DIFFERENTIABLE OPEN MAPS ON MANIFOLDS[1]

BY
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Introduction. This paper contains a detailed discussion with proofs of results announced in [4].

Let M^n and N^n be n-manifolds without boundary, and let $f: M^n \to N^n$ be continuous. The map f is *open* if, whenever U is open in M^n, $f(U)$ is open in N^n; it is *light* if, for every $y \in N^n$, $\dim (f^{-1}(y)) \leqq 0$. For $n \geqq 2$ there is a canonical light open map $F_{n,d}: E^n \to E^n$ given by $F_{n,d}(x_1, x_2, \cdots, x_n) = (u_1, u_2, x_3, \cdots, x_n)$, where

$$u_1 + iu_2 = (x_1 + ix_2)^d \qquad (i = \sqrt{-1};\ d = 1, 2, \cdots).$$

For $n = 2$ it is well known that a nonconstant complex analytic function is open and light. Conversely, Stoilow [12] proved that every light open map is locally topologically equivalent to an analytic map, and thus to some $F_{2,d}$ $(d = 1, 2, \cdots)$. In fact (1.10), if M^2 is compact and f is C^2 and open, then f has this canonical structure. The main object of this paper is to prove (2.1) that the corresponding conclusion holds for arbitrary n $(n \geqq 2)$, if we first remove an exceptional set of dimension at most $n - 3$. Examples are given, especially in §3, showing that the exceptional set and some of the hypotheses used are necessary.

DEFINITION. As in [5] the branch set B_f is the set of points in M^n at which f fails to be a local homeomorphism.

NOTATION. If $f: E^n \to E^p$ is C', then f_i will be the ith component real-valued function, and $D_j f_i$ will be the first partial derivative of f_i with respect to its jth coordinate. If y is a point in E^n, then y_i will be its ith coordinate. The symbols M^n and N^p will refer to manifolds of dimensions n and p, respectively. The statement that $f: M^n \to N^p$ is C^m will imply that the manifolds are also C^m. The set of points in M^n at which the Jacobian matrix of f has rank at most q will be denoted by R_q.

The closure of a set X is denoted by $\mathrm{Cl}[X]$ or $\bar{X}$, its interior by int X, and the restriction of f to X by $f|X$. A map is a continuous function, the distance between the points x and y is $d(x, y)$, and $S(x, \varepsilon) = \{y: d(x, y) < \varepsilon\}$.

1. General results.

1.1. LEMMA. *Let $h: E^n \to E^p$, $h \in C^m$ $(m = 1, 2, \cdots)$, and let the rank of the Jacobian matrix of h at $\bar{x}$ be at least q $(q = 1, 2, \cdots, n-1)$. Then there exist open neighborhoods U of $\bar{x}$ and V of $h(\bar{x})$, and C^m diffeomorphisms (onto)*

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k^1: $E \to U$ and $k^2: V \to E^p$ such that k^2hk^1, call it g, has the following properties:

(1) *For each $(p-q)$-plane α given by g_i constant $(i = 1,2,\cdots,q)$, $g^{-1}(\alpha)$ is a (single) $(n-q)$-plane given by x_j constant $(j = 1,2,\cdots,q)$.*

(2) *For each x in $g^{-1}(\alpha)$, the rank of the Jacobian matrix of g at x is s if and only if the rank of the Jacobian of $g \mid g^{-1}(\alpha)$ at x is $s-q$ $(s = q, q+1,\cdots,$* $\min(n,p))$.

Proof. By reordering the variables, both dependent and independent, we may suppose that the determinant $\det[D_jh_i](\bar{x}) \neq 0$ $(i,j = 1,2,\cdots,q)$. Let W be a neighborhood of $\bar{x}$ such that $\det[D_jh_i] \neq 0$ on all of W, and let $\bar{h}: W \to E^q$ be defined by $\bar{h}_i = h_i$ $(i = 1,2,\cdots,q)$. Since $\bar{h}$ has maximal rank at every point of W, we may apply the rank theorem [7, pp. 273-274]. Thus, there exists an open n-cell U in W about $\bar{x}$ and C^m diffeomorphisms $k^1: E^n \to U$ and $\bar{k}: h(U) \to E^q$ such that $(\bar{k}\bar{h}k^1)_i(x_1,x_2,\cdots,x_n) = x_i$ $(i = 1,2,\cdots,q)$. Using $V = \bar{h}(U) \times E^{p-q}$, $k_i^2(x) = \bar{k}_i(x_1,x_2,\cdots,x_q)$ $(i = 1,2,\cdots,q)$, and $k_i^2(x) = x_i$ $(i = q+1, q+2,\cdots,p)$, conclusion (1) follows.

Given $x \in E^n$, let s be the rank of the Jacobian matrix of g at x. If α is the $(p-q)$-coordinate plane containing $g(x)$, then $J = (D_jg_i(x))$ $(i = q+1, q+2,\cdots,p;$ $j = q+1, q+2,\cdots,n)$ is the Jacobian matrix of $g \mid g^{-1}(\alpha)$ (as a map into α) at x. Since (by (1)) $D_jg_i(x) = 0$ $(i = 1, 2, \cdots, q;$ $j = q+1,$ $q+2,\cdots,n)$, and $\det(D_jg_i(x)) \neq 0$ $(i,j = 1,2,\cdots,q)$, J has rank $s-q$, yielding (2). (Clearly, the same result holds if we interpret $g \mid g^{-1}(\alpha)$ as a map into E^p.)

Remark. If $p = n$ and if $f \mid U$ has Jacobian determinant non-negative or nonpositive, then, for each $(n-q)$-cell γ given by conclusion (1), $f \mid \gamma$ (i.e., $g \mid \gamma$) has Jacobian determinant non-negative or nonpositive (not "respectively," in general). In particular, if $q = n-1$, then each map $f \mid \gamma$ is monotone.

