ON THE EXISTENCE AND CHARACTERIZATION OF BEST NONLINEAR TCHEBYCHEFF APPROXIMATIONS

BY

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1. **Introduction.** This paper is concerned with the following approximation problem:

Tchebycheff approximation problem. Let $f(x)$ be continuous on $[0,1]$ and let $F(A,x)$ be a continuous approximating function depending on n parameters, $A=(a_1,a_2,\cdots,a_n)$. Denote by P the domain of the parameters. Given $f(x)$ determine $A^*\in P$ such that

$$\max_{x\in[0,1]} |F(A^*,x)-f(x)| \leqq \max_{x\in[0,1]} |F(A,x)-f(x)|$$

for all $A\in P$. Such an $F(A^*, x)$ is a best approximation to $f(x)$.

Relative to this problem there are three principal statements to be investigated. They are

STATEMENT A. *$f(x)$ possesses a best approximation.*

STATEMENT B. *Best approximations are characterized as those $F(A^*, x)$ for which $F(A^*, x)-f(x)$ alternates at least n times on $[0,1]$.*

STATEMENT C. *The best approximation is unique.*

Statement B gives the usual characteristic property of best Tchebycheff approximations. The function $F(A^*, x)-f(x)$ is said to *alternate n times* if there are $n+1$ points

$$0\leqq x_1<x_2<\cdots<x_{n+1}\leqq 1$$

such that

$$F(A^*, x_j)-f(x_j)=-[F(A^*, x_{j+1})-f(x_{j+1})]=\pm\max|F(A^*, x)-f(x)|.$$

All maxima in this paper are taken over $x\in[0,1]$ unless otherwise stated.

The usual effort on a problem of this type is to be given a particular $F(A,x)$ and then to attempt to establish one or more of Statements A, B and C. The effort presented in this paper is one of a more general and partially converse nature. This study concerns the following question: What conditions on $F(A,x)$ are *both necessary and sufficient* for a certain combination of Statements A, B and C to be valid for all continuous functions?

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Such a question was first posed and answered by Haar [2] where, for linear approximating functions

$$F(A,x) = \sum_{i=1}^{n} a_i\phi_i(x),$$

he asked what are necessary and sufficient conditions for Statement C to be valid for all continuous functions. More recently Rice [4] has posed and answered the question (for linear approximating functions) for Statement A and for Statements A and B. Also Rice [5] has answered this question for Statement B and for a continuous nonlinear approximating function.

In this paper conditions on a nonlinear approximating function $F(A,x)$ are found which are necessary and sufficient for both Statements A and B to be valid for all continuous functions.

The conditions found to be equivalent to Statements A and B are closure under pointwise convergence (Definition 4) and local unisolvence (Definition 3).

In the final section it is noted that if Statement B is valid for all continuous functions, then so is Statement C. This implies that the condition found here for Statements A and B are also the conditions for Statements A, B and C.

2. **The theorem and proof.** In order to describe the properties of F that characterize Statements A and B, and in order to facilitate the discussion, the following definitions are made.

DEFINITION 1. *F has Property Z if $A^* \neq A$ implies that*

$$F(A^*,x) - F(A,x)$$

has at most $n-1$ zeros in $[0,1]$.

DEFINITION 2. *F is said to be locally solvent if given $0 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$, $A^* \in P$ and $\varepsilon > 0$ then there is a $\delta(A^*, \varepsilon, x_1, x_2, \cdots, x_n) > 0$ such that*

$$|F(A^*,x_j) - y_j| < \delta$$

implies the existence of a solution $A \in P$ to

$$F(A,x_j) = y_j \tag{2.1}$$

with $\max|F(A,x) - F(A^*,x)| < \varepsilon$.

DEFINITION 3. *F is said to be locally unisolvent if F is locally solvent and has Property Z.*

The final property is closely related to unisolvence [3]. In both cases Property Z is present. For F unisolvent one is assured of solving (2.1) for any set of values $\{y_j\}$, the present definition only assures a solution of (2.1) if the points $\{(x_j,y_j)\,|\,j = 1,2, \cdots, n\}$ lie in a neighbohood of some curve $F(A^*, x)$.

