THE ADMISSIBLE MEAN VALUES OF A STOCHASTIC PROCESS

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If $x(t)$ is a real valued stochastic process on the linear set T and f is a function on T, we will write P_x and P_{x+f} for the probability measures induced on sample space by the processes $x(t)$ and $x(t)+f(t)$ respectively. Thus, if g is a measurable function of $x(t_1),\cdots,x(t_n)$,

$$\int g(x(t_1),\cdots,x(t_n))\,P_{x+f}(dx)=\int g(x(t_1)+f(t_1),\cdots,x(t_n)+f(t_n))\,P_x(dx).$$

We will say that f is an *admissible mean value for x over the field S* if P_{x+f} is absolutely continuous with respect to P_x (which we will hereafter write $P_{x+f}<P_x$) over S. The purpose of this paper is to investigate the size and structure of $M(x,S)$, the set of admissible mean values.

The field of most interest and the one generally referred to in this paper is that generated by all the random variables $x(t)$ for t in T. We will write S' for this field but will often suppress it in the notation and when no field is explicitly mentioned S' is meant. All the fields considered will be subfields of S'. A knowledge of $M(x,S)$ is important in setting up "signal plus noise" problems (i.e., problems of discriminating between x and $x+f$) since a realistic set of assumptions should usually lead to a set of f's contained in $M(x,S)$ for the relevant S.

I. **The size of $M(x)$ in the L_2 case.** Throughout this section we will assume that $x(t)$ is a measurable process, that T is a finite or infinite interval, that $\int x(t)P_x(dx)=0$ and $\int x^2(t)P_x(dx)<\infty$ for all t, and that $\int_T x^2(t)dt<\infty$ with probability one. The assumption that the mean value vanishes is harmless since, as is easily seen, $P_{x+g+f}<P_{x+g}$ if and only if $P_{x+f}<P_x$. We will also assume that the correlation function $R(s,t)=\int x(s)x(t)P_x(dx)$ satisfies $\int_T\int_T R^2(s,t)dsdt<\infty$ so that the integral transform $R:Rf(s)=\int_T R(s,t)f(t)dt$ is a compact operator on $L_2(T)$.

EXAMPLE 1. If $x(t)$ is a Gaussian process with continuous sample functions, then $M(x)=R^{1/2}(L_2(T))$ (where for each f in $R^{1/2}(L_2(t))$ its continuous version is chosen). This is proved in [2]. Theorem 1 below shows that this is as large as $M(x)$ can get.

EXAMPLE 2. If $x(t)$ is a Poisson process, then any f in $M(x)$ must be a jump function and since the probability of a jump at any point is zero, $M(x)=(0)$.

Received by the editors August 8, 1962.

EXAMPLE 3(1). Let $x(t)$ be a process with independent increments whose Levy representation is

$$\log \int e^{i\lambda x(t)} P_x(dx) = -\frac{1}{2}\sigma^2(t)\lambda^2 + \int_{|u|\leqq 1;0\leqq\tau\leqq t} (e^{i\lambda u} - 1 - i\lambda u)Q(d\tau,du)$$

$$+\int_{|u|>1;0\leqq\tau\leqq t} (e^{i\lambda u} - 1)Q(d\tau,du).$$

$\sigma^2(t)$ is nondecreasing and represents the Gaussian part of $x(t)$, that is, we can write $x(t) = x_G(t) + y(t)$ where $x_G(t)$ is a Gaussian process with correlation function $R(s,t) = \sigma^2(\min(s,t))$ and $y(t)$ is independent of $x_G(t)$. Then, [6, Theorem 4, p. 421], $M(x) = M(x_G) = R^{1/2}(L_2(T))$.

THEOREM 1. $M(x) \subset R^{1/2}(L_2(T))$.

