

A GENERALIZED KOSZUL COMPLEX. I

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Introduction. In [1], the Koszul complex was used to study the relationship between codimension and multiplicity. It also helped us investigate Macaulay modules and rings, and provided a context in which to prove the Cohen-Macaulay Theorem concerning the unmixedness of complete intersections.

Now there is a generalization of the Cohen-Macaulay Theorem (known, we believe, as the generalized Cohen-Macaulay Theorem) which has to do with the unmixedness of an ideal generated by the minors of a matrix. Since the Koszul complex can be thought of as a complex associated with a map of $R^n \rightarrow R$, where R is a commutative ring, it seemed likely that there should be a complex associated with a map of $R^m \rightarrow R^n$. In fact, as long as one is willing to go that far, why not look for some complex associated with an arbitrary map of modules over any commutative ring.

In this paper, we define a complex associated with a map of modules. In §1, we discuss this complex in complete generality.

In §2, we pretty much restrict our attention to maps of $R^m \rightarrow R^n$, and establish many of the formal properties of the complexes attached to this map which will be needed in the rest of this paper as well as in subsequent ones.

§§3 and 4 are included here to give an indication of how the notion of E -sequence may be generalized, and to show that over local rings this generalized notion of E -sequence is (as in the usual case) independent of order.

In subsequent papers we shall investigate these more general ideas, relating them to each other (as in [1]) and also applying them to the case of the singular variety of a variety which is defined by the ideal generated by the minors of the Jacobian matrix.

It might also be mentioned that the usual Koszul complex plays a role in studying the invariant factors [2] of certain special modules, and we shall subsequently show the connection between the homology groups of the complex associated with a map $f: R^m \rightarrow R^n$, and the invariant factors of the cokernel of f .

Another reason for looking at complexes associated with maps $f: R^m \rightarrow R^n$ is the following. If R is noetherian, and $\text{coker } f$ has finite length, then $\text{coker } S_p(f)$ has finite length for every p , where $S_p(f): S_p(R^m) \rightarrow S_p(R^n)$ denotes the induced map

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on the p th symmetric product. Thus a Hilbert characteristic function may be defined which leads to the notion of multiplicity of coker f . Under suitable conditions, this multiplicity may be computed as the Euler-Poincaré characteristic of the complex associated with the map, f .

Throughout this paper, all rings will be assumed to be commutative with an identity element, and all modules will be unitary. We will also adopt the following notation. Instead of subscripts, we will use functional notation (e.g., $\omega(\lambda(1) \wedge \cdots \wedge \lambda(q))$, $b(j)$, etc.). We will denote the ordered set $\{1, 2, \dots, p\}$ by $\{p\}$, and $\mathcal{F}(p; q)$ will denote the set of strictly increasing functions from $\{p\}$ to $\{q\}$. If σ is in $\mathcal{F}(p; q)$, the symbol $|\sigma|$ will denote the sum of the values of σ . Finally, we will let ∂ stand for the deletion operator, i.e., $b(1) \wedge \cdots \wedge \partial b(j) \wedge \cdots \wedge b(p)$ will mean the product of all the $b(i)$ except for $b(j)$.

If we are given elements $b(1), \dots, b(p)$ in a set B , and a function σ in $\mathcal{F}(q; p)$, we may of course regard $b: \{p\} \rightarrow B$ as a function, and the composite notation $b\sigma: \{q\} \rightarrow B$ therefore makes sense.

1. The Koszul complex of a map. Let B be an arbitrary R -module, and let B^* denote $\text{Hom}_R(B, R)$. If $\lambda(1), \dots, \lambda(q)$ are in B^* , and $b(1), \dots, b(p)$ are in B , we denote by $\omega(\lambda(1) \wedge \cdots \wedge \lambda(q)) [b(1) \wedge \cdots \wedge b(p)]$ the element

$$\sum_{\sigma} (-1)^{|\sigma|+1} \det[\lambda(i)(b\sigma(k))] b(1) \wedge \cdots \wedge \partial b\sigma(1) \wedge \cdots \wedge \partial b\sigma(q) \wedge \cdots \wedge b(p)$$

in $\wedge^{p-q} B$, where σ runs over the functions in $\mathcal{F}(q; p)$. If $q > p$, we of course get zero.

