ON AMITSUR'S COMPLEX AND RESTRICTED LIE ALGEBRAS(1)

BY

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1. **Introduction.** Given a commutative ring C and a commutative C-algebra A, Amitsur introduced a complex which will be described in §2. We will denote the cohomology of this complex by $H^n(A)$.

If F is a separable field extension of C, the groups $H^n(F)$ have been described in [7]. However, in the inseparable case no general results on $H^n(F)$ were known. Our main results deal with that case.

Let C be an imperfect field of characteristic p and F a finite purely inseparable extension of exponent one. Hochschild [6] introduced the notion of a *regular restricted Lie algebra extension of F by T*, where T is the restricted Lie algebra of derivations of F over C. He also showed that the group of regular restricted Lie algebra extensions of F by T is equivalent to the Brauer group of F over C. In [7] Rosenberg and Zelinsky introduced for each Amitsur 2-cocycle t of F a regular restricted extension of F by T, denoted by $\Theta(t)$, and showed that under the correspondence induced by $t \to \Theta(t)$, $H^2(F)$ is isomorphic to the group of extensions introduced by Hochschild. In §4 we define regular restricted Lie algebra extensions of M by T where M is an abelian restricted Lie algebra satisfying some additional hypotheses. We also define for each n a restricted Lie algebra K^n which serves as an M. We then show that for certain classes of kernels, including the K^n for $n > 2$, any regular restricted Lie algebra extension of the kernels by T must split. In §5, by a definition similar to that of the $\Theta(t)$, we define for each n-cocycle t of the Amitsur complex, a restricted Lie algebra extension $\Theta_n(t)$ of K^n by T. This extension is regular in the sense of §4 and has the property that if $\Theta_n(t)$ splits then t must be a coboundary. Then we use the results of §4 to prove that for F as above and $n > 2$, $H^n(F) = 0$.

We begin by showing that if C is a field and A an arbitrary algebra, $H^1(A) = 0$. This may be viewed as a generalization of Hilbert's Theorem 90 (cf. [1, Theorem 6.1; 7, Theorem 1]).

2. **Notations.** Let A be a commutative C-algebra with unit, where C is a field. Let $A^n = A \otimes_C \cdots \otimes_C A$ (n factors). We define C-algebra monomorphisms $\varepsilon_i : A^n \to A^{n+1}$ by

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$\varepsilon_i(a_i \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n$ for $i = 1, \cdots, n+1$(2).

As usual we let A^{n*} be the multiplicative group of units of the algebra A^n. We define

$$\Delta n : A^{n*} \to A^{n+1*}$$

by

$$\Delta_n(x) = \varepsilon_1(x)\varepsilon_2(x^{-1})\cdots\varepsilon_{n+1}(x^{\pm 1}).$$

Clearly Δ_n is a homomorphism and it was shown in [1, Theorem 5.1] by Amitsur that the sequence of groups and homomorphisms (A^{n*}, Δ_n) form a complex, i.e., $\Delta_{n+1}\Delta_n = 1$. We shall call this the Amitsur complex over A and denote it by $Q(A)$. The cohomology group $\mathrm{Ker}(\Delta_{n+1})/\mathrm{Im}(\Delta_n)$ will be denoted by $H^n(A)$. For the significance of these groups see [1] and [7].

We shall also use the following variant of $Q(A)$: Let $\delta_n = \varepsilon_2 - \varepsilon_3 + \cdots \pm \varepsilon_{n+1}$. Then the sequence of additive groups and homomorphisms (A^n, δ_n) is easily seen to be exact [7, Lemma 4.1].

3. **The first cohomology group.** Let A be a commutative C-algebra with unit. We shall show that $H^1(A) = 0$. To begin with we prove

LEMMA 3.1. *Let V and W be vector spaces over the field C. Then if $x \neq 0$ is an element of $V \otimes_C W$, we have*

$$x = \sum_1^s v_i \otimes w_i$$

where the v_i and w_i are linearly independent in V and W, respectively (*cf.* [3, §1, *Exercise* 6]).

Proof. From the definition of $V \otimes_C W$ we know that every element of $V \otimes_C W$ can be written as $\sum_{i=1}^n f_i \otimes g_i$, with f_i in V and g_i in W. We define s = rank of x as the smallest positive integer such that x is the sum of s tensor products $f \otimes g$. Then if x has rank s, $x = \sum_{i=1}^s v_i \otimes w_i$. Suppose the v_i were linearly dependent. Then we could write

$$v_r = \sum_{i \neq r} c_i v_i, \qquad \text{with } c_i \text{ in } C.$$

But this would make $x = \sum_{i \neq r} v_i \otimes (w_i + c_i w_r)$ of rank less than s, contradicting the definition of s. Similarly, the w_i's must be linearly independent.

