

COORDINATE SYSTEMS OF SOME SEMI-TRANSLATION PLANES⁽¹⁾

BY

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In [6], one of the authors gave a construction for a new class of finite affine planes. These new planes are all of order q^2 (where q is a prime power) and each has the property of admitting a translation group of order q^2 which is transitive on the points of a subplane of order q . At least one other class of planes (the Hughes planes) has this same property. Hence the term "semi-translation plane" was introduced. (See [7] for precise definitions.)

Our main purpose is to investigate circumstances under which the planes in [6] may (1) be self dual or (2) admit collineations displacing the line at infinity. A properly chosen coordinate system must be what we call "automorphic" (see the definition immediately preceding Corollary 1) if either (1) or (2) is to hold. We are thus led to the more general question of automorphic algebraic systems and the planes coordinatised by them.

In Theorem 2, multiplication for automorphic algebras is given a relatively explicit form. In Theorem 5, we show that there are severe restrictions on the associativity of multiplication. In Theorem 7, we show that the right distributive law cannot hold and, in Theorem 8, that the corresponding semi-translation planes are "strict," characteristic two being an exception.

The dual of a semi-translation plane coordinatised by an automorphic system is also a semi-translation plane coordinated by an automorphic system. The relation between the two coordinate systems is given quite simply in Lemma 1.

Our last theorem does not deal directly with automorphic coordinate systems as such. It shows that, in Ostrom's construction, strict semi-translation planes derived from nonisomorphic planes must themselves be nonisomorphic.

I. Summary of background information. For further details and definitions, see [7].

Whenever we use small Greek letters (excepting ρ, σ, τ) in this paper, it is to be understood that we are dealing with an algebraic system \mathfrak{L} of order q^2 which contains a subfield \mathfrak{F} of order q . The small Greek letters are to be understood as denoting elements of \mathfrak{F} . In all cases, the elements of \mathfrak{L} constitute an abelian group under the operation of addition.

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The following remarks pertain to the known semi-translation planes and their coordinate systems, provided that the coordinate system is appropriately chosen:

Lines are represented by equations of the form $y = (x - \alpha)m + \beta$, $y = x\gamma + b$ and $x = c$. If the plane is self dual or admits a collineation displacing the line at infinity, then the following "partial left distributive law" holds: $a(b + \alpha) = ab + \alpha a$ for all $a, b \in \mathfrak{T}$ and all $\alpha \in \mathfrak{F}$.

Those planes that are strict semi-translation planes can be obtained from other planes related to them in the following way:

Let π be an affine plane of order q^2 and let \mathfrak{M} be a set of $q + 1$ points on the line at infinity. Suppose that every two affine points of π which are collinear with a point of \mathfrak{M} can be embedded in an affine subplane of order q whose extension to a projective plane contains \mathfrak{M} . Then there is an affine plane π' whose affine points are the affine points of π . The affine lines of π' are (1) the lines of π which do not intersect \mathfrak{M} and (2) the proper subplanes of π which contain \mathfrak{M} .

In this case π' is said to be derived from π . The process of deriving is of order two, i.e., π is also derived from π' . Each collineation of π which carries \mathfrak{M} into itself is also a collineation of π' (in the sense that a collineation may be considered to be a permutation of the points which carries lines into lines). If all of the affine points of some line intersecting \mathfrak{M} in π are in a single transitive class under the translation group of π , then π' is a semi-translation plane.

The original plane π is coordinatised by a system \mathfrak{T} which is a right vector space of dimension two over \mathfrak{F} . \mathfrak{M} is the set of points at infinity of the subplane coordinatised by \mathfrak{F} . Using 1, t as basis elements, elements of \mathfrak{T} may be written in the form $t\alpha + \beta$. Let \mathfrak{T}' be the coordinate system for π' . The elements of \mathfrak{T}' are the same as the elements of \mathfrak{T} . Addition is the same in both systems; \mathfrak{T}' is a right vector space over \mathfrak{F} . If the point (x, y) of π has coordinates $(t\xi_1 + \xi_2, t\eta_1 + \eta_2)$ in \mathfrak{T} , then this same point has coordinates $(x', y') = (t\xi_1 + \eta_1, t\xi_2 + \eta_2)$ in \mathfrak{T}' . Let \circ denote multiplication in \mathfrak{T}' . For $\lambda_1 \neq 0$, $(t\xi_1 + \eta_1) \circ (t\lambda_1 + \lambda_2) = t\xi_2 + \eta_2$ is equivalent to $(t\xi_1 + \xi_2)(t\mu_1 + \mu_2) = t\eta_1 + \eta_2$ in \mathfrak{T} , where $\lambda_1(t\mu_1 + \mu_2) = t + \lambda_2$. If the partial left distributive law holds in $\mathfrak{T}(\mathfrak{T}')$, then $\mathfrak{T}'(\mathfrak{T})$ admits a group of automorphisms of order q such that \mathfrak{F} is elementwise fixed. If \mathfrak{T} and \mathfrak{T}' both admit the partial distributive law, then both \mathfrak{T} and \mathfrak{T}' are "automorphic," as defined in §II. The relation between \mathfrak{T} and \mathfrak{T}' is given in [7], but was originally developed by Albert [1].

In the case of the planes constructed by Ostrom [6], π is a dual translation plane and π' is a semi-translation plane. If π' is self dual or admits collineations moving the line at infinity, the coordinate systems can be chosen so that \mathfrak{T} and \mathfrak{T}' are automorphic.

II. Automorphic algebras. In this part, the argument makes no direct use of the fact that the systems under consideration are coordinate systems for planes.