1.2. Remark. *If X is a compact set contained in $E^n = E^{n-r} \times E^r$, and if* $\dim(X \cap (E^{n-r} \times \{x\})) \leqq q - r$ *for each x in E^r, then* $\dim X \leqq q$.

Let $g: E^{n-r} \times E^r \to E^r$ be the projection map, and let f be the restriction of g to X. The proof, pointed out to the author by E. Connell, follows from an application of [9, pp. 91–92].

The following result is related to Sard's theorem [11].

1.3. Proposition. *If $f: M^n \to N^p$, f and the manifolds are C^n, then* $\dim(f(R_q)) \leqq q$ *(where R_q is the set of points of M^n at which the Jacobian matrix of f has rank at most q). In particular,* $\dim(f(M^n)) \leqq n$. *If f is also light then* $\dim(R_q) \leqq q$.

Proof. Clearly, it is sufficient to prove the theorem for $f: E^n \to E^p$. If X_i is the set of critical points of f_i (the points at which all first partials are zero), then the measure of $f_i(X_i)$ is zero [10, p. 68, (4.3)] $(i = 1, 2,\cdots,p)$. Thus, $\dim(\bigcap_{i=1}^p f(X_i)) \leqq 0$. Since $R_0 = \bigcap_{i=1}^p X_i$, it follows that $\dim(f(R_0)) \leqq 0$.

(In fact, for each u in $f(R_0)$, there exists a coordinate p-cube C containing u such that the sides of C are coordinate $(p-1)$-planes, $\operatorname{diam}(C) < \varepsilon$, and $f(R_0) \cap \operatorname{bdy} C = \emptyset$.)

The proposition follows for $p = 1$ and all n and q; we proceed by induction on p.

Since each R_q is closed, $f(R_q)$ is the countable union of compact sets; thus it is sufficient to prove that $\dim(f(R_q - R_0)) \leqq q$, i.e., to prove the result in the case that the rank is at least one at each point. Furthermore, it suffices to prove the conclusion for $f|U$, where U is the open set given by (1.1) for f, $q = 1$, and an arbitrary point $\bar{x}$. For each $(n-1)$-cell γ given in (1.1) and each point x of $R_q \cap \gamma$, $f|\gamma$ (as a map into the corresponding $(p-1)$-cell) has rank at most $q-1$ at x. From the inductive hypothesis, $\dim(f(R_q \cap \gamma)) \leqq q-1$. Thus $f(R_q \cap U)$ meets each $(p-1)$-cell of (1.1) in a set of dimension at most $q-1$. Since $R_q \cap U$ is the countable union of compact sets, it follows from (1.2) that $\dim(f(R_q \cap U)) \leqq q$; thus $\dim(f(R_q)) \leqq q$.

If f is also light, then by [9, pp. 91-92] $\dim(R_q) \leqq \dim(f(R_q))$. The condition that $f \in C^n$ is necessary [17] in the above result for $p = 1$.

The following result is, for open maps, an extension of the inverse function theorem.

1.4. Theorem. *Let $f: E^n \to E^n$ be open and C'. If the rank of the Jacobian matrix of f at $\bar{x}$ is at least $n-1$, then f is locally a homeomorphism at $\bar{x}$. In other words, $B_f \subset R_{n-2}$.*

Proof. For $n=1$, the openness alone implies that f is a homeomorphism into. For $n > 1$, let U be the neighborhood of $\bar{x}$ given by (1.1) for $q = n-1$, and let γ be one of the 1-cells. Since $f(\gamma)$ is contained in a 1-cell, and since the restriction $f|\gamma$ is open [20, p. 147, (7.2)], $f|\gamma$ is a homeomorphism (into). Thus, $f|U$ is one-to-one and open, so that $\bar{x} \notin B_f$.

1.5. Corollary. *If $f: M^n \to N^n$, f open and C^n, then* $\dim(f(B_f)) \leqq n-2$. *If f is also light, then* $\dim(B_f) \leqq n-2$.

The proof follows from (1.2) and (1.4).

1.6. Corollary. *Let $f: M^n \to N^n$, f light and C^n. Then f is open if and only if $B_f \subset R_{n-2}$.*

Proof. If $B_f \subset R_{n-2}$, then $\dim(f(B_f)) \leqq n-2$ by (1.2); thus f is open [5, p. 531, (2.4)].

A sufficient condition for openness was given in [14] by Titus and Young. We observe that (for $f \in C^n$) the condition is necessary, and give an independent proof of the sufficiency.

1.7. Corollary. *Let $f: E^n \to E^n$ be C^n and light. Then f is open if and only if the Jacobian determinant J is non-negative or nonpositive everywhere.*

Proof. If f is open, then, by (1.3) and (1.6), $\dim(B_f) \leqq n-2$; thus J does not change sign.

Suppose that $J \geqq 0$ (or $J \leqq 0$). If the rank of the Jacobian matrix of f at $\bar{x}$ is (at least) $n-1$, then (from the remark after (1)) $f|U$ is one-to-one. For each closed n-cell C in $U, f|C$ is a homeomorphism onto its image; thus $f|\text{int } C$ is a homeomorphism onto its image, which is open by the theorem on invariance of domain. It follows that $x \notin B_f$. Since $B_f \subset R_{n-2}$, the conclusion follows from (1.6).

1.8. Theorem. *If $f: M^n \to N^n$, M^n compact, f open and C^n, then f is light.*

In fact, f is a pseudo-covering map [5, pp. 529 and 531, (2.4)].

Proof. By (1.5) $\dim(f(B_f)) \leqq n-2$. By the second paragraph of the proof of [5, p. 531, (2.4)] the restriction of f to $M^n - f^{-1}(f(B_f))$ is a k-to-1 covering map for some k.

Suppose that for some y in $f(B_f), f^{-1}(y)$ contains at least $k+1$ distinct points y^i $(i = 1, 2, \cdots, k+1)$. Then there exist disjoint open neighborhoods U^i of these points and $\bigcap_{i=1}^{k+1} f(U^i)$ is an open set; thus it meets $N^n - f(B_f)$, yielding a contradiction.