THEOREM 1. *Let C be a field and let A be any commutative C-algebra with unit. Then $H^1(A) = 0$.*

Proof. Let x be a cocycle of rank s in A^{2*}. We may then write $x = \sum_1^s v_i \otimes w_i$ with the v_i, w_i elements of A and the sets $\{v_i\}$ and $\{w_i\}$ linearly independent over C. Since x is a cocycle we have $\Delta_2(x) = 1$, or

(2) We shall not put superscripts n on the ε_i to indicate that the domain of ε_i is F^n since the domain will be apparent from context.

$$\left(\sum_1^s 1\otimes v_i\otimes w_i\right)\left(\sum_1^s v_j\otimes w_j\otimes 1\right) = \sum_1^s v_i\otimes 1\otimes w_i$$

which can be written as

$$(3.1) \qquad \sum_{i,j=1}^{s} v_j\otimes(v_iw_j-\delta_{ij})\otimes w_i = 0 \qquad \text{(Kronecker delta)}.$$

Since for $\{a_\alpha\}$ any basis of A, the elements $\{v_j\otimes a_k\otimes w_i\}$ are linearly independent over C [3, §1, Corollary 2 to Proposition 7, p. 11, used twice], then if $v_iw_j-\delta_{ij}=\sum_k c_{ijk}a_k$, we have that

$$\sum_{i,j,k} c_{ijk}v_j\otimes a_k\otimes w_i = 0.$$

Hence each c_{ijk} must be 0, and so $v_iw_j-\delta_{ij}=0$. Thus each v_i is a unit with inverse w_i. But $i\neq j$ implies $v_iw_j=0$, hence $s=1$ and $x=v\otimes w$ with $vw=1$. But $\Delta_1(w)=(1\otimes w)(w\otimes 1)^{-1}=v\otimes w=x$, so that $H^1(A)=0$.

In [2, Theorem 3.8] Amitsur showed that $H^1(A)=0$ if C is a direct sum of local rings and A is a *finitely generated* free C-module. Thus the two theorems are different.

4. **Split regular extensions.** Now let A be a finite-dimensional purely inseparable extension field F of exponent 1 over C. Then $F=C[\alpha_1,\cdots,\alpha_n]$ and α_1^p is in C, where p is the characteristic of C, [4, Proposition 1, p. 190] or [1, p. 107]. Let T be the set of derivations of F over C. If D_1,D_2 lie in T it is easily verified that for all f in F, $fD_1, D_1D_2-D_2D_1$ and D_1^p (i.e., D_1 iterated p times) are again in T [6, Theorem 1, p. 478]. Thus T is a left F-vector space and a restricted Lie algebra over C. For f_1,f_2 in F and D_1, D_2 in T, these structures are related by

$$(4.1) \qquad [f_1D_1, f_2D_2] = f_1D_1(f_2)\cdot D_2 - f_2D_2(f_1)\cdot D_1 + f_1f_2[D_1,D_2].$$

$$(4.2) \qquad (f_1D_1)^p = f_1^pD_1^p + (f_1D_1)^{p-1}(f_1)\cdot D_1$$

as shown in [6, p. 481].

A left F-vector space M will be called a *regular T-module* if we are given a map of T into the endomorphisms of the abelian group M such that for all D, D' in T, m in M, and f in F

$$(4.3) \qquad \begin{aligned}(D+D')(m) &= D(m)+D'(m),\\ [D,D'](m) &= D(D'(m))-D'(D(m)),\\ D^p(m) &= D(\cdots(D(m))\cdots) \qquad (p \text{ times}),\\ (fD)(m) &= f\cdot D(m),\\ D(fm) &= D(f)\cdot m + f\cdot D(m).\end{aligned}$$

In [4, Proposition 3, p. 194] it was shown that if M is a regular T-module and N is the subset of elements of M such that

$$Dn = 0, \qquad \text{for all } D \text{ in } T,$$

then M is isomorphic as a regular T-module to $F \otimes_C N$ via the map $f \otimes n \to fn$. Here $F \otimes_C N$ is a regular T-module with the usual action of T on F, i.e., $D(\sum_i f_i \otimes n_i) = \sum_i D(f_i) \otimes n_i$.

We shall need something of the general theory of restricted Lie algebra extensions, and we recollect here those facts which we shall use. First let us note that if L is a restricted Lie algebra then for any x, y in L we have

$$[x^p, y] = [x[\cdots[x, y]\cdots]] \quad (p \text{ times}), \tag{4.4}$$

where x^p is the image of x under the pth power map of L [5, p. 559].

Next let M be an abelian restricted Lie algebra, i.e., $[M, M] = 0$. By an extension E of M by T is meant a restricted Lie algebra E containing M and a restricted Lie algebra homomorphism ϕ of E onto T, i.e., an exact sequence of restricted Lie algebras and restricted Lie algebra homomorphisms

$$0 \to M \xrightarrow{i} E \xrightarrow{\phi} T \to 0.$$

As usual, an action of T on M is defined by

$$Dm = [e, m] \text{ with } \phi(e) = D.$$

From the axioms for restricted Lie algebras it then follows that the action of T on M satisfies the first three conditions of (4.3). The pth power map on E satisfies

$$(e + m)^p = e^p + \phi(e)^{p-1}(m) + m^p \tag{4.5}$$

by [5, p. 564]. We shall need this relation several times.

If M is not only a restricted Lie algebra but also an F-vector space satisfying

$$(fm)^p = f^p m^p \tag{4.6}$$

we may consider the '*regular*' restricted extensions of M by T as introduced by Hochschild [6, pp. 481–482] (3). These are restricted extensions in which E is a left F-space, i and ϕ are F-linear, and which satisfy for f_1, f_2 in F and e_1, e_2 in E,

$$(f_1e_1)^p = f_1^p e_1^p + \phi(f_1e_1)^{p-1}(f_1) \cdot e_1, \tag{4.7}$$

$$[f_1e_1, f_2e_2] = f_1\phi(e_1)(f_2) \cdot e_2 - f_2\phi(e_2)(f_1) \cdot e_1 + f_1f_2[e_1, e_2]. \tag{4.8}$$

(3) Hochschild treats only the case $M = F$ but the definition extends readily to our M.