The next definition describes a property to be associated with Statement A.

DEFINITION 4. *F is said to be closed if P is arcwise connected and if F is closed under pointwise limits, i.e.,*

$$\lim_{k\to\infty} F(A_k, x) = G(x), \qquad x \in [0,1], \qquad |F(A_k, x)| \leq M \tag{2.2}$$

implies there is an $A_0 \in P$ *such that*

$$F(A_0, x) \equiv G(x). \tag{2.3}$$

At this point it is appropriate to make a remark on the topology of the parameter space P. Since $F(A,x)$ depends on n parameters, one naturally associates the parameters A with a point in Euclidean n-space E_n. However the topology of E_n may not be suitable for P and indeed it may be extremely difficult to imbed P in E_n in such a way that the E_n topology has any meaning at all for F. Thus P is considered to be an abstract space with its topology derived from the uniform norm on the set of functions $\{F(A,x)\}$. In this way the statement

$$\lim_{k\to\infty} A_k = A^* \tag{2.4}$$

is defined to be equivalent to

$$\lim_{k\to\infty} \max \; |F(A_k, x) - F(A^*, x)| = 0. \tag{2.5}$$

If F is locally solvent then one may show that pointwise closure (2.2) becomes uniform closure (2.5).

The above definitions were originally made in [5]. The definition of closure here has been made slightly less restrictive. For the application of the results of [5] in this paper, the definitions are equivalent.

The main thorem of this paper answers the question posed in the introduction, namely, what does the validity of Statements A and B for every continuous $f(x)$ imply about the properties of the approximating function $F(A,x)$, and vice versa.

THEOREM 1. *Statements* A *and* B *are valid for every continuous function if and only if F is closed and locally unisolvent.*

A portion of the proof of this theorem is found in [5]. There are two points which remain to be established. The first and simplest is that the closure of F (along with Property Z) implies the existence of best approximations for every continuous function. The second is that the validity of Statements A and B imply that F is closed.

LEMMA 1. *If F is closed and has Property* Z *then Statement* A *is valid for every continuous function.*

Proof. Let $f(x)$ be a given function continuous on $[0,1]$ and $F(A', x)$ an approximating function. Denote by P' the parameters

$$P' = \{A \,|\, \max |F(A,x) - f(x)| \leq \max |F(A',x) - f(x)|\}.$$

This set is not empty since it contains A'. Furthermore it is a bounded set, i.e., there exists an $M < \infty$ such that $|F(A,x)| \leqq M$ for $A \in P'$.

There is a sequence $\{A_k\}$ in P' such that

$$\operatorname*{Lim}_{k\to\infty} \max |F(A_k,x) - f(x)| = \inf_{A\in P} \max |F(A,x) - f(x)|.$$

It is known [5, Theorem 2] that Property Z implies the existence of a pointwise convergent subsequence of every infinite sequence in P'. If P' contains only a finite number of parameter sets then Statement A is clearly valid for $f(x)$. Thus the sequence $\{F(A_k,x)\}$ contains a pointwise convergent subsequence and, by the hypothesis of closure, this subsequence possesses a limit $F(A_0,x)$ which is a best approximation to $f(x)$.

In order to establish the second point, one would like to construct for $G(x)$ a continuous function $f(x)$ such that $f(x) - G(x)$ alternates n times at $n+1$ specified points in $[0,1]$. This would imply (after some arguments) that $G(x)$ is a best approximation to $f(x)$ and hence (by Statement A) that $G(x) \equiv F(A_0,x)$. However it is not possible at this point to construct such an $f(x)$ since $G(x)$ is an unknown and possibly highly discontinuous function. This difficulty is circumvented in Lemma 2 where two functions associated with $G(x)$ are introduced for which one may construct the required continuous function $f(x)$. These two functions are

$$G^+(x) = \max\left[G(x),\ \operatorname*{Lim\,sup}_{|x-y|\to 0} G(y)\right], \tag{2.6}$$

$$G^-(x) = \min\left[G(x),\ \operatorname*{Lim\,inf}_{|x-y|\to 0} G(y)\right]. \tag{2.7}$$

Since $G(x)$ is a bounded function, both of these functions are well defined.