Proof. $x(t)$ is a measurable process and its sample functions are in $L_2(T)$ with probability one so, for any h in $L_2(T)$, $\theta(h,x) = \int_T x(t)h(t)dt$ is a P_x-measurable function of x. An easy computation shows that $\int \theta^2(h,x)P_x(dx) = \int_T (R^{1/2}\,h)^2(t)dt$. Hence, if $R^{1/2}h_n$ converges to $R^{1/2}h$, $\theta(h_n,x)$ is $L_2(P_x)$ convergent and some subsequence converges to $\theta(h,x)$ almost everywhere. Now the distribution of $\theta(h_n,x)$ with respect to P_{x+f} is the same as the distribution of $\theta(h_n,x) + \int_T f(t)h_n(t)dt$ with respect to P_x, so if $P_{x+f} < P_x$, $\int_T f(t)h_n(t)dt$ must converge to $\int_T f(t)h(t)dt$ whenever $R^{1/2}h_n$ converges to $R^{1/2}h$, and this implies that f is in $R^{1/2}(L_2(T))$.

If $M(x)$ is a measurable set, then $P_x(M(x))$ would give another, more intrinsic measure of the size of $M(x)$. The only result of this kind known to the author, except for cases where $M(x) = (0)$, is in the Gaussian case where $P_x(M(x)) = 1$ or 0 depending on whether R has finitely or infinitely many nonzero eigenvalues [4, p. 42].

II. **The semigroup of isometries associated with** $M(x)$. We now replace the assumptions of the previous section by a separability assumption. $x(t)$ is measurably separable if there exists a countable subset (t_i) of T such that the smallest σ field with respect to which all the $x(t_i)$ are measurable is equivalent to S' under P_x. Any process whose sample functions are continuous with probability one is measurably separable. We shall write x_i for $x(t_i)$ throughout this section and S_n for the field generated by $x_1,\cdots,x_n$. If $\pi_n\phi$ is the conditional expectation of ϕ on S_n, then π_n has a unique extension to a projection in $L_p(P_x)$ whenever $\infty > p \geqq 1$ and π_n converges strongly to the identity. It follows that $L_p(P_x)$ is separable for $\infty > p \geqq 1$.

If f is in $M(x)$ and ϕ is a bounded S_n-measurable function, we define for each $\infty > p \geqq 1$

(1) Another example related to this, but not of L_2 type, is afforded by the symmetric stable processes of index $\alpha < 1$. Here again $M(x) = 0$ since for every t_0 in T and $\beta > \alpha$, $(t-t_0)^{-1/\beta}(x(t)-x(t_0)) \to 0$ with probability one [1, Theorem 3.1, p. 497].

$$u_f^p\phi(x) = \left[\frac{dP_{x+f}}{dP_x}\right]^{1/p} \phi(x_1 - f_1, \cdots, x_n - f_n)$$

(writing f_i for $f(t_i)$). u_f^p is densely defined and preserves positivity and norm in $L_p(P_x)$, hence has a unique extension to a positivity preserving isometry in $L_p(P_x)$ which we also denote by u_f^p. This definition is clearly independent of the particular sequence (t_i) which was used.

THEOREM 2. *If $x(t)$ is measurably separable, then $f+g$ is in $M(x)$ whenever f and g are, and $u_{f+g}^p = u_f^p u_g^p$ for all p. The semigroup $U_p(x) = [u_f^p | f \in M(x)]$ is strongly closed.*

Proof. Let ϕ be a bounded S_n-measurable function and ψ be a nonnegative bounded S_m-measurable function. Then if f is in $M(x)$, $\int \phi(x)[u_f^p(\psi)(x)]^p P_x(dx) = \int \phi(x+f)[\psi(x)]^p P_x(dx)$ and this relation extends by continuity to all nonnegative ψ in $L_p(P_x)$. In particular, if g is in $M(x)$. we can apply this equation with $\psi = dP_{x+g}/dP_x$ and $p = 1$ to get

