

INVARIANT MEASURES FOR MARKOV PROCESSES⁽¹⁾

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1. **Introduction.** Let (X, \mathcal{B}, m) be a finite or σ -finite measure space. Suppose a temporally homogeneous discrete Markov process is given on this space. We shall consider this process to be given in terms of its transition probability $P(x, B)$. In this work we are interested in finding necessary and sufficient conditions for the existence of a finite measure ν which is invariant under the Markov process in the sense that

$$\int_x P(x, B) \nu(dx) = \nu(B) \quad \text{for every } B \text{ in } \mathcal{B},$$

and which is stronger than the given measure m in the sense that m is absolutely continuous with respect to ν .

Various sufficient conditions for the existence of a finite or σ -finite measure invariant under the given Markov process have been obtained by several authors. Notable among them are the conditions obtained by W. Doeblin [3], N. Kryloff-N. Bogoliouboff [13; 14] and T. E. Harris [9]. However, none of these conditions is necessary for the existence of invariant measures, and furthermore, the measures obtained under the conditions of the first two authors may not be stronger than m , though they are finite, and the measure obtained by the third author may not be finite, though it is σ -finite and is stronger than m .

On the other hand, E. Hopf [10], A. P. Calderón [1], Y. N. Dowker [4; 5], A. B. Hajian [7], and A. B. Hajian and S. Kakutani [8] have obtained several necessary and sufficient conditions for the existence of a finite invariant measure which is equivalent to m for a one-to-one measurable transformation ϕ of the space (X, \mathcal{B}, m) onto itself. Now, if we define

$$P(x, B) = \chi_{\phi(B)}(x),$$

where $\chi_A(x)$ denotes the characteristic function of a set A , then this defines a transition probability of a Markov process of a special type (we shall call such a process 'deterministic'), and the notion of invariance of a measure under the

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measurable transformation coincides with our notion of invariance for this process. Therefore, the problem investigated by the authors in the second group can be regarded as a special case of our problem.

Our results obtained in this work will be direct extensions of the results of these authors, and we shall give a complete solution to the problem in the sense that we find necessary and sufficient conditions, though our search for invariant measures is restricted to the ones which are finite and stronger than the given measure m .

Our method of approach to the problem will be operator-theoretic rather than probabilistic. In §2, basic definitions and concepts necessary for this work will be introduced. In §3, we shall first show that in attacking our problem we may, without loss of generality, make certain assumptions on the given measure m , the transition probability $P(x, B)$, and the nature of invariant measures to be found. In this way, we shall reduce our problem to the one which is easier to handle. Then, in the remainder of §3, we shall introduce various conditions most of which will be shown in §4 to be necessary and sufficient for our problem. Finally, in §5, two simple examples of Markov process will be discussed to show that among the conditions introduced in §3, some are not sufficient for our problem, though they are necessary and are closely related to the ones which are necessary and sufficient. After the examples, a few remarks will be made concerning the relationship between our problem and the validity of the mean and the pointwise ergodic theorems for the operators which are associated with the given Markov process. In this connection, two more necessary and sufficient conditions for our problem will be obtained.

2. Basic definitions and concepts. Let (X, \mathcal{B}, m) be a finite or σ -finite measure space: $X = \{x\}$ is an abstract set of elements x , $\mathcal{B} = \{B\}$ is a σ -field of measurable subsets B of X , and m is a nonnegative, countably additive measure defined on \mathcal{B} which is either finite or σ -finite.

A nonnegative function $P(x, B)$ of two variables (x, B) defined for $x \in X$, $B \in \mathcal{B}$, is called the *transition probability* of a *temporally homogeneous discrete Markov process* on X if it satisfies the following conditions:

(2.1) for every fixed $x \in X$, $P(x, B)$ is a countably additive set function of B defined on \mathcal{B} and satisfies $P(x, X) = 1$,

(2.2) for every fixed $B \in \mathcal{B}$, $P(x, B)$ is a \mathcal{B} -measurable function of x defined on X .

We shall define the n th *transition probability* $P^n(x, B)$ of the process recurrently by

$$(2.3) \quad \begin{aligned} P^n(x, B) &= \int_X P^{n-1}(y, B) P(x, dy), \quad n = 2, 3, \dots, \\ P^1(x, B) &= P(x, B), \end{aligned}$$

where the integration is of Radon-Stieltjes type. Then, it is easy to see that

(2.4) for any fixed pair (x, B) , $x \in X$, $B \in \mathcal{B}$, and for any pair of positive integers n, k ,

$$P^{n+k}(x, B) = \int_X P^n(y, B) P^k(x, dy) = \int_X P^k(y, B) P^n(x, dy).$$

It is also clear that for each positive integer n , the n th transition probability $P^n(x, B)$ satisfies the same conditions as (2.1) and (2.2).

The transition probability $P(x, B)$ is called *m-nonsingular* (or, equivalently, the Markov process is *m-nonsingular*) if the condition $m(B) = 0$ implies $P(x, B) = 0$ *m*-almost everywhere. It is easy to verify that if the transition probability $P(x, B)$ is *m-nonsingular*, then so is the n th transition probability for each positive integer n .

An *m-nonsingular* Markov process will be called *deterministic* if its transition probability is given by a one-to-one, measurable, nonsingular transformation ϕ of (X, \mathcal{B}, m) onto itself by means of the formula:

$$P(x, B) = \chi_{\phi(B)}(x).$$

By $M(\mathcal{B})$ we shall denote the Banach space of all real-valued countably additive set functions defined on \mathcal{B} furnished with the usual total variation norm. An element μ of $M(\mathcal{B})$ will be called *positive* (and denoted by $\mu \geq 0$), if $\mu(B) \geq 0$ for all $B \in \mathcal{B}$.

By $L^1(m) = L^1(X, \mathcal{B}, m)$ and $L^\infty(m) = L^\infty(X, \mathcal{B}, m)$ we shall denote, as usual, the Banach space of all real-valued \mathcal{B} -measurable *m*-integrable functions defined on X and the Banach space of all real-valued \mathcal{B} -measurable *m*-essentially bounded functions defined on X , respectively, with the usual norms.

Two functions in $L^1(m)$ or $L^\infty(m)$ which coincide with each other *m*-a.e. will be identified. A function $f(x)$ in $L^1(m)$ or $L^\infty(m)$ will be called *positive* (and denoted by $f \geq 0$), if $f(x) \geq 0$, *m*-a.e.

A transition probability of a Markov process induces a functional operator S of $M(\mathcal{B})$ into itself by means of the formula:

$$(2.5) \quad \mu \rightarrow S\mu: S\mu(B) = \int_X P(x, B) \mu(dx).$$

It is easy to see that S is linear and bounded with norm 1. Furthermore, the nonnegativity of $P(x, B)$ implies that S is a *positive operator* in the sense that it maps a positive element of $M(\mathcal{B})$ onto a positive element.

We say that an element μ of $M(\mathcal{B})$ is *invariant under* $P(x, B)$ (or, equivalently, invariant under the Markov process) if $S\mu(B) = \mu(B)$ for all $B \in \mathcal{B}$.

Let now $M_0(\mathcal{B})$ be the linear subspace of $M(\mathcal{B})$, consisting of all elements in $M(\mathcal{B})$ which are absolutely continuous with respect to the measure *m*. It is easy to see that $M_0(\mathcal{B})$ is a closed subspace of $M(\mathcal{B})$ and hence is itself a Banach space with the same norm. The proof of the following proposition is clear.

PROPOSITION 1. *A transitional probability $P(x, B)$ is m -nonsingular if and only if the induced operator S on $M(\mathcal{B})$ maps the subspace $M_0(\mathcal{B})$ into itself.*

Now, since the space $M_0(\mathcal{B})$ is isometrically isomorphic with the Banach space $L^1(m)$ by virtue of the Radon-Nikodym theorem, an m -nonsingular transition probability $P(x, B)$ induces, in view of the proposition above, a linear, bounded, positive operator on $L^1(m)$ into $L^1(m)$. We shall denote this induced operator on $L^1(m)$ by T . Then,

$$(2.6) \quad f \rightarrow Tf: Tf(x) = \frac{d}{dm} (S\mu_f)(x),$$

where $\mu_f(B) = \int_B f(x) m(dx)$ for $B \in \mathcal{B}$ and d/dm denotes the Radon-Nikodym derivative with respect to m . Clearly, the norm of T is the same as that of S , i.e. $\|T\| = 1$.

An m -nonsingular transition probability $P(x, B)$ defines also an operator U of $L^\infty(m)$ into itself by

$$(2.7) \quad f \rightarrow Uf: Uf(x) = \int_x f(y) P(x, dy).$$

It is easy to check that U is linear, positive and bounded with norm 1, and furthermore, that U is the adjoint operator of the operator T .

We remark that if T and U are defined by an m -nonsingular transition probability $P(x, B)$ as in (2.6) and (2.7), then the iterations T^n and U^n of T and U are given by the n th transition probability $P^n(x, B)$ in the following way:

$$(2.8) \quad T^n f(x) = \frac{d}{dm} (S^n \mu_f)(x),$$

where

$$S^n \mu_f(B) = \int_B P^n(x, B) \mu_f(dx) = \int_B P^n(x, B) f(x) m(dx)$$

and

$$(2.9) \quad U^n f(x) = \int_x f(y) P^n(x, dy).$$

For every positive integer n , the iterations T^n and U^n are linear, bounded, positive operators of $L^1(m)$ into itself and $L^\infty(m)$ into itself, respectively, and they both have the norm equal to 1.