Lemma 2. *Given $G(x)$ bounded on $[0,1]$, $M > 0$, $\delta_0 > 0$ and $x_0 \in [0,1]$ there exists a continuous function $f(x)$ such that*

$$f(x) - G^+(x)$$

has a minimum $-M$ at x_0 in the interval $|x - x_0| \leqq \delta_0$. Further

$$f(x_0 \pm \delta_0) - G^+(x_0 \pm \delta_0) = 0.$$

Proof. Set

$$\omega^+(\delta) = \sup[G^+(y) - G^+(x_0)], \qquad 0 < |x_0 - y| < \delta. \tag{2.8}$$

This is an "upper modulus of continuity" of $G^+(x)$ at x_0. It is also the uppersemicontinuous function $u(x)$ (upper boundary function) described in [6, Chapter 7]. If

$$G(x_0) = G^+(x_0) > \operatorname*{Lim\,sup}_{|x_0-y|\to 0} G(x) \tag{2.9}$$

then clearly

$$\lim_{\delta\to 0} \omega^+(\delta) < 0.$$

When (2.9) does not hold then we have the following assertion: *If*

$$G(x_0) \leqq G^+(x_0)$$

then

(2.10) $$\lim_{\delta\to 0} \omega^+(\delta) = 0.$$

The basic reason that this assertion is true is that $G^+(x)$ itself is upper-semicontinuous Assume (2.10) to be false, then there is an $\varepsilon > 0$ and a sequence $\{x_i \mid i = 1,2,\cdots\}$ tending to x_0 such that

$$G^+(x_0) < G^+(x_i) - \varepsilon, \qquad i = 1,2,\cdots.$$

This contradicts the fact

$$G^+(x_0) \geqq \operatorname*{Lim\,sup}_{|x_0-y|\to 0} G^+(y)$$

which may be established by a straightforward argument.

A construction is now given to establish the following

ASSERTION. *There exists a continuous function* $\omega(\delta)$ *such that for* $0 \leqq \delta \leqq \delta_0$

(2.11) $$\omega^+(\delta) \leqq \omega(\delta)$$

and if (2.10) *holds then* $\omega(0) = 0$.

Note that $\omega^+(\delta)$ is a monotonic nondecreasing function. Define

$$\omega(\delta) = \frac{1}{\delta}\int_\delta^{2\delta} \omega^+(x)\,dx, \qquad \delta > 0,$$

$$\omega(0) = \operatorname*{Lim\,sup}_{|x_0-y|\to 0} G(y) - G^+(x_0).$$

It follows immediately from the mean value theorem that

$$\omega^+(\delta) \leqq \omega(\delta) \leqq \omega^+(2\delta), \qquad \delta > 0.$$

It is clear that $\omega(\delta)$ is a continuous function and further if (2.10) holds then $\omega(0) = 0$.

The function $f(x)$ required in this proof is now constructed. If $\omega(\delta_0) \leqq 0$ set

$$f(x) = G^+(x_0) - M + 2|x - x_0|.$$

Then if $|x - x_0| \leqq \delta_0$ one has

$$f(x) - G^+(x) = G^+(x_0) - G^+(x) - M + 2|x - x_0| \geqq -M + 2|x - x_0|.$$

If $\omega(\delta_0) > 0$ set $\omega'(\delta) = \max[\omega(\delta), 0]$ and

$$f(x) = G^+(x_0) - M + 2\omega'(|x - x_0|).$$

Then

$$\begin{aligned} f(x) - G^+(x) &= G^+(x_0) - G^+(x) - M + 2\omega'(|x - x_0|) \\ &\geqq -M + 2\omega'(|x - x_0|) - \omega^+(|x - x_0|) \\ &\geqq -M + \omega'(|x - x_0|) \geqq -M. \end{aligned}$$