$$\int \phi(x) u_f^1 \left(\frac{dP_{x+g}}{dP_x}\right)(x)\; P_x(dx) = \int \phi(x+f) \frac{dP_{x+g}}{dP_x}(x) P_x(dx) = \int \phi(x+f+g) P_x(dx)$$

which proves that $f+g$ is in $M(x)$ and that $dP_{x+f+g}/dP_x = u_f^1(dP_{x+g}/dP_x)$. Now applying the equation to $u_g^p(\psi)$ we have

$$\int \phi(x)[u_f^p(u_g^p(\psi))(x)]^p P_x(dx) = \int \phi(x+f)[u_g^p(\psi)(x)]^p P_x(dx)$$

$$= \int \phi(x+f+g)[\psi(x)]^p P_x(dx) = \int \phi(x)[u_{f+g}^p(\psi)(x)]^p P_x(dx)$$

which shows that $u_f^p u_g^p$ and u_{f+g}^p agree on nonnegative functions and hence on all functions in $L_p(P_x)$.

Now suppose V is in the strong closure of $U_p(x)$. For each t in T and real number λ, there is a sequence (f_n) from $M(x)$ with $V(1) = \lim u_{f_n}^p(1)$ and

$$V(\exp i\lambda x(t)) = \lim u_{f_n}^p(\exp i\lambda x(t)) = \lim u_{f_n}^p(1) \exp i\lambda(x(t) + f_n(t))$$

$$= \lim V(1) \exp i\lambda(x(t) + f_n(t))$$

which shows that $f_n(t)$ converges to a limit which we call $f(t)$. $f(t)$ clearly does not depend on the particular sequence (f_n) and since a common sequence (f_n') can be found for both $\exp i\lambda x(t)$ and $\exp i\mu x(t)$, it does not depend on the choice of λ. For any real numbers $\lambda_1, \cdots, \lambda_k$ a sequence (f_n) can be found so that $u_{f_n}^p$ will approximate V on all the functions $1, \exp i\lambda_1 x_1, \cdots, \exp i\lambda_k x_k$ and $\exp i\sum_{j=1}^k \lambda_j x_j$ and thus $V(\exp i\sum_{j=1}^k \lambda_j x_j) = V(1) \exp i\sum_{j=1}^k \lambda_j(x_j + f(t_j))$. If $\hat{\phi}$ is any continuous function of k real variables with compact support and N is any positive number, there is a trigonometric function $\hat{\psi}_N$ with period $2N$ such that $|\hat{\psi} - \hat{\phi}| < 1/N$ whenever $|x_i| < N$, $i = 1, \cdots, k$. Setting

$\phi(x) = \hat{\phi}(x_1 + f(t_1), \cdots, x_k + f(t_k))$ and $\psi_N(x) = \hat{\psi}_N(x_1 + f(t_1), \cdots, x_k + f(t_k))$,

we have $V\phi(x) = \lim_{N\to\infty} V\psi_N(x) = V(1)\hat{\phi}(x_1, \cdots, x_k)$. Hence,

$$\int V(1)^p(x)\hat{\phi}^p(x_1, \cdots, x_k)P_x(dx) = \| V\phi\| = \|\phi\|^p,$$

$$= \int \phi^p(x_1 + f(t_1), \cdots, x_k + f(t_k))P_x(dx)$$

$$= \int \hat{\phi}^p(x_1, \cdots, x_k)P_{x+f}(dx)$$

and, since functions of this form are dense in $L_1(P_x)$, this implies that f is in $M(x)$ and that $V(1)^p = dP_{x+f}/dP_x$. Finally, since V agrees with u_f^p on a dense subset, $V = u_f^p$.

