INSTANTANEOUS STATES OF MARKOV PROCESSES

BY
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1. This paper answers a question suggested by Professor Kai Lai Chung concerning an instantaneous state of a countable state, continuous time, stationary Markov process. A countable state, continuous time, stationary Markov process is defined by its probability transition functions $p_{ij}(t)$, $i,j = 1,2,3,\cdots$, $t \geqq 0$, which satisfy the equations

$$p_{ij}(t) \geqq 0, \qquad \sum_{j=1}^{\infty} p_{ij}(t) = 1,$$

$$\sum_{k=1}^{\infty} p_{ik}(s)p_{kj}(t) = p_{ij}(s+t).$$

$p_{ij}(t)$ is interpreted to be the probability that the process will be in state j at time $s + t$ given that it is in state i at time s.

An additional condition which is usually imposed on the $p_{ij}(t)$, and which we will impose, is that $\lim_{t\to 0} p_{ii}(t) = 1$. This implies (see Chung [1]) the continuity of all the $p_{ij}(t)$, and the existence of all $p'_{ij}(t)$, which may be $-\infty$ for $i = j$, $t = 0$. Letting $p'_{ij}(0) = q_{ij}$, $i \neq j$, and $p'_{ii}(0) = -q_i$, we have $0 \leqq q_{ij} < \infty$ and $0 \leqq q_i \leqq \infty$. In the case $q_i = \infty$ we say i is an instantaneous state. An equivalent definition (see Chung) of an instantaneous state is a state which the process cannot remain in for any positive time interval.

In §2 an example is presented of a Markov process with $\limsup_{t\to 0} p'_{ii}(t) = +\infty$ for an instantaneous state i, answering negatively the question of whether $\lim_{t\to 0} p'_{ii}(t) = -\infty$ always if i is instantaneous.

2. A Markov process of a certain type, containing exactly one instantaneous state and a countable number of stable states will first be constructed, by the probabilistic construction of its sample functions.

Consider as in Chung [1, p. 255], a sequence of independent Poisson processes, all starting at time $t = 0$, with parameter λ_i for the ith process, where $\sum_{i=1}^{\infty} \lambda_i = \infty$. As in Chung, assign a different color to each process. Let all the jumps of a given process be denoted by marks of the assigned color at the points of the jumps. Then superimpose all the processes with their different colored marks, with $t = 0$ coinciding for all of them.

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DEFINITION 1. F_c is a distribution function defined by $F_c(x) = 0$ for $x < 0$, and $F_c(x) = 1 - e^{-cx}$ for $x \geqq 0$.

Choose for each i an $a_i > 0$ so that $\sum_{i=1}^{\infty}(a_i)^{-1} < \infty$ and for each mark of the ith color associate an ordered pair of numbers, each number of the pair being chosen probabilistically and independently from the distribution function $F_{\lambda_i a_i}$. Do this for all i, choosing the numbers independently for each mark. The result shall be called an L-process. More formally, we have

DEFINITION 2. Let $\lambda_i > 0$, $a_i > 0$, $i = 1, 2, \cdots$. We shall call the set $(\omega, \{c_j^{(1)}\}, \{d_j^{(1)}\}, \{c_j^{(2)}\}, \{d_j^{(2)}\}, \cdots)$ a sample function with assigned values of an an L-process if ω is a sample function defined on $t \geqq 0$ taking values $\gamma, 1, 2, \cdots$, such that for all $t > 0$ and integers i the set of $s \geqq 0$ such that $\omega(s) = i$ and $s < t$ consists of at most a finite number of points, and the $\{c_j^{(i)}\}$ and $\{d_j^{(i)}\}$ are sequences of positive numbers for each integer i. A $\{\{\lambda_i\}; \{a_i\}\}$, or more briefly, a $\{\lambda_i; a_i\}$ L-process is a probability space of sample functions of assigned values of an L-process with the probabilistic construction as follows. Let $X_{\alpha_i}^{(j)}$, $X_{\beta_i}^{(j)}$ and $Y_i^{(j)}$, $i, j = 1, 2, \cdots$, be independent random variables, $X_{\alpha_i}^{(j)}$ and $X_{\beta_i}^{(j)}$ having distribution function $F_{\lambda_i a_i}$ for all j, and $Y_i^{(j)}$ having distribution function F_{λ_i} for all j. Choose a point of the space of random variables. Associate with this point a sample function with assigned values of an L-process with $\omega(t) = i$ if $\sum_{j=1}^{n} Y_i^{(j)} = t$ for i and some n, $\omega(t) = \gamma$ otherwise, and $\{c_j^{(i)}\} = \{X_{\alpha_i}^{(j)}\}$, $\{d_j^{(i)}\} = \{X_{\beta_i}^{(j)}\}$. (Note that with probability one $\omega(t)$ can equal at most one i for each t, so the ω is well-defined.) If $\omega(t) = i$ and $\sum_{j=1}^{n} Y_i^{(j)} = t$ we say ω has an nth mark of the ith color at t, and we say the ordered pair $(X_{\alpha_i}^{(j)}, X_{\beta_i}^{(j)})$ is assigned to this mark. We set $\text{Prob}\{g(A)\} = \text{Prob}\{A\}$, where A is any measurable set in the space of random variables and $g(A)$ is the set of sample functions of assigned values associated with all members of A.