Extending this operation linearly, we obtain a map $\wedge^q B^* \otimes_R \wedge^p B \rightarrow \wedge^{p-q} B$ with the image of $\lambda \otimes \beta$ denoted by $\omega(\lambda)[\beta]$.

LEMMA 1.1. *Let $\lambda(1), \dots, \lambda(n)$ be elements in $\wedge B^*$, with $\lambda(i) \in \wedge^{q(i)} B^*$, and let $\beta \in \wedge^p B$. Then*

$$\omega(\lambda(1) \wedge \cdots \wedge \lambda(n)) [\beta] = (-1)^{n+1} \omega(\lambda(1)) \omega(\lambda(2)) \cdots \omega(\lambda(n)) [\beta].$$

Proof. It clearly suffices to prove this when $n = 2$ and $\lambda(i)$ are in B^* . We may also assume that $\beta = b(1) \wedge \cdots \wedge b(p)$ with $b(i)$ in B .

When $p = 1$ or 2 , the result is trivially true. Proceeding, then, by induction on p we obtain:

$$\begin{aligned} \omega(\lambda(1) \wedge \lambda(2)) [\beta] &= \sum_{\sigma} (-1)^{|\sigma|+1} \det[\lambda(i)(b\sigma(k))] b(1) \wedge \cdots \wedge \partial b\sigma(1) \wedge \cdots \wedge \partial b\sigma(2) \wedge \cdots \wedge b(p) \\ &= b(1) \wedge \omega(\lambda(1) \wedge \lambda(2)) [b(2) \wedge \cdots \wedge b(p)] \\ &\quad + \sum_{\tau} (-1)^{|\tau|+1} \det[\lambda(i)b\tau(k)] b(2) \wedge \cdots \wedge \partial b\tau(2) \wedge \cdots \wedge b(p), \end{aligned}$$

where σ runs over the functions in $\mathcal{F}(2; p)$ and τ runs over those functions in

$\mathcal{F}(2; p)$ with the property that $\tau(1) = 1$. By applying the induction hypothesis to $\omega(\lambda(1) \wedge \lambda(2)) [b(2) \wedge \cdots \wedge b(p)]$, we have

$$\begin{aligned} \omega(\lambda(1) \wedge \lambda(2))[\beta] \\ = -b(1) \wedge \omega(\lambda(1))\omega(\lambda(2)) [b(2) \wedge \cdots \wedge b(p)] \\ + \lambda(1)(b(1))\omega(\lambda(2)) [b(2) \wedge \cdots \wedge b(p)] - \lambda(2)(b(1))\omega(\lambda(1)) [b(2) \wedge \cdots \wedge b(p)] \end{aligned}$$

On the other hand, using the fact that $\omega(\lambda(i))$ is a derivation, one verifies easily that this last expression is equal to $-\omega(\lambda(1))\omega(\lambda(2))[\beta]$.

DEFINITION. Let B be an R -module. We denote by $T(B)$ the complex whose n -chains, $T(B; n)$, are defined as follows:

$$T(B; n) = \sum \wedge^{s(1)} B^* \otimes \cdots \otimes \wedge^{s(n)} B^* \otimes \wedge^{t+|s|} B$$

where the sum is taken over all $t, s(i) \geq 0$, and where $|s|$ denotes $\sum s(i)$. $T(B; 0)$ is then just $\wedge B$, and $T(B; n) = 0$ for $n < 0$.

The map $d(n): T(B; n) \rightarrow T(B; n-1)$ is defined as follows:

$$\begin{aligned} d(n)[\lambda(1) \otimes \cdots \otimes \lambda(n) \otimes \beta] \\ = \lambda(1) \otimes \cdots \otimes \lambda(n-1) \otimes \omega(\lambda(n))[\beta] \\ + (-1)^n \sum_{i=1}^{n-1} (-1)^{i+1} \lambda(1) \otimes \cdots \otimes \lambda(i) \wedge \lambda(i+1) \otimes \cdots \otimes \lambda(n) \otimes \beta. \end{aligned}$$

REMARK. If $\lambda(1) \otimes \lambda(2) \otimes \gamma$ is in $T(B; n)$, then $d(n)$ can be defined inductively by

$$\begin{aligned} d(n)[\lambda(1) \otimes \lambda(2) \otimes \gamma] &= (-1)^n \lambda(1) \wedge \lambda(2) \otimes \gamma + \lambda(1) \otimes d(n-1)[\lambda(2) \otimes \gamma]; \\ d(0)[\lambda \otimes \beta] &= \omega(\lambda)[\beta]. \end{aligned}$$

Lemma 1.1 immediately gives us the fact that $T(B)$ is a complex, i.e., that $d(n-1)d(n) = 0$. That this complex is acyclic comes from the following stronger fact.