COROLLARY 1. *If $M(x) = -M(x)$, in particular, if P_x is symmetric about* 0, *then $U_2(x)$ is a strongly closed group of unitary operators.*

COROLLARY 2. *Suppose that the joint distribution of $x_1, \cdots, x_k$ is given by a density $p_k(\xi_1, \cdots, \xi_k)$. Then if (f_n) is a sequence from $M(x)$ such that* $\lim f_n(t) = f(t)$ *exists for all t in T, and if*

$$A(n, m) = \limsup_{k\to\infty} \int \cdots \int | p_k(\xi_1 - f_n(t_1), \cdots, \xi_k - f_n(t_k)) - p_k(\xi_1 - f_m(t_1), \cdots, \xi_k - f_m(t_k)) | d\xi_1 \cdots d\xi_k$$

goes to zero as $\inf(n,m)$ *goes to infinity, f is also in $M(x)$ and dP_{x+f}/dP_x is the $L_1(P_x)$ limit of dP_{x+f_n}/dP_x.*

Proof. If $\hat{\phi}$ is a bounded function of j real variables with bounded first and second derivatives and $\phi(x) = \hat{\phi}(x_1, \cdots, x_j)$, then

$$\begin{aligned}
\|u^1_{f_n}(\phi) - u^1_{f_m}(\phi)\| &= \lim_{k\to\infty} \int \cdots \int \Bigg| \frac{p_k(\xi_1 - f_n(t_1), \cdots, \xi_k - f_n(t_k))}{p_k(\xi_1, \cdots, \xi_k)} \hat{\phi}(\xi_1 - f_n(t_1), \cdots, \xi_k - f_n(t_k)) \\
&\quad - \frac{p_k(\xi_1 - f_m(t_1), \cdots, \xi_k - f_m(t_k))}{p_k(\xi_1, \cdots, \xi_k)} \hat{\phi}(\xi_1 - f_m(t_1), \cdots, \xi_k - f_m(t_k)) \Bigg| \\
&\quad \cdot p_k(\xi_1, \cdots, \xi_k) d\xi_1 \cdots d\xi_k \\
&\leqq C \int \cdots \int | p_k(\xi_1 - f_n(t_1), \cdots, \xi_k - f_n(t_k)) - p_k(\xi_1 - f_m(t_1), \cdots, \xi_k - f_m(t_k)) | d\xi_1 \cdots d\xi_k \\
&\quad + \int \cdots \int p_k(\xi_1 - f_n(t_1), \cdots, \xi_k - f_n(t_k)) \\
&\quad \cdot | \hat{\phi}(\xi_1 - f_n(t_1), \cdots, \xi_j - f_n(t_j)) - \hat{\phi}(\xi_1 - f_m(t_1), \cdots, \xi_j - f_m(t_j)) | d\xi_1 \cdots d\xi_k \\
&\leqq C A(n, m) + C \max_{1 \leqq i \leqq j} |f_n(t_i) - f_m(t_i)|,
\end{aligned}$$

and this goes to zero by hypothesis as n and m go to infinity. It follows that $u^1_{f_n}$ is strongly convergent, hence converges to some u^1_g in $U_1(x)$ and $g(t) = \lim f_n(t) = f(t)$ for all t in T.

THEOREM 3. *u^p_f has no eigenvectors if $f \neq 0$. $U_p(x)$ has no almost periodic vectors.*

Proof. The subspace of almost periodic vectors of an abelian semigroup of operators is spanned by its one-dimensional invariant subspaces [3, Satz 1.7.5, pp. 30 ff.] so the second assertion will follow from the first. If $u^p_f(\psi) = \lambda\psi$, then

$$u^p_f(|\psi|) = |u^p_f(\psi)| = |\lambda||\psi| = |\psi|.$$

Let t be a point of T at which f does not vanish, say $f(t) > 0$, and define A_n to be the set where $nf(t) \leqq x(t) < (n+1)f(t)$.

$$\int_{A_n} |\psi|^p dP = \int_{A_n} |u^p_f(|\psi|)|^p dP = \int_{[x|x(t)+f(t) \in A_n]} |\psi|^p dP = \int_{A_{n-1}} |\psi|^p dP$$

so $\int_{A_n} |\psi|^p dP = c$, and since $\int |\psi|^p dP = \sum_n \int_{A_n} |\psi|^p dP < \infty$ we must have $c = 0$ and hence, $\psi = 0$.