Suppose V is a linear, bounded operator on some Banach space \mathbf{E} . Then, we say that the *mean ergodic theorem holds for the operator V* if the sequence of averages $A_n f = (n^{-1}) \sum_{k=0}^{n-1} V^k f$ converges in the norm topology of \mathbf{E} for every f in \mathbf{E} . (V^0 denotes the identity operator I on \mathbf{E} .)

A subset K of a Banach space \mathbf{E} is called *weakly sequentially compact* if every sequence $\{f_n\}$ in K contains a subsequence which converges weakly to an element

in **E**. The following characterization for a weakly sequentially compact subset K of the Banach space $L^1(m)$ is well known. (See, e.g., p. 292 in [6].)

THEOREM I. *A subset K of $L^1(m)$ is weakly sequentially compact if and only if it is bounded and the countable additivity of the integrals $\int f(x)m(dx)$ is uniform with respect to f in K .*

A subset K of a Banach space **E** is called *fundamental* if the norm closure of the subspace spanned by K is the whole space **E**.

The following criterion for the validity of the mean ergodic theorem for a linear bounded operator V defined on a Banach space **E** was given by K. Yosida and S. Kakutani [15].

THEOREM II. *If there exists a constant C such that $\|V^n\| \leq C$ for $n = 1, 2, 3, \dots$, then the mean ergodic theorem holds for the operator V if and only if the set $\{A_n f; n = 1, 2, 3, \dots\}$ is weakly sequentially compact in **E** for every f belonging to some fundamental subset of **E**.*

3. Formulation of the problem. Let (X, \mathcal{B}, m) be a finite or σ -finite measure space and suppose a temporally homogeneous Markov process with the transition probability $P(x, B)$ is given on it. From here on, we shall not distinguish the process and its transition probability. Our object in this work is to find necessary and sufficient conditions for the existence of a finite measure ν defined on (X, \mathcal{B}) which is invariant under $P(x, B)$ and is stronger than the given measure m in the sense that m is absolutely continuous with respect to ν .

We can, first of all, make a few remarks in order to simplify our problem.

REMARK 1. Suppose (X, \mathcal{B}, m) is an infinite (but σ -finite) measure space. Then, it is easy to construct a probability measure \bar{m} on (X, \mathcal{B}) such that \bar{m} is equivalent to m (i.e., \bar{m} and m are mutually absolutely continuous). Now, if ν is any measure on (X, \mathcal{B}) which is stronger than m , then ν is also stronger than \bar{m} , and the converse is also true. Therefore, for our problem we may assume that the given measure space (X, \mathcal{B}, m) is a probability space.

DEFINITION 1. For every nonnegative integer n , Q_n denotes a measure on (X, \mathcal{B}) given by the following formulae:

$$(3.1) \quad \begin{aligned} Q_n(B) &= \int_X P^n(x, B) m(dx), \quad n = 1, 2, 3, \dots, \\ Q_0(B) &= m(B). \end{aligned}$$

Since we are now assuming m to be a probability measure on (X, \mathcal{B}) , each Q_n is also such a measure.

REMARK 2. Consider a measure m^* on (X, \mathcal{B}) defined by setting

$$m^*(B) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} Q_k(B)$$

for every B in \mathcal{B} . Then, it is easy to see that m^* is a probability measure on (X, \mathcal{B}) and m is absolutely continuous with respect to m^* . We shall see that the given Markov process $P(x, B)$ is m^* -nonsingular. For this purpose, let B be any set in \mathcal{B} with $m^*(B) = 0$. Then, from the definition of m^* it follows that $Q_k(B) = 0$ for $k = 0, 1, 2, \dots$. But then,

$$\int_x P(x, B) m^*(dx) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} Q_{k+1}(B) = 0,$$

so that we have $P(x, B) = 0$ m^* -almost everywhere, which shows that $P(x, B)$ is indeed m^* -nonsingular. Now, suppose that the given measure m is absolutely continuous with respect to some measure ν which is invariant under $P(x, B)$, and suppose that B is a set in \mathcal{B} with $\nu(B) = 0$. Then, the absolute continuity of m implies that $m(B) = 0$, while the invariance of ν implies that for each positive integer k ,

$$\int_x P^k(x, B) \nu(dx) = 0,$$

so that $P^k(x, B) = 0$, ν -a.e. It follows, then, $P^k(x, B) = 0$, m -a.e. for each k , and hence $Q_k(B) = 0$ for each k . Therefore, we must have $m^*(B) = 0$, and this implies that the measure m^* is also absolutely continuous with respect to the invariant measure ν .

From the remark made above it follows that the problem of finding such an invariant measure ν when m is given is the same as the problem of finding ν when m^* is given. Therefore, we may and do assume that the given Markov process is m -nonsingular.

Now let ν be a measure on (X, \mathcal{B}) which is invariant under $P(x, B)$ and suppose the given measure m is absolutely continuous with respect to ν . Denote by \mathcal{C} the class of sets in \mathcal{B} which have positive ν -measure but zero m -measure. Clearly, \mathcal{C} is closed under the formation of countable unions. Now, let $\alpha = \sup_{B \in \mathcal{C}} \nu(B)$, then, it is easy to show that there exists a set B_0 in \mathcal{C} such that $\alpha = \nu(B_0)$. It is also easy to see that if the Markov process is m -nonsingular, then, $P(x, B_0) = \chi_{B_0}(x)$ holds ν -a.e. for such a set B_0 . Define a new measure ν_0 on (X, \mathcal{B}) by setting

$$\nu_0(B) = \nu(B - B_0) = \nu(B - (B \cap B_0))$$

for every $B \in \mathcal{B}$. Then, we have the following

PROPOSITION 2. ν_0 is invariant under $P(x, B)$ and is equivalent to m .

Proof of this proposition is quite simple and hence omitted.

REMARK 3. Because of Proposition 2, we see that it is sufficient to restrict the search for an invariant measure ν to the class of measures equivalent to the given measure m .

In view of the remarks made above, we can now state our problem in the following form:

Given a probability space (X, \mathcal{B}, m) and an m -nonsingular, temporally homogeneous Markov process $P(x, B)$ defined on it, find necessary and sufficient conditions for the existence of a finite measure ν which is invariant under $P(x, B)$ and is equivalent to m .

As we mentioned earlier, the problem in this form has been investigated by E. Hopf, Y. N. Dowker, A. P. Calderón, A. B. Hajian, and S. Kakutani for a special case in which the Markov process is deterministic. These authors showed, among other things, that the following conditions are necessary and sufficient for our problem in the deterministic case:

(I) For any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$m(B) < \delta \text{ implies } \sup_n Q_n(B) < \varepsilon.$$

(II) $m(B) > 0$ implies $\liminf_{n \rightarrow \infty} Q_n(B) > 0$.

(II)* For any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$m(B) > \varepsilon \text{ implies } \liminf_{n \rightarrow \infty} Q_n(B) > \delta.$$

(III) $m(B) > 0$ implies $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > 0$.

(III)* For any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$m(B) > \varepsilon \text{ implies } \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > \delta.$$

(IV) $m(B) > 0$ implies $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > 0$.

(IV)* For any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$m(B) > \varepsilon \text{ implies } \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > \delta.$$

We shall show in §4 that each of conditions (II), (II)*, (III), (III)*, (IV), (IV)* is necessary and sufficient for our problem even in the case of a general m -nonsingular Markov process.

As for condition (I), we shall prove that it is necessary for our problem, but in §5 we shall give an example of a Markov process which shows that it is not sufficient. Therefore, the situation in the general case is not quite the same as in the special case of deterministic processes.

In connection with condition (I), let us consider the following two conditions:

(I)' For any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$m(B) < \delta \quad \text{implies} \quad \sup_n \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) < \varepsilon.$$

(M) The mean ergodic theorem holds for the operator T defined in (2.6) in the Banach space $L^1(m)$.

It is clear that condition (I) implies condition (I)'. We shall show in §4 that condition (I)' is equivalent to condition (M), and consequently, implies the existence of a nontrivial finite measure ν , which is invariant under $P(x, B)$ and is absolutely continuous with respect to m . This measure ν will be given by the formula:

$$\nu(B) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B).$$

Observe that each one of conditions (II), (II)*, (III), (III)*, (IV), (IV)* would imply that the measure ν given in this way has to be equivalent to m . Thus, we shall establish the sufficiency for our problem of the six conditions (II), (II)*, (III), (III)*, (IV), (IV)* by showing that they imply condition (I) (and hence (I)').

DEFINITION 2. A set $F \in \mathcal{B}$ is called *transient* for a Markov process $P(x, B)$ if $\sum_{n=1}^{\infty} P^n(x, F) < \infty$ holds m -a.e.

DEFINITION 3. A set $F \in \mathcal{B}$ is called *m-transient* for a Markov process $P(x, B)$ if $\sum_{n=1}^{\infty} Q_n(F) < \infty$ holds.

DEFINITION 4. A set $F \in \mathcal{B}$ is called *dissipative* for a Markov process $P(x, B)$ if $\sum_{n=1}^{\infty} T^n 1(x) < \infty$ holds m -a.e. on the set F , where T is the operator defined in (2.6) and $1(x)$ is the function in $L^1(m)$ which takes the constant value 1, m -a.e.