Remark. This result contrasts with the examples by R.D. Anderson [1; 2] of monotone open (not C^n) maps. The compactness of the domain is necessary, as we see in (3.6).

1.9. Corollary. *If $f: E^n \to E^n$ is a C^n light open map, then point inverses are isolated. Moreover, if $f: S^n \to S^n$ is C^n open with (Brouwer) degree d, then, for each $p \in S^n, f^{-1}(p)$ has at most $|d|$ points and $|d|$ is the least such number.*

Proof. The first conclusion follows from (1.3), (1.6), and [5, p. 530, (2.2)]. For the second, since the Jacobian determinant is non-negative or nonpositive [17], $f^{-1}(p)$ has precisely $|d|$ points, for each $p \in S^n - f(B_f)$. In fact, f is a $|d|$-to-1 pseudo-covering map, and the rest of the conclusion follows from the proof of (1.8).

The corollary is related to [16, p. 329, Theorem A and p. 335, (6a)].

1.10. Stoilow ([12]; cf. [20, p. 198, (5.1)]) proved that a light open map $f: M^2 \to N^2$ is locally at each point topologically equivalent to the complex analytic map $g(z) = z^d$ $(d = 1, 2, \cdots)$. (Manifolds are assumed to be without boundary.) For completeness we give now an independent proof in the case that $f \in C^2$. In particular, from (1.7) follows the apparently new result that: If $f: M^2 \to N^2$, M^2 compact, f open and C^2, then f has that local structure.

By (1.5) $\dim(f(B_f)) \leqq 0$. Given any x in B_f, by restriction [5, p. 529, (1.4) and its proof] there exists a pseudo-covering map g such that its domain V is a compact connected neighborhood of x in E^2, $g(V)$ is a closed topological disk D, and $(\text{bdy } D) \cap g(B_g) = \emptyset$. We may also suppose [5, p. 530, (2.2), conclusion (1)] that $g^{-1}(g(x)) = x$. Since each component of $f^{-1}(\text{bdy } D)$ ($=$ bdy V) is a simple closed curve, V is a disk-with-holes.

Let U be an open 2-cell about x in int V. Let h be a pseudo-covering map given, as above, for $g|U$ and x; call its domain E and its range disk D'. If bdy E $(=f^{-1}$ (bdy D')) had two or more components (simple closed curves), then U would contain a disk whose image under g contained $D - \text{int}(D')$, contradicting the fact that $U \subset \text{int } V$. Thus E is a topological closed disk itself.

If B_h contains a point $y \neq x$, let γ be an arc in $D' - h(B_h)$, separating int(D') into two components X and Y such that $h(x) \in X$ and $h(y) \in Y$. Then $h^{-1}(\gamma)$ consists of k mutually disjoint arcs, where k is the degree of h, and thus it separates int E into $k+1$ components. Precisely one of these components has image X (since $h^{-1}(h(x)) = x$), so that the other k have image Y, contradicting the fact that Y meets $h(B_h)$.

Thus $\{x\} = B_h$, and the conclusion is evident.

2. **The structure theorem.** In this section we give a structure theorem for differentiable open maps defined on compact manifolds, or (more generally) differentiable light open maps defined on arbitrary manifolds, comparing them with the maps $F_{n,d}$ defined in the introduction.

2.1. THEOREM. *Let $f: M^n \to N^n$ be C^n and open ($n \geqq 2$); let M^n be compact, or let f be light. Then there exists a closed set E,* dim $E \leqq n-3$, *such that for each x in $M^n - E$ there exists a neighborhood U of x on which f is topologically equivalent to one of the canonical maps $F_{n,d}$ ($d = 1, 2, \cdots$). Moreover, E is nowhere dense in B_f unless f is a local homeomorphism.*

Trivial examples show that "topologically equivalent" cannot be replaced by "diffeomorphically equivalent." The hypothesis that f is C^n results from the use of (1.3).

Proof. Since f is light (1.8), $\dim(R_{n-3}) \leqq n-3$ (1.3); thus the set E may as well include R_{n-3}. To prove the first part of the theorem we may suppose that $n \geqq 3$ and that the rank of the Jacobian matrix is at least $n-2$ at every point. For each $\bar{x}$ in B_f the restriction $f|U$ of f to some neighborhood U of $\bar{x}$ has the structure of (1.1), where $q = n-2$ and $p = n$. (We may as well suppose that $f|U$ is the g of (1.1).) Thus the domain and range of $f|U$ are $E^n = E^{n-2} \times C$, where C is the complex plane; for each $v \in E^{n-2}$ the restriction of f to the plane $\{v\} \times C$ is light and open [20, p. 147, (7.2)]. By (1.6) $B_f \subset R_{n-2}$, and by (1.1) $R_{n-2} \cap (\{v\} \times C)$ is the set of points at which $f|(\{v\} \times C)$ has rank 0; thus (1.3)

$$\dim(f(B_{f|(\{v\} \times C)})) \leqq 0$$

and $\dim(f(B_f)) \leqq n-2$.

The rest of the proof of the first conclusion uses only the topological properties of $f|U$ found above, and not the differentiability of $f|U$.

Let A be a closed n-cell such that $\bar{x} \in \text{int } A$ and $A \subset U$. Since $\dim(f(B_f)) \leqq n-2$, there exists [5, p. 529, (1.4)] a connected open neighborhood V of $\bar{x}$ such that the restriction of f to V is a pseudo-covering map g, and $V \subset \text{int } A$. Choose $\bar{w} \in E^{n-2}$ so that $f(\bar{x})$ is in the plane $\{\bar{w}\} \times C$ of (1.1). Since $\text{Cl}[g(B_g)] \subset f(B_f \cap A)$, $\text{Cl}[g(B_g)]$ meets $\{\bar{w}\} \times C$ in a compact set of dimension 0. Let G be an open disk with center $g(\bar{x})$,

$$G \subset g(V) \cap (\{w\} \times C).$$

Let L be any straight line in $\{\bar{w}\} \times C$ through $g(\bar{x})$, and [9, p. 22, (D)] let a and b be points on opposite sides of $L \cap G$ from $g(\bar{x})$, a and b disjoint from $(\{w\} \times C) \cap \text{Cl}[g(B_g)]$. It follows from [9, p. 48, Corollary 1] that there exist arcs Γ_i joining a to b, Γ_i disjoint from $(\{\bar{w}\} \times C) \cap \text{Cl}[g(B_g)]$ $(i = 1,2)$, $\Gamma_1 - \{a, b\}$ contained in one component of $G - L$, and $\Gamma_2 - \{a, b\}$ in the other.