Let \mathfrak{T} be a set of q^2 elements (where q is a power of a prime p) with the operations of addition and multiplication defined in \mathfrak{T} that:

- (1) Addition is an abelian group.
- (2) Nonzero elements of \mathfrak{T} form a loop under multiplication.
- (3) \mathfrak{T} contains a subsystem \mathfrak{F} which is a field of order q with respect to addition and multiplication.
- (4) The additive group of \mathfrak{T} is a right vector space of dimension two over \mathfrak{F} with respect to multiplication on the right by elements of \mathfrak{F} .
- (5) $a(b + \alpha) = ab + a\alpha$ for all $a, b \in \mathfrak{T}$ and all $\alpha \in \mathfrak{F}$.

In the rest of this paper we shall consider systems which satisfy properties (1)–(5). We shall take $1, t$ as basis elements so that every element of \mathfrak{T} may be written in the form $t\alpha + \beta$.

THEOREM 1. *Under the conditions (1)–(5), let σ be an automorphism of addition and multiplication in \mathfrak{T} such that (i) each element of \mathfrak{F} is fixed by σ , (ii) the order of σ divides q , (iii) $t\sigma = t\alpha + \beta$. Then $\alpha = 1$.*

Proof. If $t\sigma = t\alpha + \beta$ and σ is of order k , then by induction $t = t\sigma^k = t\alpha^k + (\alpha^{k-1} + \dots + \alpha + 1)\beta$. Hence, $\alpha^k = 1$, and since k is a power of the prime p , we have $\alpha = 1$.

DEFINITION. We shall say that a system \mathfrak{T} satisfying conditions (1)–(5) is automorphic if \mathfrak{T} admits a group Σ of automorphisms such that (i) Σ is of order q ; and (ii) each element of \mathfrak{F} is left fixed by each element of Σ .

COROLLARY 1. *\mathfrak{T} is automorphic if and only if for each $\delta \in \mathfrak{F}$ there is an automorphism σ in Σ such that $t\sigma = t + \delta$.*

THEOREM 2. *If \mathfrak{T} is automorphic and if $(t\alpha_1)(t\alpha_2) = th(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2)$ where $\alpha_1 \neq 0$, $h(\alpha_1, \alpha_2)$ and $k(\alpha_1, \alpha_2)$ are in \mathfrak{F} , then*

$$\begin{aligned} (t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2) \\ = t[h(\alpha_1, \alpha_2) - \beta_1\alpha_2 + \alpha_1\beta_2] + \beta_1\alpha_1^{-1}h(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) \\ - \beta_1^2\alpha_1^{-1}\alpha_2 + \beta_1\beta_2. \end{aligned}$$

Proof. By the partial distributive law we have

$$\begin{aligned} (t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2) &= (t\alpha_1 + \beta_1)t\alpha_2 + (t\alpha_1 + \beta_1)\beta_2 \\ &= (t\alpha_1 + \beta_1)t\alpha_2 + t\alpha_1\beta_2 + \beta_1\beta_2. \end{aligned}$$

Now let us consider the automorphism σ in Σ such that $t\sigma = t - \alpha_1^{-1}\beta_1$.

$$\begin{aligned} [(t\alpha_1 + \beta_1)(t\alpha_2)]\sigma &= (t\alpha_1)(t\alpha_2 - \alpha_1^{-1}\beta_1\alpha_2) \\ &\equiv (t\alpha_1)(t\alpha_2) - t\beta_1\alpha_2 \\ &= t[h(\alpha_1, \alpha_2) - \beta_1\alpha_2] + k(\alpha_1, \alpha_2). \end{aligned}$$

Now $t\sigma^{-1} = t + \alpha_1^{-1}\beta_1$. Hence

$$\begin{aligned} (t\alpha_1 + \beta_1)(t\alpha_2) &= [(t\alpha_1 + \beta_1)t\alpha_2]\sigma\sigma^{-1} \\ &= \{t[h(\alpha_1, \alpha_2) - \beta_1\alpha_2] + k(\alpha_1, \alpha_2)\}\sigma^{-1} \\ &= (t + \alpha_1^{-1}\beta_1)[h(\alpha_1, \alpha_2) - \alpha_2\beta_1] + k(\alpha_1, \alpha_2) \\ &= t[h(\alpha_1, \alpha_2) - \alpha_2\beta_1] + \beta_1\alpha_1^{-1}h(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) - \beta_1^2\alpha_1^{-1}\alpha_2. \end{aligned}$$

Hence

$$\begin{aligned} (t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2) &= t[h(\alpha_1, \alpha_2) - \alpha_2\beta_1 + \alpha_1\beta_2] + \beta_1\alpha_1^{-1}h(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) - \beta_1^2\alpha_1^{-1}\alpha_2 + \beta_1\beta_2. \end{aligned}$$

THEOREM 3. Let $\gamma(t\alpha + \beta) = tP(\gamma, \alpha) + \beta\gamma + R(\gamma, \alpha)$ where $P(\gamma, \alpha)$ and $R(\gamma, \alpha)$ are in \mathfrak{F} , then under the hypothesis of Theorem 2, $P(\gamma, \alpha) = \alpha\gamma$.

Proof. Consider $\gamma(t\alpha + \beta) = tP(\gamma, \alpha) + \beta\gamma + R(\gamma, \alpha)$ where $\beta \neq 0$ and consider the automorphism σ such that $t\sigma = t - \alpha^{-1}\beta$. Apply σ and we have $[\gamma(t\alpha + \beta)]\sigma = \gamma t\alpha = tP(\gamma, \alpha) + R(\gamma, \alpha)$. On the other hand, $\gamma(t\alpha + \beta) = tP(\gamma, \alpha) + \beta\gamma + R(\gamma, \alpha)$. Hence,

$$[\gamma(t\alpha + \beta)]\sigma = tP(\gamma, \alpha) - \alpha^{-1}\beta P(\gamma, \alpha) + \beta\gamma + R(\gamma, \alpha),$$

and on equating the two expressions for $[\gamma(t\alpha + \beta)]\sigma$, we get that $P(\gamma, \alpha) = \alpha\gamma$.