If a Markov process $P(x, B)$ admits a finite invariant measure ν which is equivalent to m , then, it is easy to see that every transient set or m -transient set for $P(x, B)$ has to be of m -measure zero. In fact, we even have the following assertion:

PROPOSITION 3. Suppose there exists a finite measure ν which is invariant under $P(x, B)$ and is equivalent to m . Suppose for some set $F \in \mathcal{B}$ there exists an infinite sequence of positive integers $\{n_i\}$ such that either one of the following conditions holds:

- (i) $\sum_{i=1}^{\infty} P^{n_i}(x, F) < \infty$, m -a.e.,
- (ii) $\sum_{i=1}^{\infty} Q_{n_i}(F) < \infty$.

Then, we must have $m(F) = 0$.

Proof. Suppose (i) is satisfied. It follows, then, that we have $\lim_{i \rightarrow \infty} P^{n_i}(x, F) = 0$, m -a.e. so that $\lim_{i \rightarrow \infty} P^{n_i}(x, F) = 0$, ν -a.e. By virtue of the dominated convergence theorem, we get

$$\lim_{i \rightarrow \infty} \int_X P^{n_i}(x, F) \nu(dx) = 0,$$

which implies, by the invariance of ν , that $\nu(F) = 0$, and hence $m(F) = 0$.

Suppose condition (ii) is satisfied instead. Then, by virtue of the monotone convergence theorem,

$$\int_X \sum_{i=1}^{\infty} P^{n_i}(x, F) m(dx) = \sum_{i=1}^{\infty} Q_{n_i}(F) < \infty$$

so that $\sum_{i=1}^{\infty} P^{n_i}(x, F) < \infty$, m -a.e. Thus, this case reduces to the previous one. Q.E.D.

Motivated by the proposition above, we now make the following

DEFINITION 5. A set $F \in \mathcal{B}$ is called *weakly transient* for a Markov process $P(x, B)$ if there exists a sequence of positive integers $\{n_i\}$ tending to ∞ such that $\sum_{i=1}^{\infty} P^{n_i}(x, F) < \infty$ holds m -a.e.

DEFINITION 6. A set $F \in \mathcal{B}$ is called *weakly m -transient* for a Markov process $P(x, B)$ if there exists a sequence of positive integers $\{n_i\}$ tending to ∞ such that $\sum_{i=1}^{\infty} Q_{n_i}(F) < \infty$.

We shall also consider the following

DEFINITION 7. A set $F \in \mathcal{B}$ is called *weakly dissipative* for a Markov process $P(x, B)$ if there exists a sequence of positive integers $\{n_i\}$ tending to ∞ such that $\sum_{i=1}^{\infty} T^{n_i}1(x) < \infty$ holds m -a.e. on F .

Let us now look at the following list of conditions:

- (V) Every transient set for $P(x, B)$ has m -measure zero.
- (V)^{*} Every weakly transient set for $P(x, B)$ has m -measure zero.
- (VI) Every m -transient set for $P(x, B)$ has m -measure zero.
- (VI)^{*} Every weakly m -transient set for $P(x, B)$ has m -measure zero.
- (VII) Every dissipative set for $P(x, B)$ has m -measure zero.
- (VII)^{*} Every weakly dissipative set for $P(x, B)$ has m -measure zero.

We have already seen that each one of conditions (V), (V)^{*}, (VI), (VI)^{*}, is necessary for our problem. We shall show in §4 that both of conditions (V)^{*} and (VI)^{*} are actually equivalent to condition (II) cited before, and hence they are sufficient for our problem as well. If we assume that the given measure space (X, \mathcal{B}, m) is atomless, then we can also show that condition (VII)^{*} is equivalent to (VI)^{*} so that (VII)^{*} is another necessary and sufficient condition and (VII) is a necessary condition. However, in §5 we shall give an example of Markov process which shows that none of conditions (V), (VI) and (VII) is sufficient for our problem.

Finally, we shall show that the following two conditions are also necessary and sufficient.

(VIII) $m(B) > 0$ implies $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} P^k(x, B) > 0$ for all x in some F such that $m(F) > 0$.

(IX) $m(B) > 0$ implies $\limsup_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} P^k(x, B) > 0$ for all x in some set F such that $m(F) > 0$.

It will be shown, in fact, that conditions (III), (IV), (VIII) and (IX) are mutually equivalent.

4. Proofs of main results.

PROPOSITION 4. *Condition (I)' is equivalent to condition (M).*

Proof. We have already observed that our operator T has the property that $\|T^n\|_1 = 1$ for $n = 1, 2, 3, \dots$; therefore, in view of Theorem II quoted in §2, in order that condition (M) may hold, it is necessary and sufficient that for every element f belonging to a fundamental set in $L^1(m)$ the set $\{A_n f; n = 1, 2, 3, \dots\}$ be weakly sequentially compact, where $A_n f = n^{-1} \sum_{k=0}^{n-1} T^k f$. For the fundamental set in question we may take the set of all characteristic functions $\chi_B(x)$ of sets in \mathcal{B} . Thus, it suffices to consider the weak sequential compactness of the set $\{A_n \chi_B; n = 1, 2, 3, \dots\}$ where $B \in \mathcal{B}$. Now, for a fixed B , it is clear that the set $\{A_n \chi_B; n = 1, 2, 3, \dots\}$ is bounded in $L^1(m)$; therefore, in view of Theorem I in §2, this set is weakly sequentially compact in $L^1(m)$ if and only if the countable additivity of the integrals $\int_E A_n \chi_B(x) m(dx)$ is uniform in n , or, in other words, if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $m(E) < \delta$ implies $\sup_n \int_E A_n \chi_B(x) m(dx) < \varepsilon$. But, since we have for every n and every B in \mathcal{B}

$$\begin{aligned} \int_E A_n \chi_B(x) m(dx) &= \frac{1}{n} \sum_{k=0}^{n-1} \int_X T^k \chi_B(x) \chi_E(x) m(dx) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \int_B U^k \chi_E(x) m(dx) \leq \frac{1}{n} \sum_{k=0}^{n-1} Q_k(E), \end{aligned}$$

our condition (I)' implies the weak sequential compactness of the set $\{A_n \chi_B; n = 1, 2, 3, \dots\}$ for each B in \mathcal{B} .

Conversely, if the set $\{A_n \chi_B; n = 1, 2, 3, \dots\}$ is weakly sequentially compact for each set B in \mathcal{B} , then, in particular, it is true for $B = X$ so that the countable additivity of the integrals $\int_E A_n 1(x) m(dx)$ is uniform in n , i.e., condition (I)' is satisfied. Q.E.D.

PROPOSITION 5. *Condition (M) implies the existence of a finite measure ν on (X, \mathcal{B}) which is invariant under $P(x, B)$ and is absolutely continuous with respect to the measure m .*

Proof. Suppose condition (M) is satisfied. Then, we can define an operator \bar{T} on $L^1(m)$ by

$$(4.1) \quad f \rightarrow \bar{T}f: \bar{T}f(x) = s\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x),$$

where $s\text{-}\lim$ denotes the limit taken in the sense of the norm in $L^1(m)$. It is clear that \bar{T} is linear and positive, and furthermore, by virtue of the uniform boundedness theorem, \bar{T} is bounded (in fact, $\|\bar{T}\|_1 = 1$). \bar{T} satisfies the identity $T\bar{T} = \bar{T}$, since for every f in $L^1(m)$

$$\begin{aligned} \left\| T\left(\frac{1}{n} \sum_{k=0}^{n-1} T^k f\right) - \frac{1}{n} \sum_{k=0}^{n-1} T^k f \right\|_1 &= \frac{1}{n} \|T^n f - f\|_1 \\ &\leq \frac{2\|f\|_1}{n} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Now, let us define a positive measure ν on (X, \mathcal{B}) by setting

$$(4.2) \quad \nu(B) = \int_B \bar{T}1(x) m(dx)$$

for every B in \mathcal{B} . Then, clearly, ν is a measure absolutely continuous with respect to m . Since the norm convergence implies the weak convergence, we have

$$\nu(B) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_B T^k 1(x) m(dx) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B)$$

which shows that ν is finite as well as nontrivial since

$$\nu(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(X) = 1.$$

Finally, ν is invariant under $P(x, B)$ because for every B in \mathcal{B}

$$\begin{aligned} S\nu(B) &= \int_X P(x, B) \nu(dx) = \int_X U\chi_B(x) \bar{T}1(x) m(dx) \\ &= \int_B T\bar{T}1(x) m(dx) \\ &= \int_B \bar{T}1(x) m(dx) = \nu(B). \end{aligned} \quad \text{Q.E.D.}$$

We shall next show that condition (III)* implies condition (I). To obtain this implication, we shall follow the argument used by A. B. Hajian and S. Kakutani [7; 8] in obtaining the same implication for the special case of deterministic processes; however, since we cannot shift sets by using the point transformations as they did, our argument has to be more complicated. We shall need the following lemmas.

LEMMA 1. *Suppose for some $\varepsilon > 0$, there exists a sequence of measurable sets $\{B_i\}$ such that*

$$m(B_i) \rightarrow 0 \text{ as } i \rightarrow \infty,$$

but

$$\sup_n Q_n(B_i) \geq \varepsilon \text{ for all } i.$$

Then, there exists a monotone decreasing sequence of measurable sets $\{C_i\}$ such that

$$m(C_i) \rightarrow 0 \text{ as } i \rightarrow \infty,$$

and

$$\sup_n Q_n(C_i) \geq \varepsilon \text{ for all } i.$$

Proof. Obvious.