This construction gives an $f(x)$ satisfying the minimum requirement of the lemma. The construction may be easily modified so that

$$f(x_0 \pm \delta_0) = G^+(x_0 \pm \delta_0).$$

It is clear that the same type of construction is applicable to the

COROLLARY. *Given* $G(x)$ *bounded on* $[0,1]$, $M > 0$, $\delta_0 > 0$ *and* $x_0 \in [0,1]$ *then there exists a continuous function* $f(x)$ *such that*

$$f(x) - G^-(x)$$

has a local maximum of M *at* x_0 *in the interval* $|x - x_0| \leqq \delta_0$. *Further*

$$f(x_0 \pm \delta_0) - G^-(x_0 \pm \delta_0) = 0.$$

The next lemma is required to establish the second point of the proof of Theorem 1. This lemma is a restatement of some results in [5], particularly Lemma 4 and Theorem 1.

LEMMA 3. *Assume Statement* B *is valid for every continuous function. If*

$$\max_{x_i} |F(A, x_i) - f(x_i)| \leqq 2M, \qquad i = 1, 2, \cdots, n + 1$$

then either (i) $F(A,x) - f(x)$ *alternates* n *times on* $\{x_i\}$ *with deviation* $2M$ *or* (ii) *there is an* $A_0 \in P$ *such that*

$$|F(A_0, x_i) - f(x_i)| < 2M, \qquad i = 1, 2, \cdots, n + 1.$$

The next lemma establishes the second point required for the proof of Theorem 1.

LEMMA 4. *If Statements* A *and* B *are valid for every continuous function then* F *is closed.*

Proof. Assume that

$$\lim_{k \to \infty} F(A_k, x) = G(x), \qquad x \in [0,1] \tag{2.13}$$

with

$$|F(A_k, x)| \leqq M.$$

Let $n + 1$ points be given

$$0 \leqq x_1 < x_2 < \cdots < x_{n+1} \leqq 1$$

and set $\delta_0 = \frac{1}{4} \min |x_j - x_{j+1}|$.

By Lemma 2 a continuous function $f_1(x)$ may be defined by (2.12) so that

$$f_1(x_j) - G^-(x_j) = +2M, \qquad j = 1,3,5,\cdots, \tag{2.14}$$
$$f_1(x_j) - G^+(x_j) = -2M, \qquad j = 2,4,\cdots$$

and these points are local extrema of $f_1(x) - G^+(x)$ and $f_1(x) - G^-(x)$ in $[x_j - \delta_0, x_j + \delta_0]$. Further, the definition of $f_1(x)$ may be extended to the remainder of $[0,1]$ so that

$$|f_1(x) - G(x)| \leqq M, \qquad |x - x_j| > \delta_0. \tag{2.15}$$

Since $f_1(x)$ is continuous, the assumption that Statement A is valid implies that $f_1(x)$ possesses a best approximation $F(A^1, x)$. The assumption that Statement B is valid implies [5, Lemma 3] that F has Property Z. This fact is used essentially in the remainder of the proof.

The following assertion is now established:

ASSERTION 1.

$$\max |f_1(x) - F(A^1, x)| \leqq 2M. \tag{2.16}$$

Since Statement B is valid there are $n+1$ points $\{y_j \mid y_j < y_{j+1}\}$ such that

$$F(A^1, y_j) - f_1(y_j) = (-1)^j K,$$

where $K = \pm \max | f_1(x) - F(A^1, x)|$. If $|K| > 2M$ then since $|G(y_j) - f_1(y_j)| \leqq 2M < |K|$ one has

$$[F(A^1, y_j) - G(y_j)](-1)^j \operatorname{sgn}[K] \geqq |K| - 2M > 0.$$

For k sufficiently large one has

$$|F(A_k, y_j) - G(y_j)| < \frac{1}{2}(|K| - 2M)$$

and hence

$$\operatorname{sgn}[F(A^1_, y_j) - F(A_k, y_j)] = (-1)^j \operatorname{sgn}[K].$$

This implies that F does not have Property Z which contradicts the assumption that Statement B is valid for all continuous functions. This establishes the assertion (2.16).