COROLLARY. $\gamma(t\alpha + \beta) = t\alpha\gamma + \beta\gamma + R(\gamma, \alpha)$.

THEOREM 4. The full left distributive law holds in \mathfrak{L} if and only if for each fixed $\alpha_1 \neq 0$, $h(\alpha_1, \alpha_2)$, and $k(\alpha_1, \alpha_2)$ are additive on α_2 .

Proof. By calculation.

THEOREM 5. If \mathfrak{L} is automorphic and if the elements of \mathfrak{F} associate and commute with elements of \mathfrak{L} , then $q \leq 3$.

Proof. Let $t^2 = t\bar{\alpha} + \bar{\beta}$. Under the hypotheses, $(t\alpha_1)(t\alpha_2) = t\bar{\alpha}\alpha_1\alpha_2 + \bar{\beta}\alpha_1\alpha_2$. Thus, $h(\alpha_1, \alpha_2) = \bar{\alpha}\alpha_1\alpha_2$ and $k(\alpha_1, \alpha_2) = \bar{\beta}\alpha_1\alpha_2$.

Here we make use of the condition that multiplication must be a loop. Consider the equation

$$(*) \quad (t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2) = (t\gamma_1 + \gamma_2).$$

Let us fix $t\alpha_2 + \beta_2$ with $\alpha_2 \neq 0$. Then each $t\gamma_1 + \gamma_2$ must determine a unique $t\alpha_1 + \beta_1$ such that $(*)$ is satisfied. We shall see, however, that $t\gamma_1 + \gamma_2$ can determine $t\alpha_1 + \beta_1$ uniquely only in a limited number of cases.

First of all, there are at most q choices of $t\gamma_1 + \gamma_2$ which can lead to $\alpha_1 = 0$, corresponding to the q possible values for β_1 . If $\alpha_1 \neq 0$, $(*)$ is equivalent to the pair of equations

$$\begin{aligned}\bar{\alpha}\alpha_1\alpha_2 - \alpha_2\beta_1 + \alpha_1\beta_2 &= \gamma_1, \\ \beta_1\bar{\alpha}\alpha_2 + \bar{\beta}\alpha_1\alpha_2 - \beta_1^2\alpha_1^{-1}\alpha_2 + \beta_1\beta_2 &= \gamma_2.\end{aligned}$$

Eliminating β_1 , we get the quadratic in α_1 :

$$(\dagger) \quad \alpha_1^2\bar{\beta}\alpha_2 + \alpha_1(\bar{\alpha}\gamma_1 + \beta_2\alpha_2^{-1}\gamma_1 - \gamma_2) - \gamma_1^2\alpha_2^{-1} = 0.$$

Now (\dagger) is (partially) equivalent to $(*)$ in the sense that, for given $t\gamma_1 + \gamma_2$, each nonzero solution of (\dagger) for α_1 leads to a determination of β_1 , such that $t\alpha_1 + \beta_1$ is a solution of $(*)$.

There are, at most, three ways in which $t\alpha_1 + \beta_1$ can be uniquely determined:

- (1) $\alpha_1 = 0$ in $(*)$ and (\dagger) does not apply.
- (2) (\dagger) has two solutions but one of them is zero. In this case, the nonzero solution of (\dagger) might possibly correspond to a unique solution of $(*)$.
- (3) (\dagger) has a unique solution for α_1 .

In case (2), we must have $\gamma_1 = 0$. The case where $t\gamma_1 + \gamma_2 = 0$ comes under case (1); hence, there are at most $q - 1$ values of $t\gamma_1 + \gamma_2$ which correspond to case (2).

In case (3), the discriminant $\Delta = (\bar{\alpha}\gamma_1 + \beta_2\alpha_2^{-1}\gamma_1 - \gamma_2)^2 + 4\gamma_1^2\bar{\beta}$ must be zero. This cannot ever happen unless $-\bar{\beta}$ is a square. If $-\bar{\beta}$ is a square, then the equation $\Delta = 0$ determines γ_2 in terms of γ_1 .

$$\gamma_2 = \gamma_1(\bar{\alpha} + \beta_2\alpha_2^{-1} \pm 2\sqrt{-\bar{\beta}}).$$

If $\gamma_1 = 0$, then $\gamma_2 = 0$ and we are back to case (1). Otherwise, there are at most $2(q - 1)$ values of $t\gamma_1 + \gamma_2$ such that $\Delta = 0$.

Thus the total number of values of $t\gamma_1 + \gamma_2$ such that $t\alpha_1 + \beta_1$ is uniquely determined is at most $q + (q - 1) + 2(q - 1) = 4q - 3$. But there are q^2 values for $t\gamma_1 + \gamma_2$ and $q^2 > 4q - 3$ if $q > 3$. That is, we get a contradiction unless $q \leq 3$.

DEFINITION. If \mathfrak{L} is automorphic and if for each $\delta \neq 0$ there exists an automorphism σ fixing \mathfrak{F} elementwise such that $t\sigma = t\delta$, then we shall say that \mathfrak{L} is strongly automorphic.

Note that, for each $\alpha \neq 0$, β a strongly automorphic system admits an automorphism which carries t into $t\alpha + \beta$. Thus, a strongly automorphic system admits all possible automorphisms which fix \mathfrak{F} .