LEMMA 2. Let $\{a_n\}$ be a sequence of real numbers which are uniformly bounded. Let n_0 be an arbitrary but fixed positive integer. Then,

$$(4.3) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{k+n_0}.$$

Proof. Let M be a positive number such that $|a_n| \leq M$ for all n . Then, for any sequence of positive integers $\{n_j\}$ tending to ∞ ,

$$\begin{aligned} \left| \frac{1}{n_j} \sum_{k=0}^{n_j-1} a_{k+n_0} - \frac{1}{n_j} \sum_{k=0}^{n_j-1} a_k \right| &= \left| \frac{1}{n_j} (a_{n_0+n_j-1} + a_{n_0+n_j-2} + \cdots + a_{n_j}) \right. \\ &\quad \left. - \frac{1}{n_j} (a_{n_0-1} + a_{n_0-2} + \cdots + a_0) \right| \\ &\leq \frac{2n_0 M}{n_j} \rightarrow 0. \end{aligned}$$

Now, there exists a sequence of positive integers $\{n_{ij}\}$ tending to ∞ such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = \lim_{i \rightarrow \infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} a_k.$$

But, then, what we observed above shows that

$$\lim_{i \rightarrow \infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} a_{k+n_0} = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k$$

and consequently

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{k+n_0} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k.$$

Interchanging the role of $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k$ and $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{k+n_0}$ in the argument above, we get the reverse inequality

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{k+n_0} \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k. \quad \text{Q.E.D.}$$

We remark that the similar assertion for \limsup instead of \liminf is also true, but we shall not need this fact.

PROPOSITION 6. Condition (III)* implies condition (I).

Proof. Suppose false. Then, there exist an $\varepsilon > 0$ and a sequence of measurable sets $\{B_i\}$ such that

$$m(B_i) \rightarrow 0 \text{ as } i \rightarrow \infty$$

but

$$\sup_n Q_n(B_i) \geq 3\varepsilon \text{ for all } i.$$

By virtue of Lemma 1, we can suppose without loss of generality that the sequence $\{B_i\}$ is monotone decreasing. Now, condition (III)* asserts that for the above ε , (4.4) there exists a $\delta > 0$ such that if $m(B) > \varepsilon$ then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > \delta.$$

Let us put

$$(4.5) \quad \delta_1 = \frac{\varepsilon}{1-\varepsilon} \delta.$$

Then, since $\varepsilon \leq \frac{1}{3} \sup_n Q_n(X) = \frac{1}{3}$, we have $0 < \delta_1 \leq \frac{1}{2} \delta$. Since the set function $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k$ satisfies the conditions of Lemma 2 in §2 of [8], we can apply the said lemma to pick an integer i_0 with the property that for all $i > i_0$,

$$(4.6) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B_{i_0} - B_i) < \delta_1.$$

By our hypothesis, we can also choose an integer n_0 such that

$$Q_{n_0}(B_{i_0}) \geq 3\varepsilon.$$

Now, the absolute continuity of the measure Q_{n_0} with respect to m and the fact that $m(B_i) \rightarrow 0$ as $i \rightarrow \infty$ imply that we can find an integer i_1 such that

$$Q_{n_0}(B_{i_1}) < \varepsilon.$$

Obviously, $i_1 > i_0$, and consequently, we have, by (4.6),

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B_{i_0} - B_{i_1}) < \delta_1.$$

Now, write $E \equiv B_{i_0} - B_{i_1}$. Then, we have

$$(4.7) \quad Q_{n_0}(E) = Q_{n_0}(B_{i_0}) - Q_{n_0}(B_{i_1}) > 2\varepsilon,$$

and

$$(4.8) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(E) < \delta_1.$$

Define now

$$\hat{E} = \left\{ x \mid P^{n_0}(x, E) > \frac{\varepsilon}{1-\varepsilon} \right\},$$

then

$$\begin{aligned} 2\varepsilon &< Q_{n_0}(E) = \int_{\hat{E}} P^{n_0}(x, E) m(dx) + \int_{X-\hat{E}} P^{n_0}(x, E) m(dx) \\ &\leq m(\hat{E}) + \frac{\varepsilon}{1-\varepsilon} m(X - \hat{E}), \end{aligned}$$

from which it follows that

$$(4.9) \quad m(\hat{E}) > \varepsilon.$$

On the other hand, we have, for any $x \in X$,

$$\begin{aligned} P^{n_0+1}(x, E) &= \int_{\hat{E}} P^{n_0}(y, E) P(x, dy) + \int_{X-\hat{E}} P^{n_0}(y, E) P(x, dy) \\ &> \frac{\varepsilon}{1-\varepsilon} P(x, \hat{E}), \end{aligned}$$

from which it follows that

$$P(x, \hat{E}) < \frac{1-\varepsilon}{\varepsilon} P^{n_0+1}(x, E) \quad \text{for any } x \in X.$$

Therefore, for every positive integer n ,

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} Q_{k+1}(\hat{E}) &= \frac{1}{n} \sum_{k=0}^{n-1} \int_X P(x, \hat{E}) Q_k(dx) \\ &< \frac{1-\varepsilon}{\varepsilon} \frac{1}{n} \sum_{k=0}^{n-1} \int_X P^{n_0+1}(x, E) Q_k(dx) \\ &= \frac{1-\varepsilon}{\varepsilon} \frac{1}{n} \sum_{k=0}^{n-1} Q_{k+n_0+1}(E). \end{aligned}$$

Consequently,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_{k+1}(\hat{E}) \leq \frac{1-\varepsilon}{\varepsilon} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_{k+n_0+1}(E).$$

But, by virtue of Lemma 2, the right hand side of the inequality above equals

$$\frac{1-\varepsilon}{\varepsilon} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(E),$$

while the left hand side equals

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(\hat{E}).$$

Thus, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(\hat{E}) \leq \frac{1-\varepsilon}{\varepsilon} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(E) < \frac{1-\varepsilon}{\varepsilon} \delta_1 = \delta$$

by (4.8), but this is a contradiction to (4.4) since $m(\hat{E}) > \varepsilon$ as we saw in (4.9). Q.E.D.

THEOREM 1. *Each of conditions (II)* and (III)* is necessary and sufficient for the existence of a finite measure ν which is invariant under the Markov process $P(x, B)$ and is equivalent to m .*

Proof. We have seen in Proposition 6 that condition (III)* implies condition (I) (and hence (I)'). By Propositions 4 and 5, condition (I)' in turn implies the existence of a finite measure ν which is invariant under $P(x, B)$ and is absolutely continuous with respect to m . Therefore, condition (III)* implies the existence of such a measure. Since it is clear from the definition that condition (II)* implies condition (III)*, the former also implies the existence of such a measure. As we have seen in Proposition 5 this measure ν is given by the formula

$$\nu(B) = \int_B T1(x) m(dx) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B),$$

which shows that either one of conditions (II)* or (III)* implies that ν is equivalent to m .

It now remains to prove necessity. Obviously, it is enough to prove the necessity of condition (II)*. For this purpose, let us suppose that there exists a finite measure ν which is invariant under $P(x, B)$ and is equivalent to m , and suppose that condition (II)* were not satisfied. Then, we can find an $\varepsilon > 0$ and a sequence of measurable sets $\{B_i\}$ such that

$$(4.10) \quad m(B_i) > \varepsilon \quad \text{for each } i,$$

but

$$(4.11) \quad \liminf_{n \rightarrow \infty} Q_n(B_i) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

We can, therefore, find a sequence of positive integers $\{n_i\}$ such that

$$(4.12) \quad Q_{n_i}(B_i) = \int_X P^{n_i}(x, B_i) m(dx) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Consequently, we can select a subsequence $\{P^{n_j}(x, B_j)\}$ of functions from the sequence $\{P^{n_i}(x, B_i)\}$ such that

$$P^{n_j}(x, B_j) \rightarrow 0, \text{ } m\text{-a.e.} \quad \text{as } j \rightarrow \infty.$$

Equivalence of the measures m and ν implies then

$$P^{n_j}(x, B_j) \rightarrow 0, \text{ } \nu\text{-a.e.} \quad \text{as } j \rightarrow \infty,$$

and consequently, by virtue of the dominated convergence theorem,

$$\nu(B_j) = \int_X P^{n_j}(x, B_j) \nu(dx) \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

which in turn implies

$$m(B_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

But, this is a contradiction to (4.10).

Q.E.D.

Our next task is to show that condition (IV)* is also necessary and sufficient for our problem. We shall establish this fact by showing that condition (IV)* implies condition (III)* (and hence these two conditions are equivalent). A. B. Hajian and S. Kakutani also obtained this implication for the special case of the deterministic processes in [8], but their argument does not seem to yield a generalization which may be applied to our situation. In the following, we shall take an approach similar to the one used by Y. N. Dowker in [5]. However, our argument has to be more complicated than hers, since here again we are not dealing with measurable point transformations.

Let us start with a few definitions and observations. For any function $f \in L^\infty(m)$, define

$$(4.13) \quad f^+ = \max(f, 0),$$

$$(4.14) \quad f^- = -\min(f, 0).$$

Then, clearly, $f^+, f^- \in L^\infty(m)$, $f^+ \geq 0$, $f^- \geq 0$ and

$$(4.15) \quad f = f^+ - f^-.$$

Furthermore, if it is possible to express

$$f = f_1 - f_2$$

with $f_1 \geq 0, f_2 \geq 0$, then we must have

$$(4.16) \quad f^+ \leq f_1$$

and

$$(4.17) \quad f^- \leq f_2.$$

Finally, if one of f^+ and f^- is strictly positive on some set, then the other has to vanish m -almost everywhere on the same set.