If $D = \phi(e_1)$, applying (4.8) with $f_2 = 1$ and $e_2 = m$ in M, we find

$$(f_1 D)m = f_1 \cdot D(m).$$

Also (4.8) with $f_1 = 1$ and $f_2 = f$, $e_2 = m$ in M yields

$$D(fm) = D(f) \cdot m + f \cdot D(m).$$

Thus M is a regular T-module under the natural action of T on M. It is worth noting that (4.6) must be assumed if (4.7) is to be valid for e_1 in M. Conversely, if M is a regular T-module which is an abelian restricted Lie algebra such that (4.6) is valid, we shall consider regular restricted extensions of M by T which induce the given regular T-module structure and p-map on M. As usual we shall say that a regular extension E splits if there is an F-linear restricted Lie algebra homomorphism ψ of T to E such that $\phi\psi$ is the identity on T. It is the main purpose of this section to show that for a large class of regular T-modules, all regular restricted extensions split.

THEOREM 2. *Let F be a finite purely inseparable extension field of C of exponent 1 and let T be the restricted Lie algebra of derivations of F over C. Let M be a regular T-module which is a strongly abelian restricted Lie algebra, i.e., $M^p = 0 = [M,M]$. Let*

$$0 \to M \xrightarrow{i} E \xrightarrow{\phi} T \to 0$$

be a regular restricted extension of M by T which induces the given T-module structure on M. Then the extension splits as a regular extension.

Before proving Theorem 2, we recall that the generators α_i of F may be chosen in such a way that $\alpha_1^{i_1} \cdots \alpha_n^{i_n}$ with $0 \leqq i_j \leqq p-1$ are a C-basis of F (cf., e.g., [4, Proposition 1, p. 190]). Let T_0 be the restricted Lie algebra spanned over C by the derivations D_i given by

$$D_i(\alpha_j) = \delta_{ij}.$$

It is easily verified that $[D_i, D_j] = 0$ and $D_i^p = 0$ [4, p. 192, (6)] so that T_0 is itself a strongly abelian restricted Lie algebra.

Then the first part of the proof of Theorem 2 consists of

LEMMA 4.1. *Given M, T, and the extension $0 \to M \to E \xrightarrow{\phi} T \to 0$ as in Theorem 2, there is a C-linear restricted Lie algebra isomorphism of T_0 into E which is inverse to ϕ.*

Proof. If $[T_0 : C] = 1$ then we construct an isomorphism of T_0 into $E_0 = \phi^{-1}(T_0)$ as follows: Let e_0 be in $\phi^{-1}(D_1)$. Then $\phi(e_0)^p = 0$ and so $e_0^p = m_0$ in M. Let $e_1 = e_0 + \alpha_1^{p-1} m_0$. Now

$$D_1(m_0) = [e_0, m_0] = [e_0, e_0^p] = [\cdots[e_0, e_0]\cdots e_0] = 0$$

by (4.4). Hence (4.5) shows that

$$\begin{aligned} e_1^p &= (e_0 + \alpha_1^{p-1}m_0)^p = e_0^p + D_1^{p-1}(\alpha_1^{p-1}m_0) + (\alpha_1^{p-1}m_0)^p \\ &= m_0 + D_1^{p-1}(\alpha_1^{p-1})m_0 \end{aligned}$$

since $D_1(m_0) = 0$ and $(\alpha_1^{p-1}m_0)^p = 0$, M being strongly abelian. But

$$m_0 + D_1^{p-1}(\alpha_1^{p-1})m_0 = m_0 + (p-1)!\,m_0 = m_0 - m_0 = 0$$

by Wilson's theorem. Hence the map $CD_1 \to Ce_1$ is the inverse map sought if $[T_0 : C] = 1$.

Suppose for $[T_0 : C] = n - 1$ such an isomorphism exists. Let

$$V_0 = CD_1 \otimes \cdots \otimes CD_{n-1}.$$

Then we have the restricted Lie algebra extensions

$$0 \to M \to E_0 \overset{\phi_0}{\to} T_0 \to 0,$$

$$0 \to M \to G_0 \overset{\beta_0}{\to} V_0 \to 0,$$

where $E_0 = \phi^{-1}(T_0)$, $G_0 = \phi^{-1}(V_0)$ and ϕ_0 is restricted to E_0, β_0 is ϕ restricted to G_0. By the induction hypothesis there is a restricted Lie algebra isomorphism $\gamma_0 : V_0 \to G_0$ with $\beta_0\gamma_0 =$ identity on V_0. Let e_0 in E_0 be any element of ϕ_0^{-1} (D_n). Then $e_0^p = m_0$ as above. Let $e_1 = e_0 + \alpha_n^{p-1}m_0$. Just as for the case $n = 1$, we have $e_1^p = 0$ and $\phi_0(e_1) = D_n$. Now $[\beta_0(D_1), e_1] = m_1$ is in M, since

$$\phi[\beta_0(D_1), e_1] = [D_1, D_n] = 0.$$

But $D_1^{p-1}m_1 = [\beta_0(D_1), [\cdots[\beta_0(D_1), e_1]\cdots]] = [\beta_0(D_1)^p, e_1]$ by (4.4), $= 0$ since $\beta_0(D_1)^p = 0$.