DEFINITION 3. $S_i(t)$ is a function defined on sample functions with assigned values of an L-process such that $S_i(t)(\omega, \{c_j^{(1)}\}, \cdots) = \sum_{j=1}^{n}(c_j^{(i)} + d_j^{(i)})$, where n is such that $\sum_{j=1}^{n} Y_i^{(j)} < t \leqq \sum_{j=1}^{n+1} Y_i^{(j)}$.

THEOREM 1. *With probability one a sample function with assigned values of an L-process will not have $\omega(s) = \gamma$ for all $s \in (t, t')$ for some $t' > t$.*

Proof. For any $t' > t$ we have, due to the independence of the $X_{\alpha_i}^{(j)}, X_{\beta_i}^{(j)}$, and $Y_i^{(j)}$, the probability of no marks of the ith color in (t, t') equaling $e^{-\lambda_i(t'-t)}$, and the probability of no marks at all in (t, t') equaling

$$\prod_{i=1}^{\infty} e^{-\lambda_i(t'-t)} = e^{-\Sigma\lambda_i(t'-t)} = e^{-\infty} = 0.$$

Applying this to t, t' equaling each rational the result follows.

THEOREM 2. *With probability one for any $t > 0$ $\sum_{i=1}^{\infty} S_i(t)(\omega, \{c_j^{(1)}\}, \cdots)$ is finite for any L-process.*

Proof. Let $t > 0$. Since $1/\lambda_i a_i$ equals the expectation of each $c_j^{(i)}$ and $d_j^{(i)}$ and $\lambda_i \Delta s$ equals the probability of an ith colored mark occurring in an interval of length Δs for small Δs, we have

$$E(S_i(t)) = \int_0^t \lambda_i \frac{2}{\lambda_i a_i} ds = \frac{2t}{a_i}.$$

Then

$$\sum_{i=1}^{\infty} E(S_i(t)) = \sum_{i=1}^{\infty} \frac{2t}{a_i} < \infty$$

and hence $E(\sum_{i=1}^{\infty} S_i(t)) < \infty$ by the monotone convergence theorem. Letting t tend to infinity through a denumerable sequence, the result follows.

Consequently an L-process may be converted to a Markov process with one instantaneous state and a denumerable number of stable states $\{\alpha_i\},\{\beta_i\}$, $i = 1,2,3,\cdots$ as follows. Replace each mark occurring in the ω of a sample function with assigned values by two intervals of lengths equal to the two numbers associated with the mark, fixing the point $t = 0$ and shifting the rest of the sample function to the right to make room for the new intervals. More precisely, if a a mark occuring at time t of the original construction has the pair (a,b) associated with it, then on the new axis place an interval of length a with its beginning at time $t + \sum(a_\eta + b_\eta)$, where the sum is taken over all pairs (a_η, b_η) associated with marks occuring before t in the superimposed process. Place an interval of length b such that it begins at time $t + a + \sum(a_\eta + b_\eta)$ with the same sum $\sum(a_\eta + b_\eta)$ as for a. Let the first number of the ordered pairs associated with marks of the ith color represent lengths of the stable state α_i, and the second the stable state β_i. Let the time not taken up by the stable states represent the occurrence of the state γ, which is obviously instantaneous since it lasts for no interval. By the nature of the construction the process is Markovian. Formally, we have

DEFINITION 4. Let $\lambda_i > 0$, $a_i > 0$, $i = 1,2,\cdots, \sum_{i=1}^{\infty} \lambda_i = \infty, \sum_{i=1}^{\infty}(a_i)^{-1} < \infty$. A $\{\{\lambda_i\}; \{a_i\}\}$, or more briefly, a $\{\lambda_i; a_i\}$ M-process is a process defined on $t \geqq 0$ whose sample functions ω are probabilistically constructed as follows. Let $X_{\alpha_i}^{(j)}, X_{\beta_i}^{(j)}$, and $Y_i^{(j)}$ $i,j = 1,2,\cdots$ be independent random variables, with $X_{\alpha_i}^{(j)}$ and $X_{\beta_i}^{(j)}$ having the distribution function $F_{\lambda_i a_i}$, all j, and $Y_i^{(j)}$ having the distribution function F_{λ_i}, all j. $\omega(t) = \alpha_l$ if for some $0 < s < t' < t$

$$\sum_{i=1}^{\infty} \sum_{n=1}^{k(j\ i)} [X_{\alpha_i}^{(n)} + X_{\beta_i}^{(n)}] + s = t'$$

where $k(j,i)$ is the unique integer such that $\sum_{n=1}^{k(j,i)} Y_i^{(n)} \leqq s < \sum_{n=1}^{k(j,i)+1} Y_i^{(n)}$, and $\sum_{n=1}^{k(j,l)} Y_l^{(n)} = s$ and

$$(*) \qquad X_{\alpha_l}^{k(j,l)} \geqq t - t'.$$

(Note that with probability one there will not be integers m, n, p, and q such that