III. **One-parameter semigroups in** $M(x)$. In this section we investigate various conditions under which an entire half line $[\alpha f | \alpha \geqq 0]$ belongs to $M(x)$. If f is in $M(x)$, then, of course, so is kf for every positive integer k. The following simple example shows that these may be the only multiples of f in $M(x)$. We take T to consist of a single point, i.e., the stochastic process is simply a single random variable x and the possible mean values are just the real numbers. If x is distributed according to the density p,

$$p(\xi) = \begin{cases} 0 & \text{if } \xi < 2 \\ 2^{-k} & \text{if } 2 \leqq 2k \leqq \xi < 2k+1 \\ 0 & \text{if } 2k+1 \leqq \xi < 2k+2 \end{cases}$$

then $M(x)$ is just the positive integers.

THEOREM 4. *If $x(t)$ is a measurably separable process and αf is in $M(x)$ for all $\alpha \geqq 0$, then $u^p_{\alpha f}$ is a strongly continuous semigroup.*

Proof. It will be sufficient to prove the theorem for $p = 1$, since

$$\begin{aligned}\int |u^p_{\alpha f}(\phi)(x) - \phi(x)|^p P_x(dx) &\leqq \int |[u^p_{\alpha f}(\phi)]^p(x) - \phi^p(x)| P_x(dx) \\ &= \int |u^1_{\alpha f}(\phi^p)(x) - \phi^p(x)| P_x(dx)\end{aligned}$$

for nonnegative, bounded S_n-measurable ϕ and differences of such functions are dense in $L_p(P_x)$. For bounded S_n-measurable ϕ and arbitrary ψ in $L_1(P_x)$,

$$\int \phi(x) u^1_{\alpha f}(\psi)(x) P_x(dx) = \int \phi(x + \alpha f)\psi(x) P_x(dx)$$

which is a measurable function of α and for any bounded ϕ,

$$\int \phi(x) u^1_{\alpha f}(\psi)(x) P_x(dx) = \lim \int \pi_n \phi(x) u^1_{\alpha f}(\psi)(x) P_x(dx)$$

so $u^1_{\alpha f}$ is weakly measurable. This fact plus the separability of $L_1(P_x)$ implies that $u^1_{\alpha f}$ is strongly continuous for $\alpha > 0$ so that

$$\lim_{\alpha \to 0} \| u^1_{\alpha f}(\psi) - \psi \| = \lim_{\alpha \to 0} \| u^1_{\alpha f + f}(\psi) - u^1_f(\psi) \| = 0.$$

The next theorem gives the form of the infinitesimal generators of these groups in some cases. We restrict attention to the case $p = 1$ because it is easier computationally and also because it is the most important in practice (see [5, Theorem 3.1, p. 15]. Let D_0 be the set of functions ϕ of the form $\phi(x) = \hat{\phi}(x_1, \cdots, x_n)$ where $\hat{\phi}$ is bounded and has bounded first derivatives. For ϕ in D_0 we define

$$\mathscr{D}\phi(x) = \frac{\partial}{\partial \alpha} \hat{\phi}(x_1 + \alpha f_1, \cdots, x_n + \alpha f_n) \Big|_{\alpha = 0} .$$