As was noted above, a C-basis of F is given by $\prod_{i=1}^n \alpha_i^{j_i}$ with $0 \leqq j_i \leqq p-1$. Hence it is readily verified that an element of F has the property $D_i^{p-1}f = 0$, if and only if it may be written as $\sum_{j=0}^{p-2} \alpha_i^j \gamma_j$, where $D_i(\gamma_j) = 0$. Since $M \cong F \otimes_C N$ with $TN = 0$ by the map $f \otimes n \to fn$, every element of M can be written uniquely as a linear combination of the monomials $\prod_{i=1}^n \alpha_i^{j_i}$ with coefficients in N. Therefore, it again follows that an element m of M has the property $D_i^{p-1}m = 0$ if and only if $\sum_{j=0}^{p-2} \alpha_i^j \gamma_j$ where $D_i(\gamma_j) = 0$. For these elements we may define '*integrals with respect to D_i*': If $m = \sum_{j=0}^{p-2} \alpha_i^j \gamma_j$ we set

$$\int_i m = \sum_{j=0}^{p-2} \frac{\alpha_i^{j+1}}{j+1} \gamma_j,$$

Note that $\int_i m$ is in M. Clearly

$$D_i \int_i m = m \text{ for all } m \text{ with } D_i^{p-1} m = 0.$$

Hence $[\beta_0(D_1), e_1] = m_1$ has an integral with respect to D_1. Set $e_2 = e_1 - \int_1 m_1$. Then

$$[\beta_0(D_1), e_2] = m_1 - \left[\beta_0(D_1), \int_1 m_1\right] = m_1 - D_1 \int_1 m_1$$
$$= m_1 - m_1 = 0.$$

Furthermore $e_2^p = e_1^p - D_n^{p-1} \int_1 m_1$ using (4.5) and the fact that $M^p = 0$. Now from the definition of $\int_i m$ it is clear that if $i \neq j$, then $D_j \int_i m = \int_i D_j m$ since $D_i D_j = D_j D_i$. But $D_n^{p-1} m_1 = [e_1 [e_1 [\cdots [m_1] \cdots]]]$ ($p-1$ factors) $= [e_1 [e_1 [\cdots [\beta_0(D_1), e_1] \cdots]]] = -[e_1^p, \beta_0(D_1)]$ by (4.4). Since $e_1^p = 0$, then $D_n^{p-1} m_1 = 0$. Hence $D_n^{p-1} \int_1 m_1 = 0$ and so finally $e_2^p = 0$.

Next let $[\beta_0(D_2), e_2] = m_2$. Just as before, $D_2^{p-1} m_2 = D_n^{p-1} m_2 = 0$ and so if we set $e_3 = e_2 - \int_2 m_2$, we have $\phi(e_3) = D_n$, $e_3^p = 0$, and $[\beta_0(D_2), e_3] = 0$. Moreover, $[\beta_0(D_1), e_3] = [\beta_0(D_1), - \int_2 m_2] = - D_1 \int_2 m_2 = - \int_2 D_1 m_2$. Now using the Jacobi identity $D_1 m_2 = [\beta_0(D_1), [\beta_0(D_2), e_2]] = -[e_2, [\beta_0(D_1), \beta_0(D_2)]] - [\beta_0(D_2), [e_2, \beta_0(D_1)]] = 0$ since β_0 is a homomorphism and $[e_2, \beta_0(D_1)] = 0$. Hence $[\beta_0(D_1), e_3] = 0$.

Repeating this procedure will finally yield an element e_n of E_0 with $\phi(e_n) = D_n$, $e_n^p = 0$, and $[\beta_0(D_i), e_n] = 0$ for $i = 1, 2, \cdots, n-1$. Then defining $\psi_0 = \beta_0$ on V_0 and $\psi_0(D_n) = e_n$ gives the desired map, proving Lemma 4.1.

Proof of Theorem 2. Given $0 \to M \to E \xrightarrow{\phi} T \to 0$, let ψ_0 be the restricted Lie algebra map from T_0 to E_0 as given by Lemma 4.1. Since the D_i constitute a left F-basis of T [4, Proposition 2, p. 191], we may define $\psi: T \to E$ by $\psi(fD_i) = f \cdot \psi_0(D_i)$ for f in F and D_i in T_0 and additivity. This ψ is clearly F-linear, and if we can show that it is a restricted Lie algebra homomorphism, then it is the required splitting map. But by repeated applications of (4.1), (4.2), (4.7) and (4.8) we have for f, g in F, D_i, D_j in T_0

$$\psi[fD_i, gD_j] = [\psi(fD_i), \psi(gD_j)] \tag{4.9}$$

and

$$\psi((fD_i)^p) = (\psi(fD_i))^p. \tag{4.10}$$

(4.9) coupled with the additivity of ψ, shows that ψ is a Lie algebra homomorphism.

By [5, p. 559, (3)] applied n times we know that

$$\left(\sum_{i=1}^{n} f_i D_i\right)^p = \sum_{i=1}^{n} (f_i D_i)^p + \Gamma(f_1 D_1, \cdots, f_n D_n),$$

where Γ is a sum of commutators in the $f_1 D_1$. Then

$$\psi\left(\left(\sum_1^n f_i D_i\right)^p\right) = \sum_{i=1}^{n} \psi((f_i D_i)^p) + \psi(\Gamma(f_1 D_1, \cdots, f_n D_n))$$

by the additivity of ψ. By (4.10) this is

$$\sum_{i=1}^{n} (\psi(f_iD_i)^p) + \psi(\Gamma(f_1D_1,\cdots,f_nD_n)).$$

Since we already saw that ψ is a Lie algebra homomorphism, this is

$$\sum_{i=1}^{n} \psi(f_iD_i)^p + \Gamma(\psi(f_1D_1),\cdots,\psi(f_nD_n))$$

which, again by [5, p. 559] is $(\sum_{i=1}^{n}\psi(f_iD_i))^p$. Therefore $\psi: T \to E$ splits the extension as required.