$m \neq n$, and $\sum_{j=1}^{p} Y_m^{(j)} = \sum_{j=1}^{q} Y_n^{(j)} = s$ for some s.) $\omega(t) = \beta_l$ if the above equations hold with $X_{\alpha_l}^{k(j,l)} < t - t' \leqq X_{\alpha_l}^{k(j,l)} + X_{\beta_l}^{k(j,l)}$ replacing (*). $\omega(t) = \gamma$ elsewhere.

The desired process is of this type. The parameters will be chosen so that $\limsup_{t\to 0} p'_{\gamma\gamma}(t) = +\infty$. Before this can be done, however, some preliminary definitions and theorems are needed.

DEFINITION 5. Let $\lambda > 0$, $a > 0$. We shall call the set $(\omega, \{c_i\}, \{d_i\})$ a sample function with assigned values of an S-process if ω is a sample function defined on $t \geqq 0$ taking values γ and 1 such that for all $t > 0$ the set of $s \geqq 0$ such that $\omega(s) = 1$ and $s < t$ consists of at most a finite number of points, and $\{c_i\}$ and $\{d_i\}$ are sequences of positive numbers. A (λ, a) S-process is a probability space of sample functions of assigned values of an S-process with the probability measure determined as follows. Let $X_\alpha^{(j)}$, $X_\beta^{(j)}$, and $Y^{(j)}$, $j = 1, 2, \cdots$ be independent random variables, all $X_\alpha^{(j)}$ and $X_\beta^{(j)}$ having distribution function $F_{\lambda a}$, and all $Y^{(j)}$ having distribution function F_λ. Choose a point of the space of random variables. Associate with this point a sample function with assigned values of an S-process such that $\omega(t) = 1$ if $\sum_{j=1}^{n} Y^{(j)} = t$ for some n, and $\omega(t) = \gamma$ otherwise, and $\{c_i\} = \{X_\alpha^{(i)}\}$, $\{d_i\} = \{X_\beta^{(i)}\}$. We set $\text{Prob}\{g(A)\} = \text{Prob}\{A\}$, where A is any measurable set in the space of random variables and $g(A)$ is the set of sample functions of assigned values associated with all members of A. We shall say the ordered pair $(X_\alpha^{(i)}, X_\beta^{(i)})$ is assigned to the ith mark of ω.

DEFINITION 6. Let $\lambda > 0$, $a > 0$. A (λ, a) T-process is a process defined on $t \geqq 0$ whose sample functions ω are probabilistically constructed as follows. Let $X_\alpha^{(j)}$, $X_\beta^{(j)}$, and $Y^{(j)}$, $j = 1, 2, \cdots$ be independent random variables, with all $X_\alpha^{(j)}$ and $X_\beta^{(j)}$ having distribution function $F_{\lambda a}$ and all $Y^{(j)}$ having distribution function F_λ. $\omega(t) = \gamma$ if for some n

$$\sum_{j=1}^{n-1} (X_\alpha^{(j)} + X_\beta^{(j)} + Y^{(j)}) \leqq t < \sum_{j=1}^{n-1} (X_\alpha^{(j)} + X_\beta^{(j)} + Y^{(j)}) + Y^{(n)}.$$

$\omega(t) = \alpha$ if for some n

$$\sum_{j=1}^{n-1} (X_\alpha^{(j)} + X_\beta^{(j)} + Y^{(j)}) + Y^n \leqq t < \sum_{j=1}^{n-1} (X_\alpha^{(j)} + X_\beta^{(j)} + Y^{(j)}) + X_\alpha^{(n)} + Y^{(n)}.$$

$\omega(t) = \beta$ otherwise. Whenever an M-process with parameters $\{\{\lambda_i\}; \{a_i\}\}$ is being discussed, we shall understand by an S_k-process, for fixed k, a (λ_k, a_k) S-process, and by a T_k-process a (λ_k, a_k) T-process.

Thus a (λ, a) T-process is a three state Markov process with nonabsorbing states γ, α, and β, which starts in γ at $t = 0$ and moves cyclicly in the order $\gamma \to \alpha \to \beta \to \gamma \to \alpha \to \cdots$, etc., and which has $q_\gamma = \lambda$, $q_\alpha = q_\beta = \lambda a$. Note that if the marks of the ith color in a given L-process were expanded to pairs of intervals as in the construction of an M-process, and all other marks were deleted, a (λ_i, a_i) T-process would result. γ retains its original role, and α_i and β_i play the roles of the α and β of the T-process. An S-process is to a T-process what an L-process is to an M-process. Also, an L-process may be considered to be the superposition of a countable number of independent S-processes. Analogously to $S_i(t)$, we have

DEFINITION 7. $S(t)$, $t \geqq 0$, is a function on sample functions with assigned values of an S-process such that $S(t)\ (\omega, \{c_i\}, \{d_i\}) = \sum_{i=1}^{n} (c_i + d_i)$ where n is such that $\sum_{j=1}^{n} Y^{(j)} < t \leqq \sum_{j=1}^{n+1} Y^{(j)}$.