Then $\Gamma_1 \cup \Gamma_2$ bounds a topological closed disk $D \subset C$ such that $g(\bar{x}) \in \{\bar{w}\} \times (\text{int } D)$, $\{\bar{w}\} \times D \subset g(V)$, and $\{\bar{w}\} \times (\text{bdy } D)$ is disjoint from the 0-dimensional set $(\{\bar{w}\} \times C) \cap \text{Cl}[g(B_g)]$. Thus, for all w sufficiently near $\bar{w}$, the corresponding-disks $\{w\} \times D$ will also be disjoint from $\text{Cl}[g(B_g)]$. Let T^{n-2} be such a small closed $(n-2)$-cell in E^{n-2} for which $\bar{w} \in \text{int}(T^{n-2})$ and $T^{n-2} \times D \subset g(V)$. The restriction of g to the component of $g^{-1}(T^{n-2} \times D)$ containing $\bar{x}$ is also a pseudo-covering map; for convenience we now call this map g, and its domain V.

Each set $g^{-1}(\{w\} \times D)$ is the closure of a region in the plane, each boundary component a simple closed curve. Thus, each $g^{-1}(\{w\} \times D)$ is homeomorphic to the same disk-with-holes H, and we will denote $g^{-1}(\{w\} \times D)$ by H^w.

For each w in T^{n-2}, let $g \mid H^w$ be denoted by g^w; and let its branch set be denoted by $B(g^w)$. Clearly, $\bigcup B(g^w) \subset B_g$. Suppose that $x \in \text{int } H^w$ but $x \notin B(g^w)$. Choose an open neighborhood N of x in int V such that $g^w \mid (N \cap H^w)$ is a homeomorphism. Let h be a pseudo-covering map whose domain contains x and is contained in N. Then the degree of h is one, and h is a homeomorphism; therefore, $x \notin B_g$. As a result, $\bigcup_w B(g^w) = B_g$.

For each w in T^{n-2}, the light open map g^w is topologically equivalent to a simplicial map [20, p. 198, (5.1)], and it follows from [17] that for some fixed natural number K depending only on H, $B(g^w)$ contains at most K points. Let $\alpha(w)$ be the number of branch points in H^w $(1 \leqq \alpha(w) \leqq K)$. Let Y be any open set in T^{n-2}, and let $\bar{y}$ in Y be a point at which the function α is maximal on Y. Let p^i be the points of $B(g^{\bar{y}})$, and let P^i be mutually disjoint sets open in $H^{\bar{y}}$ such that $P^i \cap g^{-1}(g(p^j)) = \{p^i\}$ $(i, j = 1, 2, \cdots, \alpha(\bar{y}))$; note that p^i may be in $g^{-1}(g(p^j))$ for $i \neq j$). There exists a disk $\{\bar{y}\} \times D^i$ such that $g(p^i) \in \{\bar{y}\} \times (\text{int}(D^i))$; $\{\bar{y}\} \times D^i \subset \{\bar{y}\} \times D$; and if J^i is the component of $g^{-1}(\{\bar{y}\} \times D^i)$ containing p^i, then $J^i \subset P^i$ [20, p. 131, (4.41)]. Since g is a pseudo-covering map, J^i is a topological 2-disk, and $g \mid J^i$ is topologically equivalent to the analytic map $\mu(z) = z^d$ $(d = 2, 3, \cdots)$.

Since $g(B_g)$ is compact and $\bigcup_w B(g^w) = B_g$, there exists a closed $(n-2)$-cell $W^i \subset Y$, $\bar{y} \in \text{int } (W^i)$, such that $g(B(g^y))$ is disjoint from $\{y\} \times \text{bdy } (D^i)$ for all

$y \in W^i$. If S^i is the component of $g^{-1}(W^i \times D^i)$ containing p^i, we may suppose that W^i is chosen small enough that $S^i \cap H^w$ is connected for all $w \in W^i$ and that the S^i are mutually disjoint $(i = 1,2,\cdots,\alpha(y))$. If $W = \bigcap_i \text{int}(W^i)$, then $B_g \cap H^w \subset \bigcup_i S^i \, (i = 1,2,\cdots,\alpha(\bar{y});\ w \in W)$.

Suppose that for some $w \in W^i$, $B(g^w) \cap S^i = \emptyset$; then since $g \mid S^i$ is a pseudo-covering map and $\{w\} \times D^i$ is simply connected, $g \mid (H^w \cap S^i)$, and thus $g \mid S^i$, has degree 1 (i.e., is a homeomorphism). Since $p^i \in B_g \cap S^i$, $B(g^w) \cap S^i \neq \emptyset$, for all $w \in W^i$. Because of the choice of $\bar{y}$ and the fact that the S^i are mutually disjoint, each set $B(g^w) \cap S^i (= B_g \cap S^i \cap H^w)$ $(i = 1,2,\cdots,\alpha(\bar{y}))$ is a single point.