THEOREM 6. *If \mathfrak{L} is strongly automorphic and if the elements of \mathfrak{F} associate on the right with elements of \mathfrak{L} , then \mathfrak{L} is a left Hall-Veblen-Wedderburn system.*

Proof. Let $t(t\xi + \eta) = tf(\xi, \eta) + g(\xi, \eta)$. If the elements of \mathfrak{F} associate on the right, then $f(\xi\delta, \eta\delta) = f(\xi, \eta)\delta$ and $g(\xi\delta, \eta\delta) = g(\xi, \eta)\delta$. In particular, $f(\xi, 0) = \xi f(1, 0)$ and $g(\xi, 0) = \xi g(1, 0)$.

If $\alpha \neq 0$, there is an automorphism carrying t into $t\alpha + \beta$. Hence, $(t\alpha + \beta)[(t\alpha + \beta)\xi + \eta] = (t\alpha + \beta)f(\xi, \eta) + g(\xi, \eta)$. But also,

$$\begin{aligned}(t\alpha + \beta)[(t\alpha + \beta)\xi + \eta] &= (t\alpha + \beta)[(t\alpha + \beta)\xi] + (t\alpha + \beta)\eta \\ &= (t\alpha + \beta)\xi f(1, 0) + \xi g(1, 0) + t\alpha\eta + \beta\eta.\end{aligned}$$

Hence,

$$\alpha f(\xi, \eta) = \alpha \xi f(1, 0) + \alpha \eta$$

and

$$\beta f(\xi, \eta) + g(\xi, \eta) = \beta \xi f(1, 0) + \xi g(1, 0) + \beta \eta.$$

Solving for f and g , we obtain

$$(t\alpha + \beta)[(t\alpha + \beta)\xi + \eta] = (t\alpha + \beta)[\xi f(1, 0) + \eta] + \xi g(1, 0).$$

Except for the interchange of right and left multiplication, this is precisely of the form given in Theorem 20.2.7 in Hall's book [4], with $f(1, 0)$ and $g(1, 0)$ replaced by r and s , respectively.

The restrictions on r and s necessary in order that multiplication be a loop are precisely those given by Hall.

Now let us consider the possibility of a right distributive law in \mathfrak{L} :

THEOREM 7. *Suppose that $(t\alpha_1 + \beta_1)(t\alpha_2) = (t\alpha_1)(t\alpha_2) + \beta_1(t\alpha_2)$ for some particular $\alpha_1, \beta_1, \alpha_2$ all distinct from zero. Then \mathfrak{L} must have characteristic 2.*

Proof.

$$\begin{aligned}(t\alpha_1 + \beta_1)(t\alpha_2) &= (t\alpha_1)(t\alpha_2) + \beta_1(t\alpha_2) \\ &= th(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) + t\alpha_2\beta_1 + R(\beta_1, \alpha_2) \\ &= t[h(\alpha_1, \alpha_2) - \alpha_2\beta_1] + \beta_1\alpha_1^{-1}h(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) - \beta_1^2\alpha_1^{-1}\alpha_2;\end{aligned}$$

hence, $t(-\alpha_2\beta_1) = t\alpha_2\beta_1$, or $2\alpha_1\beta_1 = 0$ so \mathfrak{L} is of characteristic 2.

COROLLARY. *Except possibly at characteristic 2, the right distributive law does not hold in \mathfrak{L} .*

III. Applications to the geometry. In this part we shall be concerned with planes coordinatised by automorphic systems. Whenever we refer to a semi-translation plane coordinatised by such systems, it is to be understood that the equations of lines take the form mentioned in §I unless indications are given to the contrary:

DEFINITION. A semi-translation plane is said to be a strict semi-translation plane if the translation group is exactly of order q^2 .

THEOREM 8. *Let π be a semi-translation plane coordinatised by an automorphic system. Then π is a strict semi-translation plane unless q is even.*

Proof. The mappings $(x, y) \rightarrow (x + \alpha, y + \beta)$ constitute a group of translations of order q^2 . Hence, if π is not strict, there exist some ξ, η not both zero such that $(x, y) \rightarrow (x + t\xi, y + t\eta)$ is a translation.

Consider the line $y = xt$. Its image must be $y = (x - \alpha)t + \beta$, where $(\alpha - t\xi, \beta - t\eta)$ is a point on $y = xt$. That is, $\beta - t\eta = (\alpha - t\xi)t$.

Since the image of each point (c, ct) must be on $y = (x - \alpha)t + \beta$, we have the identity

$$ct + t\eta = (c + t\xi - \alpha)t + \beta \quad \text{for all } c.$$

This is equivalent to

$$ct = (c + t\xi - \alpha)t + (\alpha - t\xi)t.$$

Taking $c = 0$, $(\alpha - t\xi)t = -(t\xi - \alpha)t$. If $\xi \neq 0$, $\alpha \neq 0$, take $c = \alpha$ and we get $(t\xi - \alpha)t = (t\xi)t - \alpha t$. By a slight modification of the argument in Theorem 7, \mathfrak{I} must have characteristic 2. Similarly, if $\xi \neq 0$ but $\alpha = 0$, take $c = \gamma \neq 0$ and apply the same argument. If $\xi = 0$, $\alpha \neq 0$ we get $ct = (c - \alpha)t + \alpha t$ and again \mathfrak{I} is of characteristic 2. If ξ and α are both zero, then $\eta = 0$ contrary to the condition that ξ and η are not both zero.

REMARK. See [7] for an example of a semi-translation plane of even order which is coordinatised by an automorphic system but is not a strict semi-translation plane.

DEFINITION. The coordinate system \mathfrak{I} will be said to be linear with respect to \mathfrak{F} if, for γ in \mathfrak{F} and b in \mathfrak{I} , $y = x\gamma + b$ is the equation of a line.

As mentioned in §I, \mathfrak{I} is linear with respect to \mathfrak{F} in the known cases. However, the situation arises in the following lemma where the coordinate system might possibly not be linear with respect to \mathfrak{F} . See Theorem 3 for the definition of R .