DEFINITION 8. A first process of one of the above types (L, M, S, or T) is said to be finer than a second of the same type if each parameter of the first is at least as large as the corresponding parameter of the second (e.g. in an L- or M-process $\lambda_i^{(1)} \geqq \lambda_i^{(2)}$ and $a_i^{(1)} \geqq a_i^{(2)}$ for all i, the superscripts referring to the first and second processes, respectively.) A parameter λ, (a), is said to be finer than another parameter λ', (a'), if $\lambda \geqq \lambda'$, $(a \geqq a')$.

DEFINITION 9. An L-process (M-process) with parameters $\{\lambda_i; a_i\}$ is said to be n-finer than a second L-process (M-process) with parameters $\{\lambda_i'; a_i'\}$ if the first process is finer than the second and if $\lambda_i = \lambda_i'$ and $a_i = a_i'$ for all $i \leqq n$.

Since M- and T-processes are stationary Markov processes, there exist transition probability functions between the states of either of these processes, and these functions will be used in Definitions 10 and 14.

DEFINITION 10. Let $t' > t > 0$. We shall say $\mathrm{DQ}(t', t) \geqq K$, for a T-process or M-process if $(p_{\gamma\gamma}(t') - p_{\gamma\gamma}(t))/(t' - t) \geqq K$. (DQ $\equiv$ difference quotient.)

DEFINITION 11. We shall say $\mathrm{DQ}(t', t) \geqq K$, n-fixed, for an M-process if all n-finer M-processes have $\mathrm{DQ}(t', t) \geqq K$.

DEFINITION 12. Let $\eta > 0$. We shall say $\mathrm{DQ}(t', t) \geqq K$, (ε, η) protected, for a T-process if

$$\frac{\mathrm{Prob}\{\omega(s) = \gamma \text{ for all } s \in [t' - \eta, t']\}}{t' - t} - \frac{\mathrm{Prob}\{\omega(s) = \gamma \text{ for some } s \in [t - \eta, t]\} + \varepsilon}{t' - t} \geqq K.$$

(If $\eta > t$ then $t - \eta$ should be replaced by 0.)

DEFINITION 13. An M-process is pre(ε, n, t) free if $n = 1$ or if

$$\mathrm{Prob}\left\{\sum_{i=1}^{n-1} S_i(t) > 0\right\} < \frac{\varepsilon}{4}.$$

DEFINITION 14. An M-process is post $(\varepsilon, n, t, \eta)$ free if

$$\mathrm{Prob}\left\{\sum_{i=n+1}^{\infty} S_i(t) > \eta\right\} < \frac{\varepsilon}{8},$$

and

$$\sum_{i=n+1}^{\infty} [p_{\gamma\alpha_i}(t') + p_{\gamma\beta_i}(t')] < \frac{\varepsilon}{8}$$

for all t', and if these two equations also hold for all n-finer M-processes.

The essential result allowing the desired process to be constructed is Lemma 1 of Theorem 3, giving the existence of a t for certain T-processes such that $p'_{\gamma\gamma}(t) > 0$. Lemmas 2 and 3 of the same theorem allow changes in the T-process so that there exists $p'_{\gamma\gamma}(t)$ as large as desired, for some t. For any integer n, an M-process would be a T_n-process if it were not for the interference of intervals α_i and β_i, $i \neq n$. Specifically, it could be a T-process having a large positive $p'_{\gamma\gamma}(t)$. Although this interference exists, it can be made sufficiently small so that the desired result is still obtainable. Interference coming from those intervals with $i < n$ is made sufficiently small by making it pre(ε, n, t') free, and interference coming from intervals with $i > n$ is made small by making it post $(\varepsilon, n, t', \eta)$ free. Then the T_n-process predominates so that $p'_{\gamma\gamma}(t)$ is sufficiently large for a suitable t. This T_n-process predomination construction will be done in turn for $n = 1, 2, 3, \cdots$ without subsequent constructions altering results already obtained. All steps will then be considered simultaneously, giving a well-defined process having the property $\limsup_{t \to 0} p'_{\gamma\gamma}(t) = +\infty$.

The following theorems concern the decreasing of interference in a T_n-process.

THEOREM 3. *Let an M-process have parameters $\{\lambda_i; a_i\}$, let $T > 0$, $K > 0$ and let n be a positive integer. There exist parameters λ'_n and a'_n finer than λ_n and a_n, respectively, $\varepsilon > 0$, $\eta > 0$, t, and t' such that $T > t' > t > 0$, $\mathrm{DQ}(t', t) \geqq K$, (ε, η) protected for the (λ'_n, a'_n) T-process, and such that the M-process which results from replacing λ_n and a_n by λ'_n and a'_n, respectively is* pre(ε, n, t') *free.*

This theorem will be proved by first proving six lemmas.