Let $\rho^i: W^i \times D^i \to W^i$ be the projection map, and let $\beta^i = \rho^i \mid g(B_g \cap S^i)$. Then β^i is continuous, and one-to-one $((\beta^i)^{-1}(w)$ is the single point of $g(B_g \cap S^i) \cap (\{w\} \times D^i))$. Since $g(B_g \cap S^i)$ is compact, β^i is a homeomorphism onto W^i. Let d^i be the distance from $g(B_g \cap S^i)$ to $W^i \times \text{bdy}(D^i)$, and let Δ^i be the closed disk of radius d^i and center 0 in C. Let $\sigma^i: W^i \times \Delta^i \to W^i \times D^i$ be the map defined by $\sigma^i(w,x) = (\beta^i)^{-1}(w) + (0,x)$, where $+$ is vector addition, 0 is the origin of E^{n-2}, and $x \in D^i$ (in $C = E^2$). Since σ^i is continuous and one-to-one, $W^i \times \Delta^i$ is compact, σ^i is a homeomorphism (into). Since $\sigma^i(W^i \times \{0\}) = g(B_g \cap S)$, it follows from the theorem on invariance of domain that $g(B_g \cap S^i)$ is a tamely embedded $(n-2)$-cell. By [5, p. 533, (4.1)] $g \mid S^i$ (i.e., $f \mid S^i$) is topologically equivalent to $F_{n,d}$, for some d $(d = 2, 3, \cdots)$.

Let $\Omega \cap \text{int}(T^{n-2})$ be the maximal open set (possibly empty) such that $g \mid g^{-1}(\Omega \times D)$ is locally, at each point, topologically equivalent to one of the maps $F_{n,d}$. To review, we have seen that, for every open set Y in $\text{int}(T^{n-2})$, there exists (of course) a point $\bar{y} \in Y$ such that $\alpha(\bar{y}) \geqq \alpha(w)$ for all $w \in Y$; moreover, that there is an open neighborhood W of $\bar{y}$ with $W \subset \Omega$. Thus $\Omega \cap Y \neq \emptyset$. Since Y is an arbitrary open set in $\text{int}(T^{n-2}), \Omega$ is a dense open set in $\text{int}(T^{n-2})$. Therefore [9, p. 44, Theorem IV 3] its complement F in $\text{int}(T^{n-2})$ has dimension at most $n-3$

Let E be the set of points of B_g in $g^{-1}(\{w\} \times \text{int}D)$ for $w \in F$, and let $\pi: \text{int}V \to \text{int}(T^{n-2})$ $(\text{int}V = g^{-1}(\text{int}(T^{n-2}) \times \text{int}D))$ be the projection map. Then $\pi(E) = F$, and, since $\dim(B(g^w)) = 0$, $\dim(\pi^{-1}(w)) = 0$ for all $w \in F$. Since $g \mid \text{int}V$ is a pseudo-covering map, π is a closed map and by [9, pp. 91-92] $\dim E \leqq n-3$ This completes the proof of the first conclusion.

For the second conclusion, we will suppose throughout that $B_f \neq \emptyset$. If E is somewhere dense in B_f, then there exists an open set Λ in M^n such that $\Lambda \cap B_f \neq \emptyset$ and $\Lambda \cap B_f \subset E$. By the preceding argument, E is nowhere dense in the set of branch points at which the Jacobian matrix has rank at least $n-2$. Then $\Lambda \cap B_f \subset R_{n-3}$. Thus, if we still denote $f \mid \Lambda$ by f, it suffices to prove that $B_f \not\subset R_{n-3}$ (if $B_f \neq \emptyset$).

First suppose that $B_f \subset R_0$. Given $\bar{x} \in B_f$, let g be a pseudo-covering map given by [5, p. 530, (2.2) and p. 529, (1.4)] on a neighborhood V of $\bar{x}$, $V \subset E^n$, such that $g \mid g^{-1}(g(B_g))$ is a homeomorphism and $g(V) = E$. Then $g(B_g) \neq g(\bar{x})$ [5, p. 535, (5.6)]. It follows (see the first paragraph of the proof of (1.3)) from A. P. Morse's theorem [10] that for every point u in $g(B_g) - g(\bar{x})$ (and therefore in $g(R_0)$), there exists a closed n-cube X such that: $q(\bar{x}) \in \text{int } X$, $u \notin X$, its faces are parallel to the coordinate $(n-1)$-planes, and those faces are disjoint from $g(R_0)$. The restriction of g to each component of $g^{-1}(\text{bdy } X)$ is a covering map onto bdy X, and therefore that map is a homeomorphism. Since $\bar{x} \in B_g$, the degree of g is at least two; since $V \subset E^n$, $g^{-1}(\text{bdy } X)$ separates V into at least three components, each of which maps *onto* one of the components of $E^n - \text{bdy } X$ $(= g(V) - \text{bdy } X)$. This contradicts the fact that $g \mid g^{-1}(g(B_g))$ is one-to-one, so that $B_f \not\subset R_0$.

Thus $B_f \not\subset R_{n-3}$, for $n = 3$. We continue by induction on n. If $n \geqq 4$ and $B_f \subset R_{n-3}$, then there exists $\bar{x}$ in B_f at which the Jacobian matrix has rank at least one. We may suppose, by restriction, that the rank is at least one everywhere, and that f is the g of (1.1) for $q = 1$. Let γ be the $(n-1)$-plane of (1.1) that contains $\bar{x}$. Let $h = f \mid \gamma$, and let Q_{n-4} be the set of points in γ at which the Jacobian matrix of h has rank at most $n-4$; then, by the second conclusion of (1.1), $\gamma \cap R_{n-3} = Q_{n-4}$. Since $B_h \subset \gamma \cap B_f$, $B_h \subset Q_{n-4}$, contradicting the inductive hypothesis. Thus $B_f \not\subset R_{n-3}$ for $n \geqq 3$ (unless $B_f = \emptyset$), yielding the second conclusion.

The following extension of the inverse function theorem was proved in [4]. For $n = 2$ the analytic function $f(z) = z^2$ is an obvious counterexample.