LEMMA 1. *Let π be a semi-translation plane coordinatised by an automorphic coordinate system \mathfrak{I} . Let π' be the dual of π . Then π' can be coordinatised by a coordinate system \mathfrak{I}' where*

- (a) *The elements of \mathfrak{I}' are the same as the elements of \mathfrak{I} .*
- (b) *If \mathfrak{I} is linear with respect to \mathfrak{F} , then $R' = 0$ in \mathfrak{I}' .*
- (c) *\mathfrak{I}' is linear with respect to \mathfrak{F} and addition in \mathfrak{I}' is the same as addition in \mathfrak{I} if and only if $R = 0$.*

(d) *The general rule for multiplication in \mathfrak{I}' differs from multiplication in \mathfrak{I} only in that $h(\alpha_1, \alpha_2)$ and $k(\alpha_1, \alpha_2)$ are replaced by $h'(\alpha_1, \alpha_2) = -h(\alpha_2, \alpha_1)$ and $k'(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1)$, respectively.*

Proof. Since π' is the dual of π , there exists a 1-1 mapping from the lines of π onto the points of π' , such that the image of the set of lines through a point of π is the set of points on a line of π' . Let us assign coordinates (x', y') in accordance with the following scheme:

$$l_\infty \rightarrow (\infty),$$

$$x = c \rightarrow (c),$$

$$y = (x - \alpha)c + \beta \rightarrow (c, c\alpha - \beta) = (x', y'),$$

$$y = x\alpha + b \rightarrow (\alpha, -b) = (x', y').$$

Consider the set of points (x', y') of π' which are collinear with (γ) and $(0, 0)$. These points are the images of the lines of π through the intersection of $x = \gamma$ and $y = 0$. The lines of π which go through $(\gamma, 0)$ are of the form $y = (x - \gamma)c$ and correspond to $(x', y') = (c, c\gamma)$. Thus $y' = x'\gamma$ is the equation of a line of π' , i.e., multiplication on the right by elements of \mathfrak{F} is the same in \mathfrak{T}' as in \mathfrak{T} .

Now consider the set of points (x', y') collinear with (γ) and $(0, b)$ in π' . These points are the images of the lines of π which go through $(x, y) = (\gamma, -b)$. The case where $x' \in \mathfrak{F}$ offers little trouble and we shall not go through the details. The line $y = (x - \alpha)c + \beta$ contains $(\gamma, -b)$ if and only if $(\gamma - \alpha)c + \beta = -b$. Thus the points of π' collinear with (γ) and $(0, b)$ are of the form $(x', y') = (c, c\alpha - \beta)$ where $(\gamma - \alpha)c + \beta = -b$. Let $c = t\xi + \eta$, then

$$(\gamma - \alpha)c = (t\xi + \eta)(\gamma - \alpha) + R(\gamma - \alpha, \xi) = c\gamma - c\alpha + R(\gamma - \alpha, \xi).$$

Putting $c\gamma - c\alpha + R(\gamma - \alpha, \xi) + \beta = -b$ we get that $c\alpha - \beta = c\gamma + b + R(\gamma - \alpha, \xi)$ which corresponds to $y' = x'\gamma + b + R$, where R depends upon x' . Taking $\gamma = 1$, we see that addition in \mathfrak{T}' is identical with addition in \mathfrak{T} if and only if R is identically zero.

If $R = 0$ we then have that \mathfrak{T}' is linear with respect to \mathfrak{F} . This establishes part (c).

Now for each α , there is a group of elations of π with center (∞) and axis $x = \alpha$. Each of these groups is of order q ; (∞) is also the center of a group of elations with axis l_∞ . Thus, π is a dual semi-translation plane with respect to (∞) and π' is a semi-translation plane with respect to l'_∞ . Thus if $*$ denotes multiplication in \mathfrak{T}' and $m \notin \mathfrak{F}$, then $y' = (x' - \alpha) * m + \beta$ is the equation of a line in π' . To determine the operation $*$, consider the set of points (x', y') collinear with $(x', y') = (0, 0)$ and $(x', y') = (1, t\mu_2 + v_2)$.

These points correspond to the lines of π through the intersection of $y = 0$ and $y = x - t\mu_2 - v_2$, i.e., the point $(x, y) = (t\mu_2 + v_2, 0)$. Note that the line $x = t\mu_2 + v_2$ corresponds to the point at infinity $(t\mu_2 + v_2)$ in π' .

If $y = (x - \alpha)c + \beta$ contains $(t\mu_2 + v_2, 0)$, then $(t\mu_2 + v_2 - \alpha)c + \beta = 0$. Thus we are concerned with the points $(x', y') = (c, c\alpha - \beta)$ such that $(t\mu_2 + v_2 - \alpha)c + \beta = 0$. Taking $c = t\mu_1 + v_1$, the equation

$$(t\mu_2 + v_2 - \alpha)(t\mu_1 + v_1) + \beta = 0$$

can be solved for α, β by using the rule for multiplication in T . Note that, if $c \notin F, \mu_1 \neq 0$, we obtain

$$\begin{aligned}\alpha &= v_2 - h(\mu_2, \mu_1)\mu_1^{-1} - \mu_2v_1\mu_1^{-1}, \\ \beta &= -k(\mu_2, \mu_1).\end{aligned}$$

Using these values of α and β in $x' = c = t\mu_1 + v_1$, $y' = c\alpha - \beta$ we get that

$$\begin{aligned}y' &= (t\mu_1 + v_1) * (t\mu_2 + v_2) \\ &= t[-h(\mu_2, \mu_1) - \mu_2v_1 + v_2\mu_1] \\ &\quad + [-h(\mu_2, \mu_1)\mu_1^{-1}v_1 + k(\mu_2, \mu_1) - v_1^2\mu_1^{-1}\mu_2 + v_1v_2].\end{aligned}$$

This establishes part (d) of the lemma. We have yet, however, to consider multiplication of the type $\alpha * (t\mu_2 + v_2)$. Here we are concerned with points $(x', y') = (x, -b)$ which are images of lines $y = x\alpha + b$ such that $(t\mu_2 + v_2)\alpha + b = 0$. We have:

$$\alpha * (t\mu_2 + v_2) = -b = t\mu_2\alpha + v_2\alpha.$$

This establishes part (b) of the lemma. We have already shown that multiplication on the right by elements of \mathfrak{F} is the same in \mathfrak{T} and \mathfrak{T}' .