We now establish

ASSERTION 2.

$$|F(A^1, x_j) - f_1(x_j)| = 2M, \qquad j = 1,2,\cdots, n+1 \tag{2.17}$$

It follows from the first assertion that

$$|F(A^1, x_j) - f_1(x_j)| \leqq 2M, \qquad j = 1,2,\cdots, n+1. \tag{2.18}$$

Lemma 3 implies that either (2.17) follows from (2.18) or there is an $A_0 \in P$ such that

(2.19) $$|F(A_0,x_j)-f_1(x_j)|<2M, \qquad j=1,2,\cdots,n+1.$$

It is now shown that the alternative (2.19) leads to a contradiction.

Set

$$\varepsilon=\min_j[2M-|F(A_0,x_j)-f_1(x_j)|]>0.$$

There is an $\eta>0$ such that $|x-x_j|<\eta$ implies

$$|F(A_0,x_j)-F(A_0,x)|<\frac{\varepsilon}{3}, \qquad j=1,2,\cdots,n+1.$$

Further, there exists a y_j with $|y_j-x_j|<\eta$ so

$$|G(y_j)-G^{\pm}(x_j)|<\frac{\varepsilon}{3}, \qquad j=1,2,\cdots,n+1.$$

One may choose k so large that

$$|G(y_j)-F(A_k,y_j)|<\frac{\varepsilon}{3}, \qquad j=1,2,\cdots,n+1.$$

With alternative (2.19) it follows from these estimates that

$$F(A_0,y_j)>F(A_k,y_j), \qquad j \text{ odd},$$

$$F(A_0,y_j)<F(A_k,y_j), \qquad j \text{ even}.$$

This implies that $F(A_0,x)-F(A_k,x)$ has n zeros which is impossible. This establishes the assertion.

These two assertions imply that

(2.20) $$F(A^1,x_j)=G^-(x_j), \qquad j \text{ odd},$$
$$F(A^1,x_j)=G^+(x_j), \qquad j \text{ even}.$$

A similar construction of a continuous function $f_2(x)$ and an analysis of a best approximation $F(A^2,x)$ to it leads to

(2.21) $$F(A^2,x_j)=G^+(x_j), \qquad j \text{ odd},$$
$$F(A^2,x_j)=G^-(x_j), \qquad j \text{ even}.$$

It follows from (2.20) and (2.21) that

(2.22) $$[F(A^2,x_j)-F(A^1,x_j)](-1)^{j+1}\geqq 0, \qquad j=1,2,\cdots,n+1.$$

Since F must have Property Z, this implies that

(2.23) $$F(A^2,x_j)\equiv F(A^1,x_j).$$

Since

$$G^+(x)\geqq G(x)\geqq G^-(x)$$

it follows that

$$G(x_j) = F(A^1, x_j), \qquad j = 1, 2, \cdots, n + 1. \tag{2.24}$$

One may fix n distinct points and let the $(n + 1)$st point be variable. Then (2.24) is valid with A^1 replaced by a new parameter set A_0. However, on the n fixed points one has $F(A^1, x_j) = F(A_0, x_j)$, $j = 1, 2, \cdots, n$ which implies that $F(A^1, x) = F(A_0, x)$. Thus one has for any x,

$$G(x) = F(A^1, x).$$

This is the approximating function required in this lemma and concludes the proof.

Proof of Theorem 1. There are two implications to be established: (i) local unisolvence and closure imply Statements A and B and (ii) Statements A and B imply local unisolvence and closure.