The decomposition obtained in the following lemma is similar to the one obtained for the operator T (rather than U) by R. V. Chacon and D. S. Ornstein in [2] (cf. also E. Hopf [12]).

LEMMA 3. Let $f \in L^\infty(m)$ and denote for some fixed positive integer N ,

$$A_N = \left\{ x \mid \max_{1 \leq n \leq N} \sum_{k=0}^{n-1} U^k f(x) > 0 \right\},$$

where the operator U is defined as in (2.7). Then, we can find two functions h and ϕ , both in $L^\infty(m)$, satisfying the following conditions:

- (i) $h^- \leq f^-$,
- (ii) $\phi \geq 0$, and $h = f + U\phi - \phi$,
- (iii) $h^-(x) = 0$ m -almost everywhere on A_N .

Proof. As in the proof of the Basic Lemma 1 in [12], define functions h_i inductively by

$$\begin{aligned} h_0 &= f, \\ h_i &= U h_{i-1}^+ - h_{i-1}^-, \quad i = 1, 2, 3, \dots \end{aligned}$$

Clearly, $h_i \in L^\infty(m)$ for each i . The same argument as that of [12] shows that with the choice of $\phi_i = \sum_{k=0}^{i-1} h_k^+$ the function h_i satisfies conditions (i) and (ii) for each i , and that

$$(4.18) \quad \phi_i \geq \sum_{k=0}^{i-1} U^k f \quad \text{holds } m\text{-a.e. for each } i.$$

Now, (4.18) implies that for m -almost all x in A_N there would exist an integer k such that $0 \leq k \leq N$ and such that $h_k^+(x) > 0$. Therefore, neglecting a set of m -measure zero, we have $\bigcup_{k=0}^N \{h_k^+(x) > 0\} \supset A_N$. But, if $h_k^+(x) > 0$ holds at a point x , then $h_k^-(x) = 0$ and hence $h_j^-(x) = 0$ for all $j \geq k$. Therefore, by choosing i sufficiently large (but not exceeding N), we can make $h_i^-(x) = 0$ on A_N . For such a choice of i , h_i thus satisfies (i), (ii) and (iii). Q.E.D.

The following lemma can be considered as a version of the maximal ergodic theorem for the operator U .

LEMMA 4. Let $f \in L^\infty(m)$ and denote for some fixed positive integer N ,

$$A_N = \left\{ x \mid \max_{1 \leq n \leq N} \sum_{k=0}^{n-1} U^k f(x) > 0 \right\}.$$

Then,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{A_N} f(y) P^k(x, dy) \geq 0,$$

for m -almost all $x \in X$. ($P^0(x, B)$ means $\chi_B(x)$.)

Proof. By the preceding lemma, we can choose two functions h and ϕ , both in $L^\infty(m)$, satisfying conditions (i), (ii) and (iii) of that lemma. Then, by virtue of the m -nonsingularity of the process, we can find a set θ in \mathcal{B} such that

(α) $m(\theta) = 0$,

(β) for every $x \in X - \theta$ and for every nonnegative integer k ,

$$\begin{aligned} - \int_{(X-A_N)} f^-(y) P^k(x, dy) &\leq - \int_{(X-A_N)} h^-(y) P^k(x, dy) \\ &= - \int_X h^-(y) P^k(x, dy) \\ (4.19) \quad &\leq \int_X h(y) P^k(x, dy) \\ &= \int_X (f(y) + U\phi(y) - \phi(y)) P^k(x, dy) \end{aligned}$$

holds.

From (4.19) it then follows that

$$(4.20) \quad \int_{A_N} f(y) P^k(x, dy) + \int_{(X-A_N)} f^+(y) P^k(x, dy) + \int_X (U\phi(y) - \phi(y)) P^k(x, dy) \geq 0$$

holds for every $x \in X - \theta$ and for every nonnegative integer k . But, on the set $X - A_N$ we have $f \leq \max_{1 \leq n \leq N} \sum_{k=0}^{n-1} U^k f \leq 0$ so that $f^+(y) = 0$ for m -almost all y in $X - A_N$. Therefore, again by the m -nonsingularity of the process, the second integral in the left hand side of (4.20) equals zero for all x except on some set of m -measure zero (which can be chosen to be independent of the particular value of k). Without loss of generality, we can suppose that this exceptional set of m -measure zero is contained in the set θ . Therefore, we have for every $x \in X - \theta$ and every nonnegative integer k ,

$$\int_{A_N} f(y) P^k(x, dy) + \int_X (U\phi(y) - \phi(y)) P^k(x, dy) \geq 0,$$

and consequently, for every $x \in X - \theta$,

$$(4.21) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{A_N} f(y) P^k(x, dy) + \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_X (U\phi(y) - \phi(y)) P^k(x, dy) \geq 0.$$

But, the second term in the left hand side of the inequality above vanishes for all x except on some set of m -measure zero, because we have, for every positive integer n ,

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=0}^{n-1} \int_X (U\phi(y) - \phi(y)) P^k(x, dy) \right| \\ = \frac{1}{n} \left| U^n \phi(x) - \phi(x) \right| \leq \frac{2}{n} \|\phi\|_{\infty}. \end{aligned}$$

We can suppose again that this exceptional set of m -measure zero is contained in θ ; consequently, (4.21) now implies that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{A_N} f(y) P^k(x, dy) \geq 0$$

for all $x \in X - \theta$.

Q.E.D.

LEMMA 5. Suppose $f \in L^{\infty}(m)$, $f \geq 0$ and $m\{x | f(x) > 0\} > 0$. Suppose also that there exists a function p in $L^{\infty}(m)$ such that p is invariant under U (i.e., $Up(x) = p(x)$ for m -almost all x) and such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) < p(x)$$

holds for m -almost all x . Then, we can find a sequence of functions $\{f_N(x)\}$ belonging to $L^\infty(m)$ such that $0 \leq f_N \leq f$, $\{f_N(x)\}$ is monotone increasing and approaches $f(x)$ as $N \rightarrow \infty$, m -a.e. and such that for each N ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f_N(x) \leq p(x)$$

holds m -a.e.

Proof. For each positive integer N , define

$$A_N = \left\{ x \mid \min_{1 \leq n \leq N} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) < p(x) \right\}.$$

Then, $\{A_N\}$ is a monotone increasing sequences of measurable sets approaching X (mod sets of m -measure zero) as $N \rightarrow \infty$. The invariance of the function p implies that for m -almost all x ,

$$\max_{1 \leq n \leq N} \frac{1}{n} \sum_{k=0}^{n-1} U^k (p - f)(x) = p(x) - \min_{1 \leq n \leq N} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x).$$

Therefore, if we denote

$$\begin{aligned} \hat{A}_N &= \left\{ x \mid \max_{1 \leq n \leq N} \frac{1}{n} \sum_{k=0}^{n-1} U^k (p - f)(x) > 0 \right\} \\ &= \left\{ x \mid \max_{1 \leq n \leq N} \sum_{k=0}^{n-1} U^k (p - f)(x) > 0 \right\}, \end{aligned}$$

then, $m(A_N \triangle \hat{A}_N) = 0$, where $A_N \triangle \hat{A}_N$ denotes the symmetric difference of the two sets A_N and \hat{A}_N . Now, for each N , let

$$f_N(x) = \begin{cases} f(x) & \text{if } x \in A_N, \\ p(x) & \text{if } x \notin A_N. \end{cases}$$

Then, clearly, $f_N \in L^\infty(m)$ and $0 \leq f_N \leq f$, since $p(x) \leq f(x)$ if $x \notin A_N$. Furthermore, $f_N(x)$ increases monotonely to $f(x)$ m -a.e. as $N \rightarrow \infty$. Now, since $m(A_N \triangle \hat{A}_N) = 0$ the m -nonsingularity of the process implies the existence of a set θ^* in \mathscr{B} such that $m(\theta^*) = 0$ and such that for every $x \in X - \theta^*$ and for every positive integer n ,

$$\frac{1}{n} \sum_{k=0}^{n-1} U^k (p - f_N)(x) = \frac{1}{n} \sum_{k=0}^{n-1} \int_{\hat{A}_N} (p(y) - f(y)) P^k(x, dy)$$

holds. But, by virtue of Lemma 4, there exists a set θ in \mathscr{B} such that $m(\theta) = 0$ and such that for all $x \in X - \theta$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\hat{A}_N} (p(y) - f(y)) P^k(x, dy) \geq 0.$$

Consequently, for all $x \in X - (\theta \cup \theta^*)$, we must have

$$p(x) - \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f_N(x) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k (p - f_N)(x) \geq 0.$$

Since $m(\theta \cup \theta^*) = 0$, we have the desired conclusion.

Q.E.D.

LEMMA 6. Suppose $f \in L^\infty(m)$, $f \geq 0$ and $m\{x | f(x) > 0\} > 0$. Suppose further that the function $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} U^k f(x)$ is invariant under U . Then, we can find a sequence of functions $\{g_N(x)\}$ belonging to $L^\infty(m)$ such that $0 \leq g_N \leq f$, $\{g_N(x)\}$ is monotone increasing and approaching $f(x)$, m -a.e. as $N \rightarrow \infty$, and such that for each N ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k g_N(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x)$$

holds m -a.e.