LEMMA 1.2. $T(B)$ is homotopically trivial.

Proof. Define a homotopy $s(n): T(B; n) \rightarrow T(B; n+1)$ by $s(n)(c) = (-1)^n 1 \otimes c$. It is easy to show, using this homotopy, that $T(B)$ is homotopically trivial.

Now suppose that we have a homomorphism $f: A \rightarrow B$. Then this gives us maps $f^*: B^* \rightarrow A^*$, $\wedge f: \wedge A \rightarrow \wedge B$, and $\wedge f^*: \wedge B^* \rightarrow \wedge A^*$. If $\lambda \in \wedge^q B^*$, and $\alpha \in \wedge^p A$, we will denote by $\omega(\lambda)[\alpha]$ the element $\omega(\wedge f(\lambda))[\alpha]$. Using this operation of $\wedge B^*$ on $\wedge A$, we may modify the complex $T(A)$ and obtain a complex $T(A, f)$ defined by $T(A, f; n) = \bigotimes_1^n \wedge B^* \otimes \wedge A$ with the boundary maps being defined as though we were still in $T(A)$, but using the operation of $\wedge B^*$ on $\wedge A$ induced by f .

As a result, we obtain a mapping, F , of $T(A, f)$ into $T(B)$ in the obvious way.

DEFINITION. The Koszul complex of the map $f: A \rightarrow B$ is the mapping cylinder of the map $F: T(A, f) \rightarrow T(B)$ [3]. We will denote this complex by $K(f)$.

Of course this complex as it now stands is a bit unwieldy and not too interesting. However, it contains many interesting subcomplexes which we will now define.

For each pair of integers $p, q \geq 0$, consider the following complexes:

$$T(B; p, q) = \{T(B; n; p, q) = \sum \wedge^{q+r} B^* \otimes \wedge^{s(1)} B^* \otimes \cdots \otimes \wedge^{s(n-1)} B^* \otimes \wedge^{v+r+|s|} B\},$$

$$T(A, f; p, q) = \{T(A, f; n; p, q) = \sum \wedge^{q+r} B^* \otimes \wedge^{s(1)} B^* \otimes \cdots \otimes \wedge^{s(n-1)} B^* \otimes \wedge^{v+r+|s|} A\},$$

where the summations run over all $r, s(i) \geq 0$, $v = p + q$, and $|s|$ again means $\sum s(i)$.

For both of these complexes, the boundary homomorphisms are just the restriction of the boundary maps of the total complexes to these submodules.

It is clear that F maps $T(A, f; p, q)$ into $T(B; p, q)$. We define $K(f; p, q)$ to be the mapping cylinder of this map.

Before proceeding with some more general facts about these complexes, let us look at some examples. As a matter of fact, these examples will be the only cases considered in this paper after this section.

We let $B = R^n$ (R^n is the direct sum of n copies of R), we let $p = 1$, and $q = n$. Then $T(R^n; 1, n)$ simply becomes:

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow R^n \rightarrow 0.$$

Thus $K(f; 1, n)$ becomes:

$$\cdots \rightarrow \sum_{s \geq 1} \wedge^n R^{n*} \otimes \wedge^s R^{n*} \otimes \wedge^{n+s+1} A \rightarrow \wedge^n R^{n*} \otimes \wedge^{n+1} A \rightarrow A \rightarrow R^n \rightarrow 0.$$

If $A = R^m$, with $m \geq n$, we see that this complex has length $m - n + 1$.

Still letting $B = R^n$, we now let $p = n$ and $q = 1$. This makes $K(f; n, 1)$ the following:

$$\begin{aligned} \cdots &\rightarrow \sum_{s(i) \geq 1} \wedge^{s(1)} R^{n*} \otimes \wedge^{s(2)} R^{n*} \otimes \wedge^{n+|s|} A \rightarrow \sum_{s \geq 1} \wedge^s R^{n*} \otimes \wedge^{n+s} A \\ &\rightarrow \wedge^n A \rightarrow \wedge^n R^n \rightarrow 0. \end{aligned}$$

Again, if $A = R^m$, with $m \geq n$, the length of this complex is $m - n + 1$.