For the first implication we have shown (Lemma 1) that local unisolvence (which includes Property Z) and closure imply Statement A. It is known [5, Theorem 3] that local unisolvence and closure imply Statement B.

For the second implication it is known [5, Lemma 3] that Statement B implies Property Z. We have shown (Lemma 4) that Statements A and B imply closure. It is known [5, Theorem 3] that Statement B and closure (and hence Statements A and B) imply local unisolvence. This concludes the proof.

3. **Example.** For linear approximating functions, it is known [4] that the classical(1) approximating functions are the only ones for which Statements A, B and C are all valid for all continuous functions. The simplest and most "classical" nonlinear approximating functions are the unisolvent functions [3]. It is known that Statements A, B and C are valid for these approximating functions. One might conjecture then that these are the only approximating functions for which Statements A, B and C are all valid. That this is not true is seen by the simple example

$$F(A, x) = \frac{a}{1 + ax}, \qquad -1 < a < +1, \qquad -1 \leqq x \leqq +1. \tag{3.1}$$

One may easily verify that the three Statements A, B and C are valid for all continuous functions. The range of this approximating function is shown in Figure 1.

4. **Remark on Haar's problem and uniqueness.** The following results are known [5, Lemma 3, and classical]:

1. If Statement B is valid for every continuous function then F has Property Z.
2. If F has Property Z then Statement C is valid for every continuous function.

These results have the obvious corollary, which has not been explicitly stated previously.

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(1) The classical linear approximating functions are $F(A,x) = \Sigma_{i=1}^{n} a_i\varphi_i(x)$, where $P = E_n$ and $\{\varphi_i(x)\}$ is a Tchebycheff set, i.e., F has Property Z.

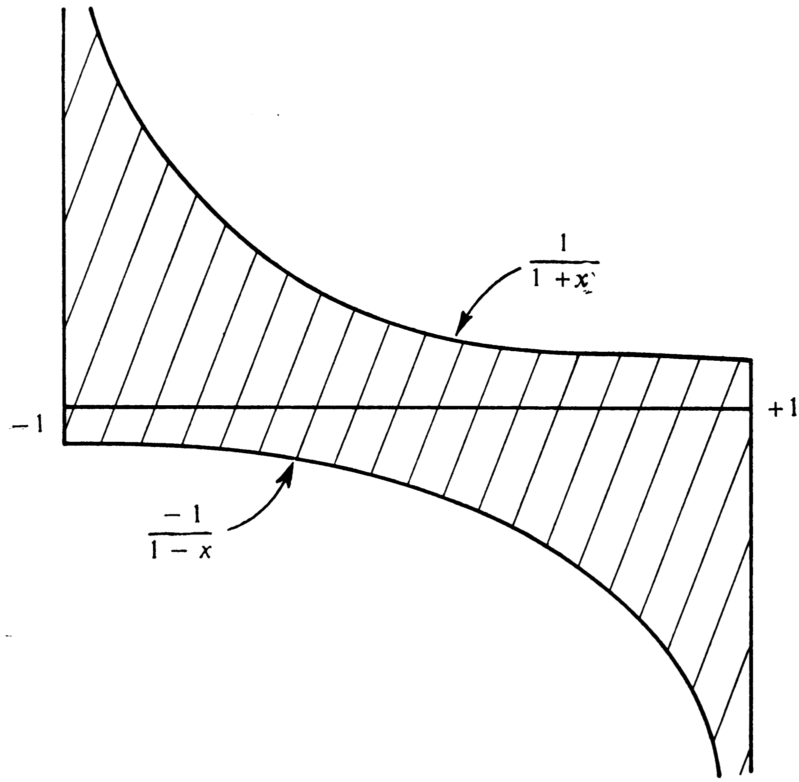

FIGURE 1. The range of $F(A, x) = a/(1 + ax)$

THEOREM 2. *If Statement* B *is valid for every continuous function then Statement* C *is valid for every continuous function.*

Thus Haar's problem [1; 2] which is the study of the implications of Statement C is subsumed by the study of the implications of Statement B.

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