LEMMA 1. *Let the parameters (λ, a) be given. There exist parameters (λ, a') and $t > 0$ such that $a' \geqq a$ and $p'_{\gamma\gamma}(t) > 0$ for the (λ, a') T-process.*

Proof. Employing the forward differential equation

$$p'_{ij}(t) = -p_{ij}(t)q_j + \sum_{k \neq j} p_{ik}(t)q_{kj}$$

for Markov transition probability functions, we have

$$p'_{\gamma\gamma}(t) = -p_{\gamma\gamma}(t)\lambda + p_{\gamma\beta}(t)\lambda a'$$

$$p'_{\gamma\alpha}(t) = -p_{\gamma\alpha}(t)\lambda a' + p_{\gamma\gamma}(t)\lambda$$

$$p'_{\gamma\beta}(t) = -p_{\gamma\beta}(t)\lambda a' + p_{\gamma\alpha}(t)\lambda a'.$$

By substitution we obtain

$$p'''_{\gamma\gamma}(t) + (2\lambda a' + \lambda)p''_{\gamma\gamma}(t) + (2\lambda^2 a' + \lambda^2 a'^2)p'_{\gamma\gamma}(t) = 0$$

solving this for $p'_{\gamma\gamma}(t)$, using the condition that $p'_{\gamma\gamma}(t)$ be real and $p'_{\gamma\gamma}(0) = -\lambda$, we obtain

$$p'_{\gamma\gamma}(t) = -\frac{\lambda}{2}e^{-(\lambda(2a'+1)/2)t}\left[e^{(\lambda(\sqrt{(1-4a')})/2)t} + e^{-(\lambda(\sqrt{(1-4a')}/2)t}\right].$$

For $a' > \frac{1}{4}$ the expression in brackets will be the sum of two complex conjugate circular functions and hence for any such a' there will be a $t > 0$ for which their sum is negative and hence for which $p'_{\gamma\gamma}(t)$ is positive.

From the definition of derivative it is obvious that there exist $t' > t > 0$ for the (λ, a') T-process for which $\mathrm{DQ}(t', t) \geqq d$ for some $d > 0$.

LEMMA 2. *If $p_{\gamma\gamma}(t)$ and $\bar{p}_{\gamma\gamma}(t)$ are the transition probability functions for (λ, a) and $(\bar{\lambda}, a)$ T-processes, respectively, then $p_{\gamma\gamma}(t/\lambda) = \bar{p}_{\gamma\gamma}(t/\bar{\lambda})$ for all t.*

Proof. λ and $\bar{\lambda}$ appear in the definitions of the T-processes merely as time-scale parameters such that the above equation is true for all t.

LEMMA 3. *If $\mathrm{DQ}(t',t) \geqq d \geqq 0$ for a (λ, a) T-process, then $\mathrm{DQ}(\lambda t'/\bar{\lambda}, \lambda t/\bar{\lambda}) \geqq \bar{\lambda}d/\lambda$ for a $(\bar{\lambda}, a)$ T-process.*

Proof. This follows from Lemma 2 and consideration of the difference quotient.

LEMMA 4. *If $\mathrm{DQ}(t',t) \geqq K + \delta$ for a T-process and some $\delta > 0$, then there is an $\varepsilon > 0$ and $\eta > 0$ such that $\mathrm{DQ}(t',t) \geqq K$, (ε, η) protected.*

Proof. There exists an $\eta > 0$ such that for the T-process

$$p_{\gamma\gamma}(t') - \mathrm{Prob}\{\omega(s) = \gamma \text{ for all } s \in [t' - \eta, t']\} \leqq \frac{\delta}{3}(t' - t)$$

and

$$\mathrm{Prob}\{\omega(s) = \gamma \text{ for some } s \in [t - \eta, t]\} - p_{\gamma\gamma}(t) \leqq \frac{\delta}{3}(t' - t).$$

Choosing $\varepsilon = \delta(t' - t)/3$, the quotient in Definition 11 will then be at least

$$\frac{p_{\gamma\gamma}(t') - \frac{\delta}{3}(t'-t) - p_{\gamma\gamma}(t) - \frac{\delta}{3}(t'-t) - \frac{\delta}{3}(t'-t)}{t'-t}$$

$$= \frac{p_{\gamma\gamma}(t') - p_{\gamma\gamma}(t)}{t'-t} - \delta \geqq K.$$

LEMMA 5. *If $\mathrm{DQ}(t', t) \geqq K$, (ε, η) protected for a (λ, a) T-process, where $K > 0$, then for any $\bar{\lambda} \geqq \lambda$ $\mathrm{DQ}(\lambda t'/\bar{\lambda}, \lambda t/\bar{\lambda}) \geqq K$, $(\varepsilon, \lambda\eta/\bar{\lambda})$ protected for a $(\bar{\lambda}, a)$ T-process.*