2.2. COROLLARY. *Suppose that* $f: E^n \to E^n$, $n \geqq 3$, $f \in C^n$ *and* $\dim(R_{n-1}) = 0$ (R_{n-1} *is the set of zeros of the Jacobian determinant*). *Then* f *is a local homeomorphism.*

2.3. COROLLARY. *If* $f: E^n \to E^n$ *is light and* C^n, *then* $B_f = \emptyset$, $\dim(B_f) = n-2$, *or* $\dim(B_f) = n-1$; *the last case occurs if and only if* f *is not open.*

Proof. Since $B_f \subset R_{n-1}$, $\dim(f(B_f)) \leqq n-1$ (by (1.3)); since f is light, $\dim(B_f) \leqq n-1$. If $\dim(f(B_f)) \leqq n-2$, then f is open [5, p. 531, (2.4)], so that either $B_f = \emptyset$ or $\dim(B_f) = n-2$ (by (2.1)). Thus, $\dim(f(B_f)) = n-1$ if and only if f is not open [5, p. 531, (2.3)]. If, in this case, $\dim(B_f) < n-1$, then the Jacobian determinant of f would be either non-negative or nonpositive everywhere; thus (1.7) f would be open. As a result, $\dim(B_f) = n-1$ if and only if f is not open.

2.4. COROLLARY. *There exists a light open map* $f: E^5 \to E^5$ *which is not topologically equivalent to any* C^5 *map.*

The map is that given by [6, p. 620, (4.3)], so that B_f is not a 3-manifold at any point. If f were equivalent to a C^5 map, then at a dense set of its points B_f would be locally a 3-manifold (2.1).

2.5. REMARKS. Given a C' map $f: E^n \to E^n$, its directional derivative at x in the direction of the nonzero vector $(a_1, a_2, \cdots, a_n)$ is the length of the vector whose jth component is $\sum_{i=1}^n a_i D_i f_j(x)$. If f is a homeomorphism, it is called quasiconformal if (*) there exists $B > 0$ such that, for every point x in E^n and pair of vectors (directions) at x, the ratio of the directional derivatives is less than B. (This definition is equivalent to that given in [8].) A nonconstant complex analytic function f satisfies condition (*) (for $B = 1$) except on B_f, which consists of isolated points. Thus, it would be natural to call quasi-conformal (or quasi-analytic) light maps in E^n ($n > 2$) which satisfy condition (*), except at those points at which all directional derivatives are zero, i.e., R_0. We now observe that the only such C^n maps are local homeomorphisms (for $n > 2$).

Suppose that f is C^n, light, and not a local homeomorphism. If $R_{n-1} \subset R_0$, then $\dim(R_{n-1}) \leqq 0$ (by (1.3)). Thus f is a local homeomorphism (2.2), contradicting the supposition. If $R_{n-1} \not\subset R_0$, then there exists x at which the rank of Jacobian matrix is k, where $0 < k < n$. It follows from the definition of rank that there exist two vectors at x for which one directional derivative is positive, and the other is zero. Thus f does not satisfy condition (*).

We also remark that, except for local homeomorphisms, no C^n light open map is generic in the sense of Thom [14].

3. **Some examples.** Examples are given now to show that the exceptional set of dimension $n-3$ in (2.1) and the compactness hypothesis in (1.8) are necessary.

3.1. Lemma. *Given $\delta_q > 0$ ($q = 1,2,\cdots$), there exists a C^∞ map $\psi: E^1 \to E^1$ with the following properties*:

(1) *ψ is an even function,*

(2) *$\psi(r) = 0$ if and only if $r = 0$,*

(3) *$\psi'(r) > 0$ for $r > 0$, and*

(4) *the ith derivative $\psi^{(i)}(r) \leqq \delta_q$ ($0 \leqq r \leqq 1/q$; $i = 0,1,\cdots,q$; where $\psi^{(0)} = \psi$).*

The proof is omitted.

3.2. Lemma. *Given $\varepsilon_q > 0$ ($q = 1,2,\cdots$), there exists a C^∞ homeomorphism $h: E^n \to E^n$ such that on each set $S(0, 1/q)$ (where 0 is the origin) all h_i and all partial derivatives of order at most q are bounded by ε_q.*

Proof. Consider the class $\mathfrak{H}$ of all functions $h: E^n \to E^n$ such that $h_i(x) = \psi(r) \cdot x_i$, where $r = x_1^2 + x_2^2 + \cdots + x_n^2$ and $\psi: E^1 \to E^1$ is any C^∞ function. On each set $S(0, 1/q)$ there exists constants $\lambda_j > 0$ such that each h_i in $\mathfrak{H}$ and all its partials of order at most q are bounded by $\sum_{j=0}^q \lambda_j |\psi^{(j)}(r(x))|$. Let $\delta_q < \varepsilon_q / \sum_{j=0}^q \lambda_j$ ($q = 1,2,\cdots$), and let ψ_0 be given by (3.1) for $\{\delta_q\}$. Let $h_i(x) = \psi_0(r) \cdot x_i$. That h is a homeomorphism follows from conclusions (2) and (3) of (3.1).

3.3. Lemma. *Let U and L be, respectively, open and closed subsets of E^n. Let $f: U \to E^n$ be continuous, C^∞ on $U - L$, and constant on $U \cap L$; let V be a bounded open subset of U such that $\bar{V} \subset U$. Then there exists a homeomorphism $h: E^n \to E^n$ such that the restriction $hf | V \in C^\infty$.*

Proof. Throughout, symbols such as $\bar{V}$ refer to closure in E^n. Suppose that $f(L)=0$. Let $X_q = V \cap f^{-1}(S(0,\ 1/q))$, let $A_q = (X_q \cap V) - X_{q+1}$, and choose α_q $(0 < \alpha_q \leqq 1)$ less than the distance $d(\bar{A}_q, L)$ $(q = 1,2,\cdots)$. We will define h so that the partial derivatives (of all orders) of $hf \mid V$, call it F, are zero on $L \cap V$; h will be given by (3.2), where we need now specify the ε_q $(q = 1,2,\cdots)$.