LEMMA 2. *If q is odd and π' of Lemma 1 is isomorphic to π and \mathfrak{T} is linear with respect to \mathfrak{F} , then $R = 0$ in \mathfrak{T} .*

Proof. We have pointed out that π' is a semi-translation plane; the centers of the translations are the points (∞) and (γ) (γ varies over \mathfrak{F}) in \mathfrak{T}' . If π' is isomorphic to π , then \mathfrak{T}' is (in effect) another coordinate system for π . We must consider the possibility that \mathfrak{T}' might correspond to another choice for the line at infinity. However, if a projective plane is a semi-translation plane with respect to two lines l_1 and l_2 , there is a collineation carrying l_1 into l_2 . This implies that we can, without loss of generality, identify the lines at infinity of \mathfrak{T} and \mathfrak{T}' .

If q is odd, π is a strict semi-translation plane. The set \mathfrak{M} of centers of translations has coordinates chosen from (∞) and elements of \mathfrak{F} in both \mathfrak{T} and \mathfrak{T}' .

Now consider the partial affine plane consisting of the affine points of π and the lines of π which intersect \mathfrak{M} . Consider also the following properties of \mathfrak{T} :

- (a) Addition is associative.
- (b) \mathfrak{T} is linear with respect to \mathfrak{F} .
- (c) Elements of \mathfrak{F} distribute on the right.
- (d) \mathfrak{F} is a field.

These properties of \mathfrak{T} imply that for each a in \mathfrak{T} and each δ in \mathfrak{F} , the mappings $(x, y) \rightarrow (x + a, y + a\delta)$ and $(x, y) \rightarrow (x, y + a)$ are translations of the partial plane. The group of translations with center (δ) is isomorphic to the additive group of \mathfrak{T} .

This implies that if \mathfrak{T}' is a new coordinate system for π with the new choices for (∞) , (0) , and (1) all in \mathfrak{M} , then \mathfrak{T}' also has properties (a), (b), and (c). Moreover, addition in \mathfrak{T}' will be isomorphic to addition in \mathfrak{T} . By Lemma 1, $R = 0$.

THEOREM 9. *Let π be a semi-translation plane coordinatised by an automorphic system \mathfrak{T} which is linear with respect to \mathfrak{F} . Then a necessary condition that π be self dual is that $R = 0$. It is sufficient that the above conditions be met and $h(\alpha_1, \alpha_2) = -h(\alpha_2, \alpha_1)$ and $k(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1)$.*

Proof. The requirement that R must be zero follows immediately from the previous two lemmas. If \mathfrak{T} is linear with respect to \mathfrak{F} , $R=0$, $h(\alpha_1, \alpha_2) = -h(\alpha_2, \alpha_1)$ and $k(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1)$, then π and its dual have isomorphic coordinate systems. Hence, π is self dual.

DEFINITION. Let \mathfrak{T} be the coordinate system for a semi-translation plane π . We shall say that \mathfrak{T} is regular if the following conditions are satisfied:

- (a) \mathfrak{T} is automorphic.
- (b) $R = 0$.
- (c) \mathfrak{T} is linear with respect to \mathfrak{F} .
- (d) Lines of π whose slopes are not in \mathfrak{F} are given by equations of the type $y = (x - \alpha)m + \beta$ where $m \notin \mathfrak{F}$.

Note that the condition that $R \equiv 0$ implies that elements of \mathfrak{F} commute with elements of \mathfrak{T} ; moreover, \mathfrak{T} is both a right and left vector space over \mathfrak{F} . It is well known that if some point at infinity is the center of a group of translations of order q^2 , then the plane admits a coordinate system in which addition is associative and the coordinate system is linear. Such planes will be called linear planes.

THEOREM 10. *For each semi-translation plane π which is coordinatised by a regular coordinate system \mathfrak{T} there is a plane $\bar{\pi}$ coordinatised by a system $\bar{\mathfrak{T}}$ such that:*

- (a) *The elements of $\bar{\mathfrak{T}}$ are the same as the elements of \mathfrak{T} .*
- (b) *If $*$ denotes multiplication in $\bar{\mathfrak{T}}$, then*

$$(t\alpha_1 + \beta_1) * (t\alpha_2 + \beta_2) = t[H(\alpha_1, \alpha_2) + \alpha_1\beta_2 + \alpha_2\beta_1] + [K(\alpha_1, \alpha_2) + \beta_1\beta_2]$$

where H and K are in \mathfrak{F} .

- (c) *$\bar{\mathfrak{T}}$ is linear, i.e., lines of π' are given by equations $y' = x' * m + b$.*

Proof. The plane $\bar{\pi}$ will be obtained from π by a chain of constructions. We shall be considering several different planes and several different coordinate systems. We shall find it convenient not to introduce new symbols for multiplications in these various systems.

First of all, we obtain the plane derived from π . The coordinate system can be obtained from \mathfrak{T} by the process given in §I. The set of points in the subplane of π coordinatised by \mathfrak{F} will be the set of points on $x' = 0$ in the new coordinate system. This set of points is in a single transitive class under the translations of π and of the derived plane. Thus lines of the derived plane will be represented by linear equations.