Proof. Letting ω and $\bar{\omega}$ be sample functions for the respective T-processes,

$$\left(\frac{\lambda t'}{\bar\lambda}-\frac{\lambda t}{\bar\lambda}\right)^{-1}\left(\text{Prob}\left\{\bar\omega(s)=\gamma \text{ for all } s\in\left(\frac{\lambda t'}{\bar\lambda}-\frac{\lambda\eta}{\bar\lambda},\frac{\lambda t'}{\bar\lambda}\right)\right\}\right.$$

$$\left.-\text{Prob}\left\{\bar\omega(s)=\gamma \text{ for some } s\in\left(\frac{\lambda t}{\bar\lambda}-\frac{\lambda\eta}{\bar\lambda},\frac{\lambda t}{\bar\lambda}\right)\right\}-\varepsilon\right)$$

$$=\left(\frac{\lambda t'}{\bar\lambda}-\frac{\lambda t}{\bar\lambda}\right)^{-1}(\text{Prob}\{\omega(s)=\gamma \text{ for all } s\in(t'-\eta,t')\}$$

$$-\text{Prob}\{\omega(s)=\gamma \text{ for some } s\in(t-\eta,t)\}-\varepsilon)\geqq K$$

the last inequality resulting from $\bar\lambda \geqq \lambda$ and $\mathrm{DQ}(t',t) \geqq K$, (ε,η) protected.

LEMMA 6. *Let an M-process have parameters $\{\lambda_i;a_i\}$, let $\varepsilon>0$, and n be a positive integer. There exists a $t>0$ such that the M-process is* pre (ε,n,t) *free.*

Proof. If $n=1$ the lemma is trivial. If not, let $t_i>0$ be such that the probability of a mark of the ith color occuring in $(0,t_i)$ in a $\{\lambda_i;a_i\}$ L-process is less than $\varepsilon/4(n-1)$. Then let $t=\min_{i=1,2,\ldots,n-1} t_i$.

The theorem follows by first increasing a_n to a_n' so that $p_{\gamma\gamma}'(t)>0$ in the T-process for some t by Lemma 1, then increasing λ_n so that $\mathrm{DQ}(t',t)\geqq K$, (ε,η) protected for another t, t', and some $\varepsilon>0$, $\eta>0$ by the remark before Lemma 2, and Lemma 4, and finally by increasing λ_n more by Lemmas 5 and 6 so that the M-process is also pre(ε,n,t') free for the t' in the difference quotient.

THEOREM 4. *Let an M-process have parameters $\{\lambda_i;a_i\}$, let $K>0$, $\varepsilon>0$, $\eta>0, t>0$, and let n be a positive integer. There exists an n-finer M-process which is* post (ε,n,t,η) *free.*

This theorem will be proved by first proving four lemmas.

LEMMA 1. *Let $\eta>0$, $\varepsilon>0$, $t>0$. There exists a (λ,a) T-process such that for all finer T-processes S(t) of the related S-process with the same parameters will be less than η with probability at least* $1-\varepsilon$.

Proof. Choosing a pair of parameters (λ,a) we see the expected length of γ intervals is $1/\lambda$. By the strong law of large numbers there exists N_1 such that with probability at least $1-\varepsilon/3$ the average length of the first n γ intervals will be greater than $1/2\lambda$ for all $n>N_1$ and λ. Since the expected length of α and β intervals is $1/\lambda a$, there exists an N_2 such that except for probability at most $\varepsilon/3$ the average length of the pairs of α and β intervals, adding the length of an α interval to the length of the β interval following it, is less than $3/\lambda a$ for all $n>N_2$, λ, and a. There exists a $\lambda>0$ such that except for probability at most $\varepsilon/3$ there will be at least $\max(N_1,N_2)$ marks appearing before t in any S-process with parameters at least as fine as (λ,a), for any a. Keep this λ fixed and choose $a>6t/\eta$. By the above, except for a set of probability at most ε there will be at most $t(1/2\lambda)^{-1}$ marks in $(0, t)$, and $S(t)$ will be at most $2\lambda t(3/\lambda a) < \eta$. By the nature of the proof it is apparant that all finer T-processes will also have this property.

LEMMA 2. *Let $\varepsilon > 0$, $\eta > 0$, $t > 0$, and let n be a positive integer. There exists an M-process with parameters $\{\lambda_i; a_i\}$ such that for all finer processes* $\mathrm{Prob}\{\sum_{i=n+1}^{\infty} S_i(t) > \eta\} < \varepsilon/8$.

Proof. By Lemma 1 make (λ_i, a_i), $i \geqq n+1$, fine enough so

$$\mathrm{Prob}\left\{ S_i(t) > \frac{\eta}{2^{n+2-i}} \right\} < \frac{\varepsilon}{8 \times 2^{n+2-i}}$$

and by addition Lemma 2 follows.