An alternate proof for Theorem 2 can be obtained by use of [5, Theorem 3.3, p. 574]. We shall outline this method of proof here. Let

$$T_1 = C\cdot\alpha_1D_1 \oplus\cdots\oplus C\cdot\alpha_nD_n,$$

where as before $D_i\alpha_j = \delta_{ij}$. Then T_1 is an abelian restricted Lie algebra such that $(\alpha_iD_i)^p = \alpha_iD_i$ and clearly the α_iD_i are an F-basis for T [6, p. 479]. Hochschild shows that his restricted Lie algebra cohomology groups $H_*^n(T_1,M)$ for T_1 in M are zero whenever M is a finite-dimensional T_1-module, since the universal restricted enveloping algebra of T_1 is semisimple [6, p. 479]. But [5, Theorem 3.3, p. 574] states that if M is a strongly abelian restricted Lie algebra on which the restricted Lie algebra L operates, then there is a canonical isomorphism of $\text{ext}_*(M,L)$ onto $H_*^2(L,M)$. Here $\text{ext}_*(M,L)$ is the set of equivalence classes of restricted Lie algebra extensions of M by L which induce the given L-module structure and p-map on M, modulo the split extensions.

Now let $0\to M\to E \xrightarrow{\phi} T\to 0$ by any regular restricted extension of M by T. Let $E_1 = \phi^{-1}(T_1)$ and ϕ_1 be ϕ restricted to E_1. Since M is strongly abelian, the restricted extension $0\to M\to E_1 \xrightarrow{\phi_1} T_1\to 0$ must split by the above remarks. Thus we have a restricted Lie algebra monomorphism $\psi_1: T_1\to E_1$ inverse to ϕ_1. The extension of ψ_1 to ψ by F-linearity can again be computed directly. Since the proof of [5, Theorem 3.3] is fairly complicated, we have preferred giving a proof avoiding this result.

For our next theorem we require the following:

LEMMA 4.2. *Let M be a regular T-module and an abelian restricted Lie algebra which contains F as an abelian restricted subalgebra. Suppose for f in F and m in M, $(fm)^p = f^pm^p$ and m^p is in C. If $0\to M\to E\xrightarrow{\phi} T\to 0$ is a regular restricted extension of M by T, then there is an F-linear Lie algebra isomorphism $\psi: T\to E$ such that $\phi\psi$ is the identity on T and $\psi(D)^p - \psi(D^p)$ is in C for each D in T.*

(This is a slight generalization of [6, Theorem 4].)

Proof. A p-semilinear map of a C-vector space A into a C^p-vector space B is an additive group homomorphism β such that $\beta(cx) = c^p\beta(x)$ for x in A. If A is an abelian restricted Lie algebra, then the p-map on A is a p-semilinear map of A into itself [5, (2), (3), p. 559].

Let $e \to e'$ be a p-semilinear map of E, as an F-vector space, into C such that $m' = m^p$. Now we define a new p-map on M and E by

$$e^{[p]} = e^p - e'.$$

It is easily seen that E is still a restricted Lie algebra with this new p-map. Moreover E remains a regular extension with the new p-maps on E and M since for f in F and e in E, we have

$$\begin{aligned}(fe)^{[p]} &= f^p e^p + (f\phi(e))^{p-1}(f) \cdot e - f^p e' \\ &= f^p e^{[p]} + (f\phi(e))^{p-1}(f) \cdot e.\end{aligned}$$

But with the new p-map, M is a strongly abelian restricted Lie algebra as well as a regular T-module. Then by Theorem 2 this extension with the new p-map splits as a regular extension: $\psi: T - E$ is an F-linear Lie algebra monomorphism inverse to ϕ such that

$$\psi(D^p) = \psi(D)^{[p]} = \psi(D)^p - \psi(D)'.$$

Thus $\psi(D)^p - \psi(D^p) = \psi(D')$ which lies in C, completing the proof.

We now introduce a specific M which will arise in connection with Amitsur's complex. We first recall here how F^n was given the structure of a regular T-module in [7, p. 347]:

By the usual properties of tensor products we may define a map

$$^{(n)}D: F^n \to F^n$$

by

$$^{(n)}D(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = D(x_1) \otimes x_2 \otimes \cdots \otimes x_n.$$

We also consider F^n as a left F-vector space via the action of F on the first factor of F^n. It is immediate with these definitions that F^n is a regular T-module. It is immediate also that it is an abelian restricted Lie algebra such that $(fx)^p = f^p x^p$ for f in F and x in F^n, where the Lie product and p-map are given by commutation and the associative pth power in F^n.

With these definitions it is clear also that the ε_i's are F-linear for $i > 1$ and that

$$(4.11) \qquad {}^{(n)}D\varepsilon_1 = 0, \text{ while for } i > 1,\ \varepsilon_i^{(n)}D = {}^{(n+1)}D\varepsilon_i.$$

Now let $K^{2n+2} = (\varepsilon_2 - \varepsilon_3 + \cdots + \varepsilon_{2n+2})F^{2n+1}$, then we have for all D in T, that ${}^{(2n+2)}D\,K^{2n+2} \subset K^{2n+2}$ and for all f in F, $f\,K^{2n+2} = K^{2n+2}$. Thus K^{2n+2} is a regular T-module.