Proof. By our hypothesis, for every positive integer j the function $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} U^k f(x) + 1/j$ is invariant under U ; therefore, we can apply Lemma 5 by taking p to be $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} U^k f(x) + 1/j$ to find, for each j , a sequence of functions $\{f_N^j(x)\}$ belonging to $L^\infty(m)$ such that $0 \leq f_N^j \leq f$, $\{f_N^j(x)\}$ is monotone increasing and approaches $f(x)$, m -a.e. as $N \rightarrow \infty$, and such that for each N ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f_N^j(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) + \frac{1}{j}$$

holds m -a.e. Now, we can find a sequence of positive integers $\{N_j\}$ such that

$$m\left\{x | f(x) - f_{N_j}^j(x) > \frac{1}{j}\right\} < \frac{1}{2j}.$$

Define $g_N(x) = \inf_{j \geq N} f_{N_j}^j(x)$. Then, the sequence $\{g_N(x)\}$ is monotone increasing and for every N , g_N belongs to $L^\infty(m)$. Furthermore, if $j \geq N$, then $0 \leq g_N \leq f_{N_j}^j$, so that we have $0 \leq g_N \leq f$. Denote

$$\Lambda = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j,$$

where $E_j = \{x | f(x) - f_{N_j}^j(x) > 1/j\}$, then, $m(\Lambda) = 0$, and for every $x \in X - \Lambda$, there exists a positive integer $i(x)$ such that $x \notin \bigcup_{j \geq i(x)} E_j$. Now, for a fixed $x \in X - \Lambda$ and for any given $\varepsilon > 0$, choose a positive integer h so large that $h > i(x)$ and $h > 1/\varepsilon$. Then, for this choice of h , we have

$$f(x) - g_h(x) \leq \frac{1}{h} < \varepsilon,$$

which shows that $g_N(x)$ approaches $f(x)$, m -a.e. as $N \rightarrow \infty$. Finally, let N be fixed, then, since $g_N \leq f_N^j$ for all $j \geq N$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k g_N(x) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f_{N,j}^j(x) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) + \frac{1}{j} \end{aligned}$$

holds m -a.e. for all $j \geq N$. Therefore, for each N ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k g_N(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x)$$

holds m -a.e.

Q.E.D.

Let us denote by $L^*(m)$ the totality of all functions f in $L^1(m)$ for which $sUf(x) = \int_X f(y)P(x, dy)$ can be defined and belong to $L^1(m)$. Then, it is clear that all functions in $L^\infty(m)$ belong to $L^*(m)$. We say that a function f in $L^*(m)$ is *ub-invariant under U* if the inequality $Uf(x) \leq f(x)$ holds m -a.e.

LEMMA 7. *The following two statements are equivalent:*

(i) *Every nonnegative function f in $L^*(m)$ which is sub-invariant under U is invariant under U .*

(ii) *If a set E is m -transient for the Markov process $P(x, B)$ (cf. Definition 3), then $m(E) = 0$.*

Proof. (i) \Rightarrow (ii): Suppose the assertion were false. Then, there should exist a set E in \mathcal{B} such that $m(E) > 0$ but $\sum_{n=0}^{\infty} Q_n(E) < \infty$. By the monotone convergence theorem,

$$\int_X \sum_{n=0}^{\infty} P^n(x, E) m(dx) = \sum_{n=0}^{\infty} Q_n(E) < \infty$$

so that we have

$$\sum_{n=0}^{\infty} P^n(x, E) < \infty, \quad m\text{-a.e.}$$

Let us define

$$f(x) = \sum_{n=0}^{\infty} P^n(x, E),$$

then, f belongs to $L^1(m)$ and by virtue of the monotone convergence theorem again,

$$\begin{aligned}
 Uf(x) &= \int_X \sum_{n=0}^{\infty} P^n(y, E) P(x, dy) = \sum_{n=0}^{\infty} P^{n+1}(x, E) \\
 &= f(x) - P^0(x, E) = f(x) - \chi_E(x) \leq f(x)
 \end{aligned}$$

with the strict inequality holding on the set E . Therefore, f is nonnegative, belongs to $L^*(m)$ and is sub-invariant but not invariant under U .

(ii) \Rightarrow (i): Suppose again that the assertion were false. Then, there exist a function f in $L^*(m)$, a positive number ε and a set E in \mathcal{B} such that $f \geq 0$, $Uf \leq f$, m -a.e., and $f - Uf \geq \varepsilon$ on E with $m(E) > 0$. Observe that the condition $Uf \leq f$, m -a.e. implies, because of the positivity of U , that for every positive integer n , $U^n f \leq f$ holds m -a.e. so that $U^n f \in L^1(m)$ and $\|U^n f\|_1 \leq \|f\|_1$. But, then, for every positive integer n ,

$$\begin{aligned}
 \varepsilon \sum_{k=0}^{n-1} Q_k(E) &\leq \int_E (f(x) - Uf(x)) \sum_{k=0}^{n-1} T^k 1(x) m(dx) \\
 &\leq \int_X (f(x) - Uf(x)) \sum_{k=0}^{n-1} T^k 1(x) m(dx) \\
 &= \int_X \sum_{k=0}^{n-1} U^k (f - Uf)(x) m(dx) \\
 &= \int_X (f(x) - U^n f(x)) m(dx) \leq 2 \|f\|_1 < \infty.
 \end{aligned}$$

Since the first term of the inequality above tends to ∞ as $n \rightarrow \infty$ by our hypothesis, we have a contradiction. Q.E.D.

PROPOSITION 7. *Condition (IV)* is equivalent to condition (III)*.*

Proof. It is obvious from the statement of these conditions that condition (III)* implies condition (IV)*. Therefore, it suffices to prove the converse implication. First, observe that for any $f \in L^\infty(m)$, Fatou's Lemma implies that

$$\begin{aligned}
 U \left(\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f \right) (x) &= \int_X \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(y) P(x, dy) \\
 &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_X U^k f(y) P(x, dy) \\
 &= \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^{k+1} f(x) \\
 &= \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x)
 \end{aligned}$$

m -a.e. so that the function $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} U^k f$ is sub-invariant under U . Now, if condition (IV)* is satisfied, it is clear that for any $B \in \mathcal{B}$, $m(B) > 0$ would imply $\sum_{n=0}^{\infty} Q_n(B) = \infty$. Consequently, by Lemma 7, the function $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} U^k \chi_B(x)$ is invariant under U , where χ_B is the characteristic function of a set B in \mathcal{B} .

Now, let ε be given. Then, by our hypothesis, there exists a $\delta^*(\frac{1}{2}\varepsilon) > 0$ such that

$$(4.22) \quad \text{if } m(B) > \frac{1}{2}\varepsilon, \text{ then } \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > \delta^*\left(\frac{1}{2}\varepsilon\right).$$

Let $\delta_*(\varepsilon) = \frac{1}{2}\delta^*(\frac{1}{2}\varepsilon)$. We shall show that

(4.23) if B' is any set in \mathcal{B} such that $m(B') > \varepsilon$, then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B') > \delta_*(\varepsilon).$$

So, let us suppose that $m(B') > \varepsilon$. Then, by the remark made above, it is clear that the function $\chi_{B'}(x)$ satisfies all the hypotheses of Lemma 6. Therefore, there exists a sequence of functions $\{g_N(x)\}$ belonging to $L^\infty(m)$ such that $0 \leq g_N \leq \chi_{B'}$, $\{g_N\}$ increases monotonely to $\chi_{B'}$, m -a.e. as $N \rightarrow \infty$, and such that for each N ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k g_N(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k \chi_{B'}(x)$$

holds m -a.e. Let us now choose a suitably large positive integer J so that

$$m\left\{x \mid x \in B', \chi_{B'}(x) - g_J(x) < \frac{1}{2}\right\} \geq \frac{1}{2}m(B') > \frac{1}{2}\varepsilon.$$

Denote the set in the curly bracket above by B_J , then, on this set, we have $g_J(x) \geq \frac{1}{2}$ and consequently, $g_J \geq \frac{1}{2}\chi_{B_J}$. Therefore,

$$\begin{aligned} \frac{1}{2} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B_J) &= \frac{1}{2} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k \chi_{B_J}(x), \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k g_J(x) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k \chi_{B'}(x) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P(x, B') \end{aligned}$$

holds m -a.e. Fatou's Lemma then asserts that

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B') &\geq \int_X \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B') m(dx) \\
&\geq \frac{1}{2} \int_X \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B_J) m(dx) \\
&\geq \frac{1}{2} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B)_J.
\end{aligned}$$

Since $m(B_J) > \frac{1}{2}\varepsilon$, (4.22) now implies $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} Q_k(B') > \delta_*(\varepsilon)$, i.e., (4.23) is satisfied. Q.E.D.

THEOREM 2. *Condition (IV)* is also necessary and sufficient for the existence of a finite measure ν which is invariant under the Markov process and is equivalent to m .*

Proof. Immediate from Theorem 1 and Proposition 7.

Our next task is to show that conditions (II), (III), and (IV) are also necessary and sufficient for our problem. This is obtained immediately from a lemma due to A. B. Hajian and S. Kakutani.