LEMMA 3. *Let $\varepsilon > 0$, $t > 0$. There exist parameters (λ, a) such that for all T-processes at least that fine* $\mathrm{Prob}\{\omega(t) = \alpha$ *or* $\beta\} < \varepsilon$.

Proof. Suppose $\omega(t') = \alpha$. There exists a $c_1 > 0$ such that the probability that ω remains entirely in α during $(t', t' + c_1/2\lambda a)$ is less than $\varepsilon/4$, whatever be λ and a. Suppose $\omega(t') = \gamma$. There exists a $c_2 > 0$ such that the probability that ω remains in γ during $(t', t' + c_2/\lambda)$ is at least $1 - \varepsilon/4$, whatever be λ. Choose $a^* > c_1/c_2$. Then $c_2/\lambda > c_1/\lambda a^*$ for all λ. The parameters (λ, a^*) meet the requirements of the lemma, for all $\lambda > 0$. For first suppose $t - c_2/\lambda > 0$. Then set $t' = t - c_2/\lambda$. For $\omega(t)$ to equal α, either $\omega(t') = \alpha$ and ω remains in α during (t', t), an event of probability less than $\varepsilon/4$, or $\omega(t'') = \gamma$ for some t'' in (t', t), and then ω enters α before t, an event of probability less than $\varepsilon/4$. Thus $\omega(t) = \alpha$ is an event of probability less than $\varepsilon/2$. If $t \leqq c_2/\lambda$, then the event $\omega(t) = \alpha$ has probability less than $\varepsilon/4$, hence less than $\varepsilon/2$. This same argument can be applied to show for any t $\mathrm{Prob}\{\omega(t) = \beta\} < \varepsilon/2$, and hence the lemma follows.

LEMMA 4. *Let $\varepsilon > 0$ and n be a positive integer. There exists an M-process with parameters $\{\lambda_i; a_i\}$ such that for all finer M-processes, and all $t > 0$,* $\mathrm{Prob}\{\omega(t) = \alpha_i$ *or* β_i *for any* $i \geqq n+1\} < \varepsilon/8$.

Proof. By Lemma 3 choose (λ_i, a_i), $i \geqq n+1$, so fine that $\mathrm{Prob}\{\omega(t) = \alpha_i$ or $\beta_i\}$ $< \varepsilon/8 \times 2^{n+2-i}$ for the (λ_i, a_i) T-process. Then this equation will also hold in the resulting M-process, as can be seen by the independence of the S_i processes and considering the point $t - c_2^{(i)}/\lambda_i$ and proceeding as in the proof of Lemma 3.

Lemmas 2 and 4 give the theorem by definition.

THEOREM 5. *Let an M-process have parameters $\{\lambda_i; a_i\}$, let $K > 0$, $\varepsilon > 0$, $\eta > 0$, $t' > t > 0$, and let n be a positive integer. If the (λ_n, a_n) T-process has* $\mathrm{DQ}(t', t) \geqq K$, *$(\varepsilon, \eta)$ protected and if the M-process is* $\mathrm{pre}(\varepsilon, n, t')$ *free and* $\mathrm{post}\,(\varepsilon, n, t', \eta)$ *free, then* $\mathrm{DQ}(t', t) \geqq K$, *$n$-fixed.*

Proof. Consider any n-finer M-process. Letting $\bar{p}$ and $\bar{\omega}$ be transition probability and sample functions for this process, and p and ω those for the (λ_n, a_n) T-process, we have

$$\begin{aligned} p_{\gamma\gamma}(t') - \bar{p}_{\gamma\gamma}(t') \leqq & \operatorname{Prob}\{\omega(t') = \gamma \text{ and } \omega(s) \neq \gamma \text{ for some } s \in (t'-\eta, t')\} \\ & + \operatorname{Prob}\left\{ \sum_{i=n+1}^{\infty} S_i(t') > \eta \right\} + \sum_{i=n+1}^{\infty} [\bar{p}_{\gamma\alpha_i}(t') + \bar{p}_{\gamma\beta_i}(t')] \\ & + \operatorname{Prob}\left\{ \sum_{i=1}^{n-1} S_i(t') > 0 \right\} \\ \leqq & \operatorname{Prob}\{\omega(t') = \gamma \text{ and } \omega(s) \neq \gamma \text{ for some } s \in (t'-\eta, t')\} + \frac{\varepsilon}{2}. \end{aligned}$$

The various interference effects are brought to light in the above inequalities. The terms to the right of the first inequality are present because for an ω sample function with $\omega(t') = \gamma$ to be changed to an $\bar{\omega}$ sample function with $\bar{\omega}(t') \neq \gamma$ by interference with intervals with subscripts $i > n$, either $\omega(s)$ must not equal γ everywhere in the interval $(t'-\eta, t')$ and it must have this point pushed forward to t' by α_i and β_i intervals $(i > n)$ of total length less than η, or it must have the sum of the α_i and β_i intervals $(i > n)$ greater than η to push an even farther back point s with $\omega(s) \neq \gamma$ up to t', or an α_i or β_i must itself cover t'. The last term represents interference from the ith coordinates with $i < n$.