The component functions (e.g., F_i) will be considered partials of order zero. Suppose h is given by (3.2) for $\varepsilon_q = \varepsilon_{q,0} = 1/q$. For $x \in A_q$, $f(x) \in S(0,\ 1/q)$, so that $|F_i(x)| < 1/q$ $(i = 1,2,\cdots,n)$.

Now suppose that numbers $\varepsilon_{q,m} > 0$ $(q = 1,2,\cdots;\ m = 0,1,\cdots,k;\ k$ fixed) have been defined so that

(1) any homeomorphism given by (3.2) for $\{\varepsilon_{q,k}\}$ will satisfy (a) $|P(x)| < 1/q$, for all $x \in A_q$ and all partials P of F with order m at most the minimum of k and q, and (b) $P(x^0) = 0$, for $x^0 \in L$;

(2) $\varepsilon_{q,m'} < \varepsilon_{q,m}$, whenever $m < m'$; and

(3) $\varepsilon_{q,m} = \varepsilon_{q,q}$, whenever $m > q$.

Let this property of the sequence $\{\varepsilon_{q,m}\}$ $(m = 0,1,\cdots,k)$ be called $\mathfrak{P}_k$ $(k=0,1,\cdots)$; we have seen that there exists $\{\varepsilon_{q,0}\}$ satisfying $\mathfrak{P}_0$. We proceed by induction. Assuming a sequence $\{\varepsilon_{q,m}\}$ $(m = 0,1,\cdots,k)$ satisfying $\mathfrak{P}_k$, we will find numbers $\varepsilon_{q,k+1}$ $(q = 1,2,\cdots)$ such that $\{\varepsilon_{q,m}\}$ $(m = 1,2,\cdots,k+1)$ satisfies $\mathfrak{P}_{k+1}$.

Given any partial P of F with order k, a natural number j $(j = 1,2,\cdots,n)$, and $x^0 \in L$,

$$D_j P(x^0) = \lim_{x_j \to x_j^0} \frac{P(x) - 0}{x_j - x_j^0}$$

(since $P(x^0) = 0$, by $\mathfrak{P}_k(1)$). Given x in A_q,

$$\frac{|P(x)|}{|x_j - x_j^0|} \leqq \frac{|P(x)|}{\alpha_q}.$$

Now each such P is on $\bar{V} - L$ a sum of products of partials of f and of h, all of orders at most k, each term having at least one partial of h as a factor. Since $\bar{A}$ is compact and $\bar{A}_q \cap L = \emptyset$, there is a uniform bound on the partials of f of order at most k. For $q \leqq k$, let $\varepsilon_{q,k+1} = \varepsilon_{q,q}$; for $q > k$, let $\varepsilon_{q,k+1}$ be chosen small enough so that $\varepsilon_{q,k+1} \leqq \varepsilon_{q,k}$ and, for any h given by (3.2) for $\varepsilon_{q,k+1}$, $|P(x)|/\alpha_q < 1/q$ (for all $x \in A_q$ and for all partials P of order at most k, a finite number of choices required for each q). Since $L \cup \bigcup_{q=1}^{\infty} A_q$ is a neighborhood of L, and since $P \mid L = 0$, all the partials of F of order at most $k+1$ are 0 on L, for h given by $\{\varepsilon_{q,k+1}\}$. It follows that $\{\varepsilon_{q,m}\}$ $(m = 1,2,\cdots,k+1)$ satisfies $\mathfrak{P}_{k+1}$.

Positive numbers $\varepsilon_{q,m}$ $(q,\ m = 1,2,\cdots)$ are defined, and the desired h is the one given by (3.2) for $\varepsilon_q = \varepsilon_{q,q}$; all its partials are zero on L.

To prove that $F \in C^\infty$ it is sufficient to prove that each partial P is continuous on L; let k be the order of P. By $\mathfrak{P}_k(1)$, $|P(x)| < 1/q$ $(x \in A_q;\ q = k, k+1, \cdots)$, so that $P(x) \to 0$ as $x \to x^0$, $x^0 \in L$.

3.4. Corollary. *There exists* $f: E^3 \to E^3$, f C^∞, *light and open, such that* B_f *has a point component.*

The map given in [6, p. 614, (3.3)] is topologically equivalent to a map simplicial except at the origin 0, and thus it is equivalent to a map C^∞ except at 0. From (3.3) we have the desired result.

Although B_f need not be locally connected, it follows from (2.1) that for $f: E^3 \to E^3$, f C^3, light and open, *each component* K *of* B_f *is locally connected.* (Suppose that K is not locally connected; then it contains [20, p. 19, (12.3)] a subcontinuum H such that K is not locally connected at any point of H. At each point x of $H - E$, where E is the exceptional set of (2.1), there exists a neighborhood U such that the restriction $f|U$ is a canonical map $F_{n,d}$. Since $H \cap U$ is a tame arc, we have a contradiction.)

The example whose branch set has a Cantor set of point components [6, p. 614] is also equivalent to a C^∞ map. (Appropriate modifications of (3.2) and (3.3) are required.)

For another example of a C^∞ (3-to-1) open map, let z be a complex variable, t real, and let $f: E^3 \to E^3$ be defined by

$$f(z,t) = (z^3 - 3ze^{-2t^{-2}} \sin^2 t^{-1}, t).$$

Then (with $z = x + iy$) B_f is the union of the curves $x = \pm e^{-t^{-2}} \sin t^{-1}$ in the $(x-t)$-plane. Still another example is given in (2.4).

The following remark answers in the negative question II of [13, p. 266].

3.5. Remark. *There exists a* C^∞ 3-*to*-1 *open map* $f: E^3 \to E^3$ *which is not topologically equivalent to any real analytic map.*

We use the map above for which B_f has a Cantor set X of point components, or one with a sequence of point components converging to a point. It follows from (1.1) (see the proof of (2.1)) that $X \subset R_0$. Suppose that f is real analytic. Then R_0 is an analytic set (the zeros of $\sum_{i,j}(D_j f_i)^2$), and thus [3, p. 141] is locally connected. Since $\dim(R_0) = 0$ (1.3), we have a contradiction.