Since $F^p \subset C$, we have $(F^n)^p \subset C$ so that $(K^{2n+2})^p \subset C$ also. Now, clearly $[(\varepsilon_i - \varepsilon_{i+1})x]^p = 0$ so that $[(\varepsilon_2 - \varepsilon_3 + \cdots + \varepsilon_{2n+2})x]^p = (\varepsilon_{2n+2}x)^p = x^p$ and so the pth power is not identically zero on K^{2n+2}. *It is this fact that necessitates a separate proof for Theorem* 3.

With these definitions we make K^{2n+2} into an abelian Lie algebra by setting $[k,k']=0$ for k, k' in K^{2n+2}. We clearly have $C \subset F \otimes 1 \otimes \cdots \otimes 1 \subset K^{2n+2}$. Hence the associative pth power map sends K^{2n+2} into itself, so that K^{2n+2} is an abelian restricted Lie algebra such that $(K^{2n+2})^p \subset C$ and for f in F and k in K^{2n+2}, $(fk)^p = f^p k^p$. If an element y of F^n is of the form $x \otimes 1 \otimes \cdots \otimes 1$, where x lies in F^r and there are $n-r$ factors of 1, we abbreviate the expression for y to $x \otimes 1^{n-r}$. Thus $F \subset K^{2n+2}$ as $F \otimes 1^{2n+1}$.

Before stating and proving Theorem 3 we recall one more concept from [7]: Let t be a cocycle in F^{3*}. For any q in F^2 let $L(q)$ be the endomorphism of F^2 given by left multiplication by q. We set

$$\Theta(t) = \left\{ {}^{(2)}D + L(q) \,\middle|\, D \text{ in } T \text{ and } q \text{ in } F^2 \text{ such that } (\varepsilon_2 - \varepsilon_3)q = \frac{{}^{(3)}Dt}{t} \right\}.$$

Clearly $\Theta(t) \subset \mathrm{Hom}_{1\otimes F}(F^2,F^2)$. It was shown in [7, Lemma 4.6, p. 350] and we shall generalize this fact later on, that $\Theta(t)$ is a regular restricted extension of F by T. The map $\phi:\Theta(t) \to T$ is given by $\phi({}^2D + L(q)) = D$. Finally we recall that Theorem 6 of [7] shows that every regular restricted extension of F by T is equivalent to $\Theta(t)$ for some cocycle t in F^{3*}.

THEOREM 3. *Let F be a finite purely inseparable extension field over C of exponent 1. Let $K^{2n+2} = (\varepsilon_2 - \varepsilon_3 + \cdots + \varepsilon_{2n+2})F^{2n+1}$. Then if $n > 0$ any regular restricted extension of K^{2n+2} by T splits as a regular restricted extension.*

Proof. Let $0 \to K^{2n+2} \xrightarrow{i} E \xrightarrow{\phi} T \to 0$ be a regular restricted extension of K^{2n+2} by T. Since K^{2n+2} is as in Lemma 4.2, there is an F-linear Lie algebra homomorphism ψ inverse to ϕ such that $\psi(D)^p - \psi(D^p)$ is in C. Let $E' = i(F \otimes 1^{2n+1}) + \psi(T)$. This is a restricted Lie subalgebra of E and a restricted regular extension of F by T: For it is clear that E' is a Lie algebra and we already saw in (4.5), for any e in E and k in K^{2n+2}, $(e+k)^p = e^p + \phi(e)^{p-1}k + k^p$ so that E' is indeed a restricted Lie subalgebra. Since E' is an F-space, the regularity properties are inherited from E. Finally ϕ restricted to E' maps E' onto T. As noted above [7, Theorem 6], E' is then equivalent to $\Theta(t)$ for some cocycle t in F^{3*}. The map from T into $\Theta(t)$ induced by ψ will also be denoted by ψ. Since $\phi\psi$ is the identity on T, we may write $\psi(D) = {}^{(2)}D + L(w(D))$ where $w: T \to F^2$. It is shown in [7, Lemma 4.4, p. 349], and indeed it is almost obvious, that w has the following properties:

$$\begin{aligned} w(fD) &= f \cdot w(D), \\ w([D,D']) &= {}^{(2)}Dw(D') - {}^{(2)}D'w(D), \\ (\varepsilon_2 - \varepsilon_3)w(D) &= \frac{{}^{(3)}Dt}{t}, \\ w(D)^p + {}^{(2)}D^{p-1}w(D) - w(D^p) &= \psi(D)^p - \psi(D^p). \end{aligned} \tag{4.12}$$

Since we are working in F^{2n+2} with $n > 0$ it turns out that $w(D)$ and ${}^{(3)}Dt/t$ are connected by another relation: By [7, Lemma 4.2] we have, if as in §2, $\delta_n = \varepsilon_2 - \varepsilon_3 + \cdots \pm \varepsilon_{n+1}$,

$$\frac{{}^{(n+1)}D\Delta_n(x)}{\Delta_n(x)} = -\delta_n \frac{{}^{(n)}Dx}{x}. \tag{4.13}$$

Moreover, keeping in mind that $\Delta_3(t) = 1$, it is easily verified that

$$\Delta_{2n+2}(t \otimes 1^{2n-1}) = t \otimes 1^{2n}.$$

Combining the last two formulae yields

$$\delta_{2n+2}\left(-\frac{{}^{(3)}Dt}{t} \otimes 1^{2n} \right) = \delta_{2n+2}\left(-\frac{{}^{(2n+2)}D(t \otimes 1^{2n-1})}{t \otimes 1^{2n-1}} \right) = \frac{{}^{(3)}Dt}{t} \otimes 1^{2n}.$$

Since $w(D)$ is in F^2, for $i > 2$, $\varepsilon_i(w(D) \otimes 1^{2n}) = \varepsilon_{i+1}(w(D) \otimes 1^{2n})$. Then $\delta_{2n+2}(w(D) \otimes 1^{2n}) = (\varepsilon_2 - \varepsilon_3) w(D) \otimes 1^{2n}$. By (4.12) this is ${}^{(3)}Dt/t \otimes 1^{2n}$. Hence $\delta_{2n+2}(w(D) \otimes 1^{2n}) = \delta_{2n+2}(-({}^{(3)}Dt/t) \otimes 1^{2n-1})$.

