \cap \mathfrak{x}(R)) = S' \cap \mathfrak{x}(S) = \mathfrak{x}(S)$$

by Lemma 6 and (1). In other words $\mathfrak{x}(S)$ is a free factor of $\mathfrak{x}(R)$.

4.3. We need only one further lemma before we are ready to prove Theorem 3.

LEMMA 7. *Let F be a free product of its subgroups U and V:*

$$F = U * V.$$

Then, if $\mathfrak{w}$ is any variety of groups,

$$U \cap \mathfrak{w}(F) = \mathfrak{w}(U).$$

Proof. Let μ be the natural mapping of F onto $U/\mathfrak{w}(U)$. If

$$u \in U \cap \mathfrak{w}(F),$$

then as the kernel of μ clearly contains $\mathfrak{w}(F)$,

$$u\mu = 1.$$

But since $u \in U$,

$$u\mu = u\mathfrak{w}(U).$$

Therefore $u \in \mathfrak{w}(U)$ and we have proved

$$U \cap \mathfrak{w}(F) \leqq \mathfrak{w}(U).$$

The reverse inequality is obvious and so the proof of Lemma 7 is complete.

4.4. We come now to the proof of Theorem 3.

THEOREM 3. *Let $\mathfrak{w}$ and $\mathfrak{x}$ be varieties of groups. Suppose $\mathfrak{x}$ is nontrivial and that the free $\mathfrak{x}$-groups are residually finite p-groups for an infinite set of primes p. Suppose also that X is a canonical set of generators of G, a free $\mathfrak{w}\mathfrak{x}$-group, and that $\{r_x\}$ is a set of nonzero integers in one-to-one correspondence with the elements $x \in X$. If H is the group generated by*

$$Y = \{x^{r_x} | x \in X\},$$

then H is a free $\mathfrak{w}\mathfrak{x}$-group and Y is a canonical set of generators for H.

Proof. It is enough, by Lemma 1, to prove Theorem 3 for finitely generated groups G. But this means we can find a finitely generated free group R, say, such that

$$R/\mathfrak{w}\mathfrak{x}(R) \cong G.$$

Consequently we may focus our attention on R. Thus we have to prove that if R is freely generated by $x_1, x_2, \cdots, x_k$, if $n_1, \cdots, n_k$ are nonzero integers and if S is the subgroup generated by $x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k}$, then

$$S\mathfrak{wx}(R)/\mathfrak{wx}(R) \cong R/\mathfrak{wx}(R).$$

But we have essentially accomplished this through Proposition 2, Lemma 6 and Lemma 7. This completes the proof of Theorem 3.

Added in proof. Siegfried Moran has pointed out to me that (although there is no duplication of results) there is a connection between this paper and the following:

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