3.6. Theorem. *There exists a* C^∞ *open map* $f: E^2 \to E^2$ *which is not light.*

Proof. The domain of f will actually be the square S given by $|x| < 1$ and $|y| < 1$; let L be the intersection of the y-axis with S. Let r^j $(j = 1, 2, \cdots)$ be any countable dense subset of $L - \{0\}$, and let $h: S \to S$ be given by $h(x,y) = (x, xy)$.

If $X_{j,k}$ $(j, k = 1, 2, \cdots)$ are the subsets of $h(S)$ defined by

$$2^{-2^{j-1}(2k-1)} \leqq x \leqq \left(\frac{3}{2}\right) \cdot 2^{-2^{j-1}(2k-1)},$$

then their closures are mutually disjoint and each

$$X_{j,k} \subset S(0, 3 \cdot 2^{-2^{j-1}(2k-1)}).$$

Let $g: h(S) \to E^2$ be a map such that

(1) $g \mid (h(S) - \{0\})$ is a C^∞ local homeomorphism,

(2) $g \mid (h(S) - \bigcup_{j,k} X_{j,k})$ is the identity map,

(3) $g(X_{j,k}) \subset S(0, 3 \cdot 2^{-2^{j-1}(2k-1)})$, and

(4) there exists a point $p^{j,k}$ common to $h^{-1}(X_{j,k})$ and the line $y = r^j$ such that $g(h(p^{j,k})) = 0$ $(j, k = 1, 2, \cdots)$.

(By (1) and (3) g is continuous at 0.)

Given any neighborhood U of r^j, there exists $p^{j,k} \in U$; since $g(h(r^j)) = 0$, it it follows from conditions (1) and (4) tha $tgh(r^j) \in \operatorname{int}(gh(U))$. Since the r^j are dense in L, and since $gh \mid (S - L)$ is a (C^∞) local homeomorphism, gh is open. The result follows from (3.3).

REMARK. Given any three natural numbers j, k, and n such that (1) $0 \leqq j \leqq \min(k-1, n-2)$, (2) $1 \leqq k \leqq n-1$, and (3) $n \geqq 2$, modifications of the above argument yield a nonlight C^∞ open map $f: E^n \to E^n$ for which $\dim(f(B_f)) = j$ and $\dim(B_f) = k$.

3.7. REMARK. If $f: E^n \to E^n$ is C^n open, but not light, then for every k $(k = 1, 2, \cdots)$ there exists x such that $f^{-1}(x)$ consists of isolated points, at least k in number. The proof is similar to that of (1.8).

4. **A counterexample to a statement of Stoilow.** In [13] S. Stoilow states that, if $f: E^3 \to E^3$ is light open, then $\dim(B_f) \leqq 1$. His proof employs the following lemma [13, pp. 263-264]: Let $x \in E^3$, and let B_ρ be the geometric ball of radius ρ and center $f(x)$. Then there exists $r > 0$ and a compact neighborhood D of x such that $f \mid D$ is open and $f(D) = B_r$. There exist positive numbers ε_1 and ε_2 such that the number of components of $f^{-1}(B_\rho)$ is the same for all ρ with $0 < \varepsilon_1 < \rho < \varepsilon_2$. Moreover, for any such set of numbers $\varepsilon_1, \varepsilon_2$, and ρ, *each component of* $f^{-1}(\operatorname{bdy}(B_\rho))$ *is a 2-manifold*. The last statement is false in general.

It appears that a modification of the proof of (2.1) using this statement would yield (2.1) for $n = 3$ and f light open but not necessarily differentiable. For this reason it seems worthwhile to give a counterexample here.

We write E^3 as $E^1 \times C$, where C is the complex plane, and let X_m be the set of (t, z) such that either $|t| \leqq 2^{-m}$ and $|z| \leqq 2^{-m}$, or $2^{-m} \leqq t \leqq 3 \cdot 2^{-m-1}$ and $|z - 2^{-m-1}| \leqq 2^{-m-1}$ $(m = 1, 2, \cdots)$. Then $X_{m+1} \subset \operatorname{int}(X_m)$, and there exists a homeomorphism $h: E^3 \to E^3$ such that $K_m = h(\operatorname{bdy}(X_m))$ is a geometric 2-sphere about the origin 0. The map $hF_{3,2}$ is the desired counterexample f, since $f^{-1}(K_m)$ $(= F_{3,2}^{-1}$ $(\operatorname{bdy}(X_m)))$ is not a 2-manifold while $f^{-1}(h(X_m))$ is connected $(m = 1, 2, \cdots)$.

Stoilow uses a characterization of compact 2-manifolds in E^3 due to Wilder [18, Theorem 21], and the sets $f^{-1}(K_m)$ fail to satisfy the first conclusion of that theorem. With a suitable modification of the sets X_m, the sets $f^{-1}(K_m)$ also fail to satisfy the second conclusion.

Added in Proof. J. Väisälä has kindly pointed out to the author the following simple example of a C^∞ map $f:E^2 \rightarrow E^2$ which is open but not light (cf. 3.6). For $z = x + iy$ and $x \neq 0, f(z) = \exp(-z/x^3); f(iy) = 0$. Except on the imaginary axis f is a local homeomorphism.

In *Images of critical sets*, Ann. of Math. (2) **68** (1958), 247–259, Arthur Sard considers maps f of $U \cdots E^n$ into E^p. He proves under very general differentiability hypotheses that if R_k is the countable union of sets of finite Hausdorff $(k+1)$-measure, then the $(k+1)$-measure of $f(R_k)$ is 0. It follows [9, p. 104] that $\dim(f(R_k)) \leqq k$. Thus, in this case (1.3) is the consequence of a more general result. In general, however, R_k need not be the countable union of sets of finite $(k+1)$-measure.

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