THEOREM 5. *If αf is in $M(x)$ for all $\alpha \geqq 0$, $u^1_{\alpha f}$ is strongly continuous and the function* 1 *is in the domain of its generator A, then for every ϕ in D_0*

$$\int A(1)(x)\phi(x) P_x(dx) = \int \mathscr{D}\phi(x) P_x(dx).$$

Moreover, D_0 is contained in the domain of A and

$$A\phi(x) = A(1)(x)\phi(x) - \mathscr{D}\phi(x).$$

Conversely, if a function $\lambda(x)$ exists in $L_1(P_x)$ satisfying $\int \lambda(x)\phi(x)P_x(dx) = \int \mathscr{D}\phi(x)P_x(dx)$ for all ϕ in D_0 and if the set $[\lambda(x)\phi(x) - \mathscr{D}\phi(x) - a\phi(x) \mid \phi \in D_0]$ is dense in $L_1(P_x)$ for some $a \neq 0$, then αf is in $M(x)$ for all real α and the generator of $u^1_{\alpha f}$ is the closure of the operator A defined on D_0 by: $A\phi(x) = \lambda(x)\phi(x) - \mathscr{D}\phi(x)$. The same conclusion holds if instead of the condition on the density of the range, λ satisfies either

$$\liminf \int_{[x \mid \lambda(x) > n]} \lambda(x) P_x(dx) = O(e^{-\varepsilon n})$$

or

$$\liminf - \int_{[x \mid \lambda(x) < -n]} \lambda(x) P_x(dx) = O(e^{-\varepsilon n}) \textit{ for some } \varepsilon > 0.$$

Proof. If 1 is in the domain of A and ϕ is in D_0, then

$$\frac{1}{\alpha}(u^1_{\alpha f}\phi - \phi)(x) = \frac{1}{\alpha}(u^1_{\alpha f}(1) - 1)(x)\phi(x)$$

$$+ \frac{1}{\alpha}(u^1_{\alpha f}(1) - 1)(\hat{\phi}(x - \alpha f) - \hat{\phi}(x)) + \frac{1}{\alpha}(\hat{\phi}(x - \alpha f) - \hat{\phi}(x))$$

which converges to $A(1)\phi - \mathscr{D}\phi$. Also,

$$\int A(1)(x)\phi(x)P_x(dx) = \lim_{\alpha\to 0} \frac{1}{\alpha}\int\left(\frac{dP_{x+\alpha f}}{dP_x} - 1\right)(x)\phi(x)P_x(dx)$$

$$= \lim_{\alpha\to 0} \int \frac{1}{\alpha}[\phi(x_1 + \alpha f_1, \cdots, x_n + \alpha f_n) - \phi(x_1, \cdots, x_n)]\ P_x(dx)$$

$$= \int \mathscr{D}\phi(x)P_x(dx).$$

The remainder of the theorem is a straightforward application of Theorems 3.3 and 3.4 of [5].

We now return to the assumptions of §1, namely, $\int x(t)P_x(dx) = 0$, $\int x^2(t)P_x(dx) < \infty$, $\int_T x^2(t)dt < \infty$ with probability one, and $R(s,t)$ square integrable on $T\times T$. Let (λ_i) and (g_i) be the eigenvalues and corresponding eigenvectors of R. We will write x_n for the random variables $x_n = \lambda_n^{-1/2}\int_T x(t)g_n(t)dt$; S_n for the field generated by $x_1,\cdots,x_n$; S_∞ for the field generated by all x_n's and $\pi_n\phi$ for the conditional expectation of ϕ on S_n. S_∞ is not larger than S' and for many applied problems is a more realistic field to deal with than S'. Since the proof of Theorem 1 was essentially an L_2 proof, it also applies to $M(x,S_\infty)$ and hence the numbers f_n, $f_n = \lambda_n^{-1/2}\int_T f(t)g_n(t)dt$, satisfy $\sum_{n=1}^{\infty} f_n^2 < \infty$ for any f in $M(x,S_\infty)$.