THEOREM 3. *Conditions (II), (III) and (IV) are also necessary and sufficient for the existence of a finite, equivalent, invariant measure.*

Proof. Observe that the set function $\limsup_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} Q_k$ is a nonnegative, monotonic and subadditive set function defined on (X, \mathcal{B}) . Therefore, we can apply Lemma 1 in §2 of [8] to get the equivalence of conditions (IV) and (IV)*. Thus, condition (IV) is necessary and sufficient for our problem. Furthermore, it is obvious from the statement of conditions (II) and (III) that they imply condition (IV), and consequently, both of them are sufficient for our problem. On the other hand, in the proof of Theorem 1 we have already established the necessity of condition (II)*. Since the latter obviously implies conditions (II) and (III), we have the necessity of (II) and (III) as well. Q.E.D.

THEOREM 4. *Conditions (V)* and (VI)* are also necessary and sufficient for our problem.*

Proof. We shall prove this theorem by showing that conditions (II), (V)* and (VI)* are all equivalent.

(II) \Rightarrow (V)*: Suppose B is a weakly transient set for the Markov process. Then, there exists a sequence of positive integers $\{n_i\}$ tending to ∞ such that

$$\sum_{i=1}^{\infty} P^{n_i}(x, B) < \infty, \text{ } m\text{-a.e.}$$

But, this would imply that $\lim_{i \rightarrow \infty} P^{n_i}(x, B) = 0$, m -a.e. The dominated convergence theorem then implies that $\lim_{i \rightarrow \infty} Q_{n_i}(B) = 0$; therefore, we must have $\liminf_{n \rightarrow \infty} Q_n(B) = 0$. Condition (II) now implies that $m(B) = 0$.

(V)[#] \Rightarrow (VI)[#]: Suppose B is a weakly m -transient set for the Markov process. Then, there exists a sequence of positive integers $\{n_i\}$ tending to ∞ such that

$$\sum_{i=1}^{\infty} Q_{n_i}(B) < \infty.$$

By virtue of the monotone convergence theorem, we must then have

$$\sum_{i=1}^{\infty} P^{n_i}(x, B) < \infty, \text{ } m\text{-a.e.}$$

Condition (V)[#] now implies that $m(B) = 0$.

(VI)[#] \Rightarrow (II): Obvious.

Q.E.D.

We say that a set B in \mathcal{B} is an *atom* if it satisfies the following conditions:

- (i) $m(B) > 0$,
- (ii) if F is a set in \mathcal{B} such that $F \subset B$, then either $m(F) = 0$ or $m(F) = m(B)$.

A measure space (X, \mathcal{B}, m) is called *atomless* if \mathcal{B} does not contain any atom.

THEOREM 5. *If the probability space (X, \mathcal{B}, m) is atomless, then condition (VII)[#] is also necessary and sufficient for our problem.*

Proof. We shall show that condition (VII)[#] is equivalent to condition (VI)[#] under our hypothesis.

(VI)[#] \Rightarrow (VII)[#]: Suppose the contrary. Then, there exist a sequence of positive integers $\{n_i\}$ tending to ∞ and a set B such that

$$\sum_{i=1}^{\infty} T^{n_i}1(x) < \infty, \text{ } m\text{-a.e. on the set } B,$$

but $m(B) = \alpha > 0$.

Then, we must have $\lim_{i \rightarrow \infty} T^{n_i}1(x) = 0$, m -a.e. on B . Let ε be an arbitrary positive number such that $\varepsilon < \alpha$. Then, since our space (X, \mathcal{B}, m) is atomless, we can find, by Egoroff's theorem, a set F in \mathcal{B} such that

$$F \subset B, \quad m(F) < \varepsilon$$

and $T^{n_i}1(x) \rightarrow 0$ uniformly in $B - F$ as $i \rightarrow \infty$. Therefore, we can choose a subsequence $\{n_j\}$ of $\{n_i\}$ tending to ∞ such that $\sum_{j=1}^{\infty} T^{n_j}1(x) \leq M$ for all $x \in B - F$ for some positive constant M . Then, by the dominated convergence theorem, we have

$$\sum_{j=1}^{\infty} Q_{n_j}(B - F) \leq Mm(B - F) < \infty.$$

Consequently, by condition (VI)[#], we get $m(B - F) = 0$. But, then,

$$\alpha = m(B) = m(F) + m(B - F) < \varepsilon,$$

which is a contradiction.

(VII)* \Rightarrow (VI)*: Proof goes exactly the same way as the proof of the implication (V)* \Rightarrow (VI)* in Theorem 4. Q.E.D.

THEOREM 6. *Conditions (VIII) and (IX) are both necessary and sufficient for our problem.*

Proof. We shall establish the proof by showing that the following cycle of implications is valid:

$$(IV) \Rightarrow (IX) \Rightarrow (VIII) \Rightarrow (III) \Rightarrow (IV).$$

(IV) \Rightarrow (IX): Obvious by Fatou's Lemma.

(IX) \Rightarrow (VIII): First, observe that under our hypothesis every set B of positive m -measure cannot be m -transient for the Markov process $P(x, B)$. This is because if there exists an m -transient set B with positive m -measure, then the monotone convergence theorem would imply

$$\sum_{n=0}^{\infty} P^n(x, B) < \infty, \text{ } m\text{-a.e.}$$

so that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B) = 0, \text{ } m\text{-a.e.},$$

which contradicts (IX). Consequently, the function $\liminf_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} P^k(x, B)$ is invariant under U by virtue of Lemma 7 and Fatou's Lemma. Thus, we can apply Lemma 6 to the function $\chi_B(x)$, and proceeding just as in the proof of Proposition 7, we can find a set B_J of positive m -measure such that

$$\frac{1}{2} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B_J) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B)$$

holds m -a.e. Our assertion now follows immediately.

(VIII) \Rightarrow (III): Obvious again by Fatou's Lemma.

(III) \Rightarrow (IV): Obvious. Q.E.D.

5. Examples and remarks. Let us first consider the following example.

EXAMPLE 1. Let (X, \mathcal{B}, m) be the Lebesgue measure space: $X = [0, 1]$, \mathcal{B} the σ -field of Lebesgue subsets of X , m the ordinary Lebesgue measure on \mathcal{B} . Define a transformation ϕ of (X, \mathcal{B}) onto itself by setting

$$\phi(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right), \\ x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, ϕ is measurable in the sense that $\phi^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$, but is not invertible. It is easy to see that the iterates $\phi^n(x)$ are given by

$$\phi^n(x) = \begin{cases} 2^n x & \text{if } x \in \left[0, \frac{1}{2^n}\right), \\ 2^{n-1} x & \text{if } x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right), \\ \vdots & \\ 2x & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right), \\ x & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Now, for every pair (x, B) , $x \in X$, $B \in \mathcal{B}$, define

$$(5.1) \quad P(x, B) = \chi_{\phi^{-1}(B)}(x) = \chi_B(\phi(x)).$$

Then, it is easy to see that this $P(x, B)$ satisfies conditions (2.1) and (2.2) so that it gives a Markov process on (X, \mathcal{B}) . Furthermore, it is clear that this process is m -nonsingular. The n th transition probability $P^n(x, B)$ is given by

$$(5.2) \quad P^n(x, B) = \chi_{\phi^{-n}(B)}(x) = \chi_B(\phi^n(x)).$$

Now, consider the set $B_1 = [0, \frac{1}{4})$. Then, $m(B_1) = \frac{1}{4} > 0$. But,

$$Q_n(B_1) = m(\phi^{-n}(B_1)) = \frac{1}{2^{n+2}}$$

so that

$$\lim_{n \rightarrow \infty} Q_n(B_1) = 0.$$

This shows that this Markov process fails to satisfy condition (II). On the other hand, let for any $\varepsilon > 0$, $\delta = \varepsilon/2$. Then, for any set B with $m(B) < \delta$, we have

$$Q_n(B) < 2m(B) \quad \text{for all } n$$

so that $\sup_n Q_n(B) < \varepsilon$, and this shows that this Markov process satisfies condition (I) (and hence (I)' and (M)).

Example 1 shows, therefore, that unlike the case of deterministic processes, conditions (I), (I)' and (M) are not sufficient in the general case, though they are necessary for our problem.

The next example will show that conditions (V), (VI) and (VII) are also necessary but not sufficient.

EXAMPLE 2. Let X be the real line, \mathcal{B} the σ -field of Lebesgue subsets of X , and μ the ordinary Lebesgue measure on \mathcal{B} . Define a measure m on (X, \mathcal{B}) by setting

$$m(B) = \frac{1}{\sqrt{(2\pi)}} \int_B e^{-x^2/2} \mu(dx)$$

for every set B in \mathcal{B} . It is clear that $m(X) = 1$ and that m is equivalent to μ . Now, for every pair (x, B) , $x \in X$, $B \in \mathcal{B}$, define

$$(5.3) \quad P(x, B) = \frac{1}{\sqrt{(2\pi)}} \int_B e^{-(x-y)^2/2} \mu(dy).$$

Then, clearly, this $P(x, B)$ satisfies conditions (2.1) and (2.2) and hence it gives a Markov process on (X, \mathcal{B}) . Furthermore, this process is obviously μ -nonsingular so that it is also m -nonsingular. Easy calculations show that the n th transition probability $P^n(x, B)$ is given by

$$P^n(x, B) = \frac{1}{\sqrt{(2\pi n)}} \int_B e^{-(x-y)^2/2n} \mu(dy)$$

and the measure $Q_n(B)$ is given by

$$Q_n(B) = \frac{1}{\sqrt{(2\pi(n+1))}} \int_B e^{-x^2/2(n+1)} \mu(dx)$$

and

$$T^n 1(x) = \frac{dQ_n}{dm} = \frac{1}{\sqrt{(n+1)}} e^{(n/2(n+1))x^2}.$$

Now, let B be any set in \mathcal{B} such that $0 < \mu(B) < \infty$. Then, $m(B) > 0$, but, for each positive integer n ,

$$Q_n(B) \leq \frac{1}{\sqrt{(2\pi(n+1))}} \mu(B)$$

so that $\lim_{n \rightarrow \infty} Q_n(B) = 0$. Therefore, this process does not satisfy condition (II).