From the above inequality

$$\bar{p}_{\gamma\gamma}(t') \geqq \operatorname{Prob}\{\omega(s) = \gamma \text{ for all } s \in (t'-\eta, t')\} - \frac{\varepsilon}{2}.$$

Also

$$\begin{aligned} \bar{p}_{\gamma\gamma}(t) - p_{\gamma\gamma}(t) \leqq & \operatorname{Prob}\{\omega(t) \neq \gamma \text{ and } \omega(s) = \gamma \text{ for some } s \in (t-\eta, t)\} \\ & + \operatorname{Prob}\left\{ \sum_{i=n+1}^{\infty} S_i(t) > \eta \right\} + \operatorname{Prob}\left\{ \sum_{i=1}^{n-1} S_i(t) > 0 \right\} \\ \leqq & \operatorname{Prob}\{\omega(t) \neq \gamma \text{ and } \omega(s) = \gamma \text{ for some } s \in (t-\eta, t)\} + \frac{\varepsilon}{2}. \end{aligned}$$

Hence

$$\bar{p}_{\gamma\gamma}(t) \leqq \operatorname{Prob}\{\omega(s) = \gamma \text{ for some } s \in (t-\eta, t)\} + \frac{\varepsilon}{2}.$$

The theorem follows from Definition 11 applied to the T-process and the above two inequalities.

The final construction is done by first defining a particular sequence $\{M_i\}$ of M-processes and sequence $\{(t_i, t_i')\}$ of pairs of points, with certain properties.

Choose a set of parameters $\{\lambda_i; a_i\}$ such that $\lambda_i > 0$, $a_i > 0$, $\sum_{i=1}^{\infty} \lambda_i = \infty$, and $\sum_{i=1}^{\infty} (a_i)^{-1} < \infty$. By Theorem 3 there exist (λ_1^*, a_1^*) finer than (λ_1, a_1) such that the (λ_1^*, a_1^*) T-process has $\mathrm{DQ}(t_1', t_1) \geqq 1, (\varepsilon_1, \eta_1)$ protected for some

$t_1' > t_1 > 0$, $\varepsilon_1 > 0$, and $\eta_1 > 0$. By Theorem 4 there exist (λ^*, a_i^*) finer than (λ_i, a_i), $i \geqq 2$, such that the M-process with parameters $\{\lambda_i^*; a_i^*\}$ is post $(\varepsilon, 1, t_1')$ free. The M_1-process is defined by the parameters $\{\lambda_i^*; a_i^*\}$. It will have $\mathrm{DQ}(t_1', t_1) \geqq 1$ 1-fixed by Theorem 5.

M_i and (t_i, t_i'), $i \geqq 2$, are defined inductively. Let the parameters of the M_{k-1}-process, $k \geqq 2$, be $\{\lambda_i; a_i\}$. By Theorem 3 there exist (λ_k^*, a_k^*) finer than (λ_k, a_k) such that the (λ_k^*, a_k^*) T-process has $\mathrm{DQ}(t_k', t_k) \geqq k$, (ε_k, η_k) protected for some $\varepsilon_k > 0$, $\eta_k > 0$, $\varepsilon_{k-1}/2 > t_k' > t_k > 0$, and so that the M-process with parameters $\{\lambda_i'; a_i'\}$, $\lambda_i' = \lambda_i$ and $a_i' = a_i$, $i \neq k$, $\lambda_k' = \lambda_k^*$, $a_k' = a_k^*$, is $\mathrm{pre}(\varepsilon_k, k, t_k')$ free. By Theorem 4 there exist (λ_i^*, a_i^*), $i > k$, such that the M-process with parameters $\{\lambda_i^*; a_i^*\}$, with $\lambda_i^* = \lambda_i$ and $a_i^* = a_i$, $i < k$, will be post $(\varepsilon_k, k, t_k', \eta_k)$ free. The M_k-process is defined by the parameters $\{\lambda_i^*; a_i^*\}$ and is $\mathrm{DQ}(t_k', t_k) \geqq k$, k-fixed be Theorem 5.

Now set $(\lambda_i^{(i)}, a_i^{(i)})$ equal to the ith parameters of the M_i-process, for all i. By the nature of the M_i, it is obvious that the parameters $\{\lambda_i^{(i)}; a_i^{(i)}\}$ define an M-process, and the M-process so defined is the desired one. For it has

$$\frac{p_{\gamma\gamma}(t_i') - p_{\gamma\gamma}(t_i)}{t_i' - t_i} \geqq i,$$

and therefore by the mean value theorem there is a sequence of points $\{t_i^*\}$ tending to zero such that $p_{\gamma\gamma}'(t_n^*) \geqq n$. Hence $\limsup_{t \to 0} p_{\gamma\gamma}'(t) = +\infty$.

References

1. K. L. Chung, *Markov chains with stationary transition probabilities*, Springer-Verlag, Berlin, 1960.

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