THEOREM 6. *If for every n the joint distribution of $x_1,\cdots,x_n$ is given by a density p_n satisfying*

(1) $p_n(\xi_1,\cdots,\xi_n) > 0$ *almost everywhere,*

(2) $\lim_{\xi_j\to\pm\infty} p_n(\xi_1,\cdots,\xi_j,\cdots,\xi_n) = 0$ *for almost all* $\xi_1,\cdots,\xi_{j-1}, \xi_{j+1},\cdots,\xi_n$, *and*

(3) $\partial p_n/\partial\xi_j$ *exists for all* $j \leqq n$ *and*

$$\int\cdots\int \frac{1}{p_n(\xi_1,\cdots,\xi_n)}\left(\frac{\partial p_n}{\partial \xi_i}\right)^2(\xi_1,\cdots,\xi_n)\,d\xi_1\cdots d\xi_n \leqq K,$$

then $M(x,S_\infty)$ is positive linear, i.e., if f is in $M(x,S_\infty)$, so is αf for all $\alpha \geqq 0$.

Proof. Assume f is in $M(x,S_\infty)$. $V_n(\alpha)$ defined on $\pi_n L_2(P_x)$ by:

$$V_n(\alpha)\phi(x_1,\cdots,x_n) = \left[\frac{p_n(x_1 - \alpha f_1,\cdots,x_n - \alpha f_n)}{p_n(x_1,\cdots,x_n)}\right]^{1/2}\phi(x_1 - \alpha f_1,\cdots,x_n - \alpha f_n)$$

is a strongly continuous semigroup of isometries in $\pi_n L_2(P_x)$. Let $J_n(\lambda)$ be the resolvent of $V_n(\alpha)$. The lower semimartingale

$$y_n(x) = \left[\frac{p_n(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n)}{p_n(x_1, \cdots, x_n)}\right]^{1/2}$$

is uniformly in $L_2(P_x)$ and hence converges in mean to some function Q_α. The operators $V(\alpha)$ defined on each $\pi_n L_2(P_x)$ by $V(\alpha)\phi(x) = Q_\alpha(x)\phi(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n)$ are norm decreasing and weakly measurable in α so the same is true of their unique extensions to $L_2(P_x)$ which we also denote by $V(\alpha)$. The operators $J(\lambda)$ defined for $\lambda > 0$ by: $J(\lambda) = \int_0^\infty e^{-\lambda\alpha} V(\alpha) d\alpha$ satisfy $\| J(\lambda) \| \leqq 1/\lambda$ and $\lim J_n(\lambda)\pi_n\phi = \lim J(\lambda)\pi_n\phi = J(\lambda)\phi$ for all bounded ϕ and hence for all ϕ in $L_2(P_x)$. According to a theorem of Trotter [7, Theorem 5.1, p. 900], if we can show

(i) $J(\lambda) - J(\mu) = (\mu - \lambda)J(\lambda)J(\mu)$,

(ii) $\| \lambda^m J^m(\lambda) \leqq \| M$ and

(iii) $\lim_{\lambda \to \infty} \lambda J(\lambda) = I$,

we can conclude that $V(\alpha)$ is a semigroup with resolvent $J(\lambda)$. (ii) is immediate and (i) follows from

$$\begin{aligned}(J(\lambda) - J(\mu))\phi &= \lim (J_n(\lambda) - J_n(\mu))\pi_n\phi \\ &= \lim (\mu - \lambda)J_n(\lambda)J_n(\mu)\pi_n\phi = \lim(\mu - \lambda)J_n(\lambda)\pi_n J(\mu)\phi \\ &= (\mu - \lambda)J(\lambda)J(\mu)\phi.\end{aligned}$$