In \mathfrak{T} we have:

$$(t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2)$$

$$= t[h(\alpha_1, \alpha_2) + \alpha_1\beta_2 - \alpha_2\beta_1] + \beta_1\alpha_1^{-1}h(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_2) - \beta_1^2\alpha_1^{-1}\alpha_2 + \beta_1\beta_2$$

when $\alpha_1 \neq 0$ and

$$\beta_1(t\alpha_2 + \beta_2) = t\alpha_2\beta_1 + \beta_2\beta_1.$$

We find that multiplication in the coordinate system of the derived plane takes the form:

$$(t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2)$$

$$= t[h(\alpha_1, \alpha_2^{-1})\alpha_2 + \alpha_1\beta_2 - \alpha_2\beta_1] + h(\alpha_1, \alpha_2)\alpha_2\beta_1\alpha_1^{-1} + k(\alpha_1, \alpha_2^{-1}) - \beta_1^2\alpha_1^{-1}\alpha_2 + \beta_1\beta_2$$

when $\alpha_1, \alpha_2 \neq 0$ and

$$(t\alpha + \beta)\gamma = \gamma(t\alpha + \beta) = t\alpha\gamma + \beta\gamma.$$

If we now interchange right and left multiplications, we obtain the coordinate system of a plane dual to the plane derived from π .

If we derive again, we get a plane with a linear coordinate system in which:

$$(t\alpha_1 + \beta_1)(t\alpha_2 + \beta_2)$$

$$= t[-h(\alpha_2^{-1}, \alpha_1^{-1})\alpha_1\alpha_2 + \beta_1\alpha_2 + \alpha_1\beta_2] + k(\alpha_2^{-1}, \alpha_1^{-1}) + \beta_1\beta_2$$

$$= t[H(\alpha_1, \alpha_2) + \beta_1\alpha_2 + \alpha_1\beta_2] + K(\alpha_1, \alpha_2) + \beta_1\beta_2.$$

As obtained, the rule for multiplication in the last case does not hold if α_1 , or α_2 is zero. However, the rule

$$\beta_1(t\alpha_2 + \beta_2) = (t\alpha_2 + \beta_2)\beta_1 = t\alpha_2\beta_1 + \beta_1\beta_2$$

holds in each of the four coordinate systems. Furthermore,

$$(t\alpha_1)(t\alpha_2) = tH(\alpha_1, \alpha_2) + K(\alpha_1, \alpha_2).$$

We can remove the exceptions by agreeing that H and K are both zero when either α_1 or α_2 are zero.

Note that the additive group is the same in each of the four coordinate systems.

Now this whole process is reversible. That is, if we start with a plane coordinatised by a "partial division ring" (we have the right and left partial distributive law) in which multiplication takes the form:

$$t(\alpha_1 + \beta_1)(t\alpha_2 + \beta_2) = t[H(\alpha_1, \alpha_2) + \alpha_1\beta_2 + \alpha_2\beta_1] + K(\alpha_1, \alpha_2) + \beta_1\beta_2,$$

we end up with a semi-translation plane having a regular coordinate system.

It is an interesting question as to whether there are partial division rings which are not division rings. If H and K are appropriate constant multiples of $\alpha_1\alpha_2$, we will have a field. (Moreover, at the second and third stages, we will have a

Hall-Veblen-Wedderburn system.) The only other system of this type known to us is a slight generalization of Dickson's commutative division rings [3]. Here we take $H(\alpha_1, \alpha_2) = 0$, $K(\alpha_1, \alpha_2) = \delta \alpha_1^\sigma \alpha_2^\rho$ where δ is a non-square in \mathfrak{F} ; ρ and σ are automorphisms of \mathfrak{F} . In the case where $\rho = \sigma$, the semi-translation plane with $h(\alpha_1, \alpha_2) = 0$, $k(\alpha_1, \alpha_2) = \delta \alpha_1^{-\sigma} \alpha_2^{-\sigma}$ is self dual by Theorem 9.

THEOREM 11. *Let π be a semi-translation plane coordinatised by a regular coordinate system. Suppose further that $(-1)(ab) = (-a)b$. Then if π^* admits collineations moving l_∞ , \mathfrak{T} satisfies the left inverse law.*

Proof. It was established in [7] that, under the hypothesis, the subplane coordinatised by \mathfrak{F} is an invariant subplane. Furthermore, since π^* admits a group of elations of order q with center (∞) and axis $x = 0$, if l_∞ is carried into any line other than $x = 0$ by some collineation, then there is some collineation which carries l_∞ into $x = 0$. This, in turn, implies the existence of a group of elations with center $(0, 0)$ and axis $x = 0$. In particular, we will have an elation σ with axis $x = 0$, center 0 which carries l_∞ into $x = 1$.

Thus the lines $y = xm$ and the points on $x = 0$ are all fixed by σ . Furthermore, since (∞) is fixed, each line $x = c$ is carried into another line through (∞) . For some value of c , $x = c \rightarrow l_\infty$. We may, in general, write $x = c \rightarrow x = c^*$ if it be understood that $c^* = \infty$ indicates that the image line is l_∞ . We shall denote the line through two given points by the symbol " \cup ". We now indicate the images of various points and lines under the mapping σ :

$$(m) \rightarrow (1, m),$$

$$y = b \rightarrow (1, 0) \cup (0, b), \text{ i.e., } y = b \rightarrow y = (x - 1)(-b),$$

$$(c, b) \rightarrow (c^*, (c^* - 1)(-b)).$$

Since $y = -x$ is fixed, this implies that $(c^* - 1)(c) = -c^*$. This, in turn, implies that $(-1)^* = \infty$, i.e., $x = -1 \rightarrow l_\infty$.