We saw in §2 that the complex (F^n, δ_n) is exact. Therefore, since

$$\delta_{2n+2}\left(-\left(\frac{{}^{3}Dt}{t}\right) \otimes 1^{2n-1} - w(D) \otimes 1^{2n} \right) = 0,$$

we have

$$\beta(D) = -\left(\frac{{}^{3}Dt}{t}\right) \otimes 1^{2n-1} - w(D) \otimes 1^{2n}$$

is in $\operatorname{Im}(\delta_{2n+1}) = K^{2n+2}$. We shall show that $(\psi + \beta): T \to E$ is an F-linear restricted Lie algebra homomorphism.

First let us note that by [7, Lemma 4.4], for t in F^{n*} and D, D' in T,

$$\begin{aligned} \frac{{}^{(n)}[D,D']t}{t} &= {}^{(n)}D\left(\frac{{}^{(n)}D't}{t}\right) - {}^{(n)}D'\left(\frac{{}^{(n)}Dt}{t}\right), \\ \frac{{}^{(n)}D^p t}{t} &= \left(\frac{{}^{(n)}Dt}{t}\right)^p + {}^{(n)}D^{p-1}\left(\frac{{}^{(n)}Dt}{t}\right). \end{aligned} \tag{4.14}$$

Combining (4.12) and (5.14) yields

$$\begin{aligned} \beta([D,D']) &= {}^{(2n+2)}D\beta(D') - {}^{(2n+2)}D'\beta(D), \quad \text{and} \\ \beta(D^p) - \beta(D)^p - {}^{(2n+2)}D^{p-1}\beta(D) &= -w(D^p) + w(D)^p + D^{p-1}w(D) \\ &= \psi(D)^p - \psi(D^p). \end{aligned} \tag{4.15}$$

Then $(\psi + \beta)([D,D']) = [\psi(D) \cdot \psi(D')] + {}^{(2n+2)}D\beta(D') - {}^{(2n+2)}D'\beta(D)$ by (4.15). But $[(\psi + \beta)D, (\psi + \beta)D'] = [\psi(D), \psi(D')] + [\psi(D), \beta(D')] + [\beta(D), \psi(D')]$

$+ [\beta(D), \beta(D')] = [\psi(D), \psi(D')] + {}^{(2n+2)}D\beta(D') - {}^{(2n+2)}D'\beta(D)$ since $\beta(D)$ is in K^{2n+2} and $[e,k] = \phi(e)(k)$.

So $(\psi + \beta)$ is a Lie algebra homomorphism.

$$(\psi + \beta)(D^p) = \psi(D^p) + \beta(D^p) = \psi(D)^p + \beta(D)^p + {}^{(2n+2)}D^{p-1}\beta(D)$$

by (4.15). But

$$[(\psi + \beta)(D)]^p = \psi(D)^p + \beta(D)^p + {}^{(2n+2)}D^{p-1}(\beta(D))$$

by (4.5).

Therefore $(\psi + \beta)$ is a restricted Lie algebra homomorphism. Since ψ and β are both F-linear, so is their sum. Finally $\beta(T) \subset \mathrm{Ker}(\phi)$ so that $\phi(\psi + \beta) = \phi\psi =$ identity on T. Then the extension splits as a regular extension.

4. **All cohomology groups but one are zero.** In this section we shall show that all cohomology groups but the second of Amitsur's complex for F, a finite purely inseparable extension field of exponent one, are zero.

We have by (4.13) that ${}^{(n+1)}D\Delta_n x/\Delta_n x = -\delta_n({}^{(n)}Dx/x)$ for any x in F^{n*}. Thus for x a cocycle, $\delta_n({}^{(n)}Dx/x) = 0$. Then. again by the exactness of the complex (F^n, δ_n), for each cocycle t in F^{n*} and each D in T, we have ${}^{(n)}Dt/t$ is in $\mathrm{Im}(\delta_{n-1})$. Thus there is a coset $V(D, t)$ modulo $\mathrm{Ker}(\delta_{n-1})$ in F^{n-1} such that $\delta_{n-1}(V(D,t)) = {}^{(n)}Dt/t$. Let $K^{n-1} = \delta_{n-2}(F^{n-2}) = \mathrm{Ker}(\delta_{n-1})$ for $n > 2$.

We shall define for each cocycle t in F^{n*} a restricted Lie algebra extension $\Theta_{n-1}(t)$ of K^{n-1} by T.

We set

$$\Theta_{n-1}(t) = \{{}^{(n-1)}D + L(q) \,|\, q \text{ in } V(D,t) \subset F^{n-1}\},$$

where ${}^{(n-1)}D$ and $L(q)$ are defined as in §4. With this definition it is clear that $\Theta_{n-1}(t) \subset \mathrm{Hom}_{1\otimes F^{n-2}}(F^{n-1}, F^{n-1})$.