It will be sufficient to prove (iii) for S_k-measurable ϕ which are bounded and have bounded derivatives since such functions are dense in $L_2(P_x)$. For such ϕ

$$\begin{aligned}\lambda J_n(\lambda)\phi(x) &= \int_0^\infty \left[\frac{p_n(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n)}{p_n(x_1, \cdots, x_n)}\right]^{1/2} \phi(x_1 - \alpha f_1, \cdots, x_k - \alpha f_k) \frac{d(-e^{-\lambda\alpha})}{d\alpha} d\alpha \\ &= \phi(x) - \int_0^\infty e^{-\lambda\alpha} \left[\frac{p_n(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n)}{p_n(x_1, \cdots, x_n)}\right]^{1/2} \\ &\qquad\qquad \cdot \left(\sum_{i=1}^k f_i \frac{\partial\phi}{\partial x_i}(x_1 - \alpha f_1, \cdots, x_k - \alpha f_k)\right) d\alpha \\ &\quad - \int_0^\infty e^{-\lambda\alpha}\left(\sum_{i=1}^n f_i \frac{\partial\sqrt{p_n}}{\partial x_i}\right)(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n) \frac{\phi(x_1 - \alpha f_1, \cdots, x_k - \alpha f_k)}{\sqrt{p_n(x_1, \cdots, x_n)}} d\alpha.\end{aligned}$$

Since

$$\left\| \left(\frac{p_n(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n)}{p_n(x_1, \cdots, x_n)}\right)^{1/2} \right\| = 1,$$

the norm of the first integral is bounded by $C\lambda^{-1}$ for some C independent of n. Also

$$\left\| \frac{\left(\sum_{i=1}^{n} f_i \frac{\partial \sqrt{p_n}}{\partial x_i}(x_1 - \alpha f_1, \cdots, x_n - \alpha f_n)\right)}{\sqrt{p_n(x_1, \cdots, x_n)}} \right\|^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} f_i f_j \int \cdots \int \frac{\partial \sqrt{p_n}}{\partial \xi_i}(\xi_1 - \alpha f_1, \cdots, \xi_n - \alpha f_n) \frac{\partial \sqrt{p_n}}{\partial \xi_j}(\xi_1 - \alpha f_1, \cdots, \xi_n - \alpha f_n) d\xi_1 \cdots d\xi_n$$

$$= \sum_{i=1}^{n} f_i^2 \int \cdots \int \left(\frac{\partial \sqrt{p_n}}{\partial \xi_i}\right)^2 (\xi_1, \cdots, \xi_n) d\xi_1 \cdots d\xi_n \leqq K^2 \sum_{i=1}^{\infty} f_i^2$$

so the norm of the second integral is also bounded by $C\lambda^{-1}$ and hence $\| \lambda J(\lambda)\phi - \phi \| = \lim \| \lambda J_n(\lambda)\phi - \phi \| \leqq C\lambda^{-1}$ which proves (iii). Now for any $0 < \alpha < 1$ and ϕ in $L_2(P_x)$, $\| \phi \| = \| u_f^2 \phi \| = \| V(1)\phi \| = \| V(1-\alpha)V(\alpha)\phi \| \leqq \| V(\alpha)\phi \| \leqq \| \phi \|$ so $V(\alpha)$ is an isometry. If ϕ is positive and S_k-measurable, then

$$\begin{aligned} \int (V(\alpha)1)^2 \phi(x_1, \cdots, x_k) P_x(dx) &= \| V(\alpha)\sqrt{\phi}(x_1 + \alpha f_1, \cdots, x_n + \alpha f_n) \|^2 \\ &= \| \sqrt{\phi}(x_1 + \alpha f_1, \cdots, x_n + \alpha f_n) \|^2 \\ &= \int \phi(x_1 + \alpha f_1, \cdots, x_n + \alpha f_n) P_x(dx) \end{aligned}$$

so that αf is in $M(x, S_\infty)$ and $(V(\alpha)1)^2 = dP_{x+\alpha f}/dP_x$. $(n+\alpha)f$ is in $M(x, S_\infty)$ because of the semigroup property of $M(x, S_\infty)$.

COROLLARY 3. *If in addition to the above properties $p_n(\xi_1, \cdots, \xi_n) = p_n(-\xi_1, \cdots, -\xi_n)$, then $M(x, S_\infty)$ is linear.*

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(2) Operated with support from the U.S. Army, Navy, and Air Force.