Now, let B be any set with $m(B) > 0$, and let M be any positive constant. Then, for every $x \in X$ and positive integer n ,

$$\begin{aligned} P^n(x, B) &\geq P^n(x, B \cap \{y \mid |y - x| \leq M\}) \\ &\geq \frac{1}{\sqrt{(2\pi)}} e^{-M^2/2} \cdot \mu(B \cap \{y \mid |y - x| \leq M\}) \cdot \frac{1}{\sqrt{n}}. \end{aligned}$$

Now, since $m(B) > 0$ implies $\mu(B) > 0$, for each fixed $x \in X$, we can always find a positive constant M such that

$$\mu(B \cap \{y \mid |y - x| \leq M\}) > 0.$$

Consequently, for each x we have

$$\sum_{n=1}^{\infty} P^n(x, B) \geq K \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

where K is some positive constant depending on x . But this implies that

$$\sum_{n=1}^{\infty} P^n(x, B) = \infty \text{ for all } x \in X, \text{ whenever } m(B) > 0.$$

From this it follows also that $\sum_{n=1}^{\infty} Q_n(B) = \infty$ for all such B . Finally, since for each positive integer n and for each $x \in X$,

$$T^n 1(x) = \frac{1}{\sqrt{(n+1)}} e^{nx^2/2(n+1)} \geq \frac{1}{\sqrt{(n+1)}}$$

we have $\sum_{n=1}^{\infty} T^n 1(x) = \infty$ for every $x \in X$. Thus, our Markov process satisfies all of conditions (V), (VI) and (VII), but fails to satisfy condition (II). Consequently conditions (V), (VI) and (VII) are not sufficient for our problem.

We have already seen in §4 that there exists a close relationship between the existence of a finite invariant measure for a Markov process $P(x, B)$ and the validity of the mean ergodic theorem for the operator T induced by the process. Even though by a recent result of R. V. Chacon and D. S. Ornstein [2] it is known that for every pair of functions f, p in $L^1(m)$ with $p \geq 0$, the sequence of ratios

$$\frac{\sum_{k=0}^{n-1} T^k f(x)}{\sum_{k=0}^{n-1} T^k p(x)}$$

converges m -a.e. on the set where $\sum_{k=0}^{\infty} T_k p(x) > 0$, it is still interesting to ask when the pointwise ergodic theorem holds for the operator T or U in the space $L^1(m)$ or $L^{\infty}(m)$. From our main results obtained in §4, it is easy to deduce the following result.

PROPOSITION 8. *Let T be the operator in $L^1(m)$ induced by a Markov process $P(x, B)$ as in (2.6). Then, both the pointwise ergodic theorem and the mean ergodic theorem hold for T in $L^1(m)$ (i.e., the sequence of averages $n^{-1} \sum_{k=0}^{n-1} T^k f$ converges m -a.e. as well as in $L^1(m)$ -norm for every function f in $L^1(m)$), if the following condition is satisfied:*

$$(X) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k 1(x) > 0, \quad m\text{-a.e.}$$

Proof. Let B be any set in \mathcal{B} such that $m(B) > 0$. Then, condition (X) along with Fatou's Lemma implies that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} Q_k(B) > 0$$

so that condition (III) is satisfied by the process. Therefore, there exists a finite invariant measure ν which is equivalent to m . By the Radon-Nikodym theorem, there exists a function p in $L^1(m)$ such that

$$\nu(B) = \int_B p(x) m(dx) \quad \text{for all } B \in \mathcal{B}$$

and the equivalence of ν and m implies that $p(x) > 0$, m -a.e. Furthermore, the invariance of ν under $P(x, B)$ implies that p is invariant under T . Therefore, by the general ergodic theorem of R. V. Chacon and D. S. Ornstein,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) = p(x) \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} T^k f(x)}{\sum_{k=0}^{n-1} T^k p(x)}$$

exists m -a.e. for every $f \in L^1(m)$.

On the other hand, we have already seen that the validity of the mean ergodic theorem for T in $L^1(m)$ (condition (M)) is necessary for the existence of such a measure ν . Hence, our assertion follows. Q.E.D.

Now, suppose there exists a finite, invariant measure ν equivalent to m . Then, as we saw above, the pointwise ergodic theorem holds for T so that, in particular,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k 1(x)$$

exists m -a.e. On the other hand, the Chacon-Ornstein ergodic theorem tells us that if $p(x)$ denotes the Radon-Nikodym derivative of ν with respect to m , then

$$\lim_{n \rightarrow \infty} \frac{n}{\sum_{k=0}^{n-1} T^k 1(x)} = \frac{1}{p(x)} \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} T^k p(x)}{\sum_{k=0}^{n-1} T^k 1(x)}$$

exists and is finite m -a.e. Therefore, $\lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} T^k 1(x) > 0$ for m -almost all $x \in X$. This proves the following

PROPOSITION 9. *Condition (X) is another necessary and sufficient condition for our problem.*

Proposition 8 is of some interest in view of the fact that it is easy to construct a positive operator T with $L^1(m)$ -norm equal to one for which the pointwise ergodic theorem holds but the mean ergodic theorem fails to hold.

As for the operator U induced by a Markov process as in (2.7), we have the following result.

PROPOSITION 10. *If either condition (VIII) or (IX) holds, then, there exists a measure ν which is equivalent to m such that the sequence of averages $n^{-1} \sum_{k=0}^{n-1} U^k f(x)$ converges ν -a.e. (and hence m -a.e.) and in $L^1(\nu)$ -norm for every f in $L^1(\nu)$. Furthermore, if $f \geq 0$ and $m\{x | f(x) > 0\} > 0$, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) > 0$$

on some set F (which depends on f) of positive m -measure.

Proof. By Theorem 6, either one of conditions (VIII) and (IX) implies the existence of a finite, equivalent, invariant measure ν . Then, it is clear that the operator U maps $L^1(\nu)$ into $L^1(\nu)$ and $L^\infty(\nu)$ into $L^\infty(\nu)$ (since $L^\infty(\nu)$ is identical with $L^\infty(m)$) and has both $L^1(\nu)$ - and $L^\infty(\nu)$ -norm equal to one. Hence, by the Hopf-Dunford-Schwartz ergodic theorem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x)$$

exists ν -a.e. as well as in $L^1(\nu)$ -norm for every f in $L^1(\nu)$. Now, suppose $f \geq 0$ and $m\{x | f(x) > 0\} > 0$, then, $\nu\{x | f(x) > 0\} > 0$ so that the invariance of ν under $P(x, B)$ together with the dominated convergence theorem implies that

$$\int_X \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) \nu(dx) = \int_X f(x) \nu(dx) > 0.$$

Therefore, we must have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f(x) > 0$$

on some set F which has positive ν -measure and hence positive m -measure. Q.E.D.

Since every characteristic function of a set in \mathcal{B} belongs to $L^1(\nu)$, Proposition 10 implies the following:

COROLLARY. *The following condition is equivalent to condition (VIII) and (IX), and consequently, to the existence of a finite, equivalent, invariant measure for the Markov process:*

(XI) *For every B in \mathcal{B}*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x, B)$$

exists m -a.e., and if $m(B) > 0$, then this limit is strictly positive on some set F of positive m -measure.

We have observed in the proof of Proposition 10 that when a Markov process $P(x, B)$ possesses a finite invariant measure ν which is equivalent to m , then the operator U induced by the Markov process on the space $L^\infty(m)$ can be extended to an operator in $L^1(\nu)$, and furthermore, this extension has both $L^1(\nu)$ - and $L^\infty(\nu)$ -norm equal to one. A positive operator V defined for functions on some measure space (X, \mathcal{B}, m) having the following properties is called a *doubly-stochastic operator* and has been investigated by many authors:

(i) V maps $L^1(m)$ into $L^1(m)$ and $L^\infty(m)$ into $L^\infty(m)$,

(ii) $\|V\|_1 = 1$ and $\|V\|_\infty = 1$.

Our results obtained in this work can be rephrased to give an answer to the following problem:

Suppose U is a positive operator defined on the space $L^\infty(m)$ having norm equal to one. When can one find a finite measure ν on (X, \mathcal{B}) which is equivalent to m so that the operator U can be extended to a doubly-stochastic operator on (X, \mathcal{B}, ν) ?

If we assume further that the operator U is induced by an m -nonsingular Markov process as in (2.7), then, all of our conditions (II), (II)*, (III), (III)*, (IV), (IV)*, (V)*, (VI)*, (VIII), (IX), (X), (XI) (and (VII)* if the space (X, \mathcal{B}, m) is atomless) are necessary and sufficient for the problem stated above. Even if the operator U may not be induced by such a Markov process, we can go through a similar argument with a slight modification by replacing $P^n(x, B)$ with $U^n \chi_B(x)$, $n = 1, 2, 3, \dots$ to prove that our conditions (II)* and (III)* are necessary and sufficient for this problem as well.

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