Lemma 5.1. *If t in $F^{n*}, n > 2$, is an Amitsur cocycle, then $\Theta_{n-1}(t)$ is a regular restricted Lie algebra extension of K^{n-1} by T.*

The proof is essentially the same as that of Lemmas 4.5 and 4.6 of [7] and will therefore be omitted.

Lemma 5.2. *If $n > 3, \Theta_{n-1}(t)$ is a split regular restricted extension of K^{n-1} by T.*

Proof. If n is even, $\delta_{n-2} = (\varepsilon_2 - \varepsilon_3) + \cdots + (\varepsilon_{n-2} - \varepsilon_{n-1})$. Since $[(\varepsilon_i - \varepsilon_{i+1})x]^p = 0$ it follows that $(K^{n-1})^p = (\delta_{n-2}F^{n-2})^p = 0$. Combining Lemma 5.1 and Theorem 2 shows that $\Theta_{n-1}(t)$ splits if n is even.

If n is odd, Lemma 5.1 shows that Theorem 3 is applicable and again $\Theta_{n-1}(t)$ splits.

Lemma 5.3. *If $\Theta_{n-1}(t)$ splits as a regular restricted Lie algebra extension of K^{n-1} by T then ${}^{(n-1)}Dt/t = \delta_{n-2}({}^{(n-2)}Dy/y)$ for some y in F^{n-1*}* (*cf.* [7, *p.* 353]).

Proof. If $\Theta_{n-1}(t)$ splits then there is an F-linear restricted Lie algebra homomorphism $\psi: T \to \Theta_{n-1}(T)$ such that $\phi\psi$ is the identity on T. Thus

$$\psi(D) = {}^{(n-1)}D + w(D),$$

where w is in $\mathrm{Hom}_F(T, F^{n-1})$ and $w(D)$ lies in $V(D,t)$. Hence F^{n-1} is a regular T-module via ψ. We can now imitate the proof of [4, Proposition 2.7] to show that w is a logarithmic derivative, just as was done in [7, p. 353]. Hence $w(D) = {}^{(n-1)}Dy/y$ for some y in F^{n-1*}, and by definition $w(D)$ lies in $V(D,t)$. Thus ${}^{(n)}Dt/t = \delta_{n-1}\,(w(D)) = \delta_{n-1}({}^{(n-1)}Dy/y)$.

We have now shown that for all $n > 3$, every cocycle t in F^{n*} is such that

$$\frac{{}^{(n)}Dt}{t} = \delta_{n-1}\left(\frac{{}^{(n-1)}Dy}{y}\right)$$

since for $n > 3$, $\Theta_{n-1}(t)$ is split. We shall show that in this case, t is cohomologous to $\Delta_{n-1}(y)$.

We first note that a routine calculation using the identities $\varepsilon_i\varepsilon_j = \varepsilon_{1+j}\varepsilon_i$ for $i \leqq j$, [7, (3.3)], shows that

$$(\varepsilon_1\Delta_n)\cdot(\Delta_{n+1}\varepsilon_1) = \varepsilon_1^2\,. \tag{5.1}$$

Lemma 5.4. *If t in $\varepsilon_1(F^{n-1*})$ is a cocycle, $t = 1 \otimes s$, then $t = \Delta_{n-1}(s)$.*

Proof. By (5.1), $\varepsilon_1(t) = \varepsilon_1^2(s) = \varepsilon_1\Delta_{n-1}(s)\cdot\Delta_n(t) = \varepsilon_1\Delta_{n-1}(s)$. Since ε_1 is a monomorphism, $t = \Delta_{n-1}(s)$.

Theorem 4. *Let F be a finite purely inseparable extension field of exponent one over the field C. Then $H^n(F)$, the nth Amitsur cohomology group, is 0 for $n = 1$ or $n > 2$.*

Proof. Theorem 1 showed $H^1(F) = 0$. For the rest, if t is a cocycle in F^{n*}, $n > 3$, Lemmas 5.2 and 5.3 show that there is an element s in F^{n-1*} with

$$\frac{{}^{n}Dt}{t} = \delta_{n-1}\left(\frac{{}^{n-1}Ds}{s}\right) = -\,\delta_{n-1}\left(\frac{{}^{n-1}Ds^{-1}}{s^{-1}}\right)$$

for all D in T. But then (4.13) shows

$$\frac{{}^{n}Dt}{t} = \frac{{}^{n}D\Delta_{n-1}(s^{-1})}{\Delta_{n-1}(s^{-1})}$$

for all D in T. But then

$$\frac{{}^{n}Dt}{t} - \frac{{}^{n}D\Delta_{n-1}(s^{-1})}{\Delta_{n-1}(s^{-1})} = \frac{{}^{n}D(t\cdot\Delta_{n-1}(s))}{t\cdot\Delta_{n-1}(s)} = 0 = {}^{n}D(t\cdot\Delta_{n-1}(s)).$$

Now the elements of F annihilated by T are precisely C [3, Proposition 4, p. 194] or [7, Theorem 1, p. 478]. Thus it is easily verified that the elements of F^n annihilated by all the ${}^nD$'s are precisely $1 \otimes F^{n-1}$. Hence we have

$$t\Delta_{n-1}(s) = 1 \otimes u \qquad \text{with } u \text{ in } F^{n-1*}.$$

But then by Lemma 5.4, $t \cdot \Delta_{n-1}(s) = \Delta_{n-1}(u)$ and $t = \Delta_{n-1}(s^{-1}u)$, a coboundary.

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