Hence,

$$(-1, 0) \rightarrow l_\infty \cap (y = 0), \text{ i.e., } (-1, 0) \rightarrow (0),$$

$$y = (x + 1)m \rightarrow (1, m) \cup (0), \text{ i.e., } y = (x + 1)m \rightarrow y = m.$$

Thus the image of $(c, (c + 1)m)$ must lie on $y = m$. That is,

$$(c^* - 1)\{(-1)[(c + 1)m]\} = m.$$

With $m = 1$, we get that $(c^* - 1)(-c - 1) = 1$, i.e., $(c^* - 1)$ is the left inverse of $-(c + 1)$. If -1 associates, this implies that

$$(-c - 1)^{-1}[(-c - 1)m] = m \quad \text{for } c \neq -1.$$

REMARK. One can now apply Theorem 10 to the planes obtained (by the chain

of constructions of Theorem 9) from the generalizations of Dickson's division rings. The left inverse law is not satisfied by the coordinate system of the semi-translation planes; these planes do not admit collineations moving l_∞ .

Next, we wish to point out some of the geometric implications of Theorem 5. Consider the left nearfields of order q^2 which satisfy conditions (1)–(5) of § II with the additional condition that the elements of \mathfrak{F} commute with elements of \mathfrak{L} . By Theorem 5, these nearfields are not automorphic for $q \neq 3$.

Each of these nearfields can be used to coordinatise two distinct planes:

- (a) a dual translation plane with lines represented by linear equations,
- (b) a semi-translation plane which is a Hughes plane.

Both of these planes are derivable. From (a) we obtain a semi-translation plane; from (b) we obtain a linear plane. These two planes are again coordinatised by a common algebraic system obtainable from the nearfield by the process described in §I. The absence of the group Σ of automorphisms in the nearfields implies the absence of the partial left distributive law in the derived algebra. It was pointed out in [7] that the planes derived from the Hughes planes [5] formed a new class of planes except in those cases where the nearfields might admit the group Σ of automorphisms.

Furthermore, dualizing a result of André [2], if π is a dual translation plane coordinatised by a nearfield, then the collineation group of π either fixes or interchanges the two lines l_∞ and $x = 0$. Suppose that π admits a collineation (whose order divides q) which fixes some subplane π_1 of order q . If π_1 contains l_∞ , it must also contain $x = 0$. If π_1 contains the points at infinity of the subplane π_0 coordinatised by \mathfrak{F} , then π_1 can be carried into π_0 by a collineation of π . This implies the existence of collineations fixing π_0 pointwise and automorphisms of the nearfield, contrary to Theorem 5.

If the plane derived from π should admit a group of elations of order q with some finite line as axis, center on the line at infinity, then π would admit a corresponding group of collineations fixing a subplane pointwise. Since this cannot be the case, the derived plane admits no such elations. This implies that the derived semi-translation planes are not self dual and admit no collineations displacing the line at infinity.

THEOREM 12. *Let π_1 and π_2 be strict semi-translation planes which are derived from the linear planes $\bar{\pi}_1$ and $\bar{\pi}_2$, respectively. Suppose that there exists an isomorphism σ such that $\pi_1\sigma = \pi_2$. Then $\bar{\pi}_1\sigma = \bar{\pi}_2$.*

Proof. Let π_1 and π_2 be derived from $\bar{\pi}_1$, and $\bar{\pi}_2$ with respect to $\bar{\mathfrak{M}}_1$ and $\bar{\mathfrak{M}}_2$, respectively. Then $\bar{\pi}_1$ and $\bar{\pi}_2$ can be derived from π_1 and π_2 with respect to the sets \mathfrak{M}_1 and \mathfrak{M}_2 , where \mathfrak{M}_1 and \mathfrak{M}_2 are the respective sets of centers of translations. If π_1 and π_2 are strict semi-translation planes, then $\mathfrak{M}_1\sigma = \mathfrak{M}_2$. Let l_1 be a line of $\bar{\pi}_1$ which does not intersect $\bar{\mathfrak{M}}_1$. Then l_1 is also a line of π_1 which does not intersect \mathfrak{M}_1 . Hence, $l_1\sigma$ is a line of π_2 which does not intersect \mathfrak{M}_2 , i.e., a line

of $\bar{\pi}_2$ which does not intersect $\bar{\mathfrak{M}}_2$. Let \bar{m}_1 be a line of $\bar{\pi}_1$ which does not intersect $\bar{\mathfrak{M}}_1$. Then \bar{m}_1 is a subplane of π_1 which contains \mathfrak{M}_1 . Hence, $\bar{m}_1\sigma$ is a subplane of π_2 which contains \mathfrak{M}_2 . It follows that $\bar{m}_1\sigma$ is a line of $\bar{\pi}_2$ which intersects $\bar{\mathfrak{M}}_2$. Thus, in any case, σ carries lines of $\bar{\pi}_1$ into lines of $\bar{\pi}_2$.

COROLLARY. *Let π_1 and π_2 be isomorphic strict semi-translation planes derived from the linear plane $\bar{\pi}$ with respect to the sets \mathfrak{M}_1 and \mathfrak{M}_2 . Then $\bar{\pi}$ admits a collineation which carries $\bar{\mathfrak{M}}_1$ into $\bar{\mathfrak{M}}_2$.*

CORRECTION ADDED IN PROOF. In Lemma 1, the condition $R \equiv 0$ is needed from the start to insure that $y' = \text{constant}$ is the equation of a line in π ! The proof of Lemma 2 thus becomes invalid; Lemma 2 is probably false. The conclusion to Theorem 9 should read, "Then π is self dual if $R \equiv 0$, $h(\alpha_1, \alpha_2) = -h(\alpha_2, \alpha_1)$ and $k(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1)$."

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