Returning now to the general case, consider a map $f: A \rightarrow B$, and two pairs of integers $(p, q), (p', q')$ with $p + q = p' + q' = v$. Suppose $p \leq p'$, and let ξ be an element of $\wedge^{p'-p} B^*$. Then we obtain a map of $K(f; p', q')$ into $K(f; p, q)$ corresponding to this element in the following way.

Since $K(f; 0; p', q')$ is $\wedge^{p'} B$ and $K(f; 0; p, q)$ is $\wedge^p B$, we simply map $\wedge^{p'} B$ into $\wedge^p B$ by $\omega(\xi)$.

Now

$$K(f; 1: p', q') = \sum_{r \geq 0} \wedge^{q'+r} B^* \otimes \wedge^{v+r} B \oplus \wedge^{p'} A$$

and

$$K(f; 1: p, q) = \sum_{r \geq 0} \wedge^{q+r} B^* \otimes \wedge^{v+r} B \oplus \wedge^p A.$$

We therefore map $K(f; 1: p', q')$ into $K(f; 1: p, q)$ by sending α in $\wedge^{p'} A$ into $\omega(\xi)[\alpha]$ in $\wedge^p A$, and sending $\lambda \otimes \beta$ in $\wedge^{q'+r} B^* \otimes \wedge^{v+r} B$ into $-\xi \wedge \lambda \otimes \beta$ in $\wedge^{q+r} B^* \otimes \wedge^{v+r} B$.

For $n > 1$, we have

$$\begin{aligned} K(f; n: p', q') &= \sum \wedge^{q'+r} B^* \otimes \wedge^{s(1)} B^* \otimes \cdots \otimes \wedge^{s(n-1)} B^* \otimes \wedge^{v+r+|s|} B \\ &\quad \oplus \sum \wedge^{q'+r} B^* \otimes \wedge^{t(1)} B^* \otimes \cdots \otimes \wedge^{t(n-2)} B^* \otimes \wedge^{v+r+|t|} A, \\ K(f; n: p, q) &= \sum \wedge^{q+r} B^* \otimes \wedge^{s(1)} B^* \otimes \cdots \otimes \wedge^{s(n-1)} B^* \otimes \wedge^{v+r+|s|} B \\ &\quad \oplus \sum \wedge^{q+r} B^* \otimes \wedge^{t(1)} B^* \otimes \cdots \otimes \wedge^{t(n-2)} B^* \otimes \wedge^{v+r+|t|} A, \end{aligned}$$

$r \geq 0, s(i)$ and $t(i) \geq 1$. We therefore multiply the first factor by $(-1)^n \xi$ or by $(-1)^{n-1} \xi$ according as we are considering the first component or the second.

2. The generalized Koszul complex.

DEFINITION. A generalized Koszul complex is the complex $K(f; 1, n)$ where $f: A \rightarrow R^n$.

REMARK. When $n=1$, we see that this is precisely the standard Koszul complex of a map $f: A \rightarrow R$. Throughout the rest of this section, n will be fixed and we will simply refer to the generalized Koszul complex as the Koszul complex.

We denote the canonical basis of R^n by $x(1), \dots, x(n)$, and the dual basis by $\xi(1), \dots, \xi(n)$. If $\rho: R \rightarrow R^n$ is a map, we define $\xi(\rho) \in \wedge^{n-1} R^{n*}$ by $\xi(\rho) = \sum (-1)^{i+1} \xi(i) \rho(1) \xi(1) \wedge \cdots \wedge \partial \xi(i) \wedge \cdots \wedge \xi(n)$. We see that

$$\omega(\xi(\rho))[x(1) \wedge \cdots \wedge x(n)] = \rho(1).$$

PROPOSITION 2.1. Let $f: A \rightarrow R^n$, and $\rho: R \rightarrow R^n$ be maps. Then $f + \rho: A \oplus R \rightarrow R^n$ is a map and $K(f + \rho; 1, n)$ is the mapping cylinder of the map $K(f; n, 1) \rightarrow K(f; 1, n)$ associated with $\xi(\rho) \in \wedge^{n-1} R^{n*}$.

Proof. The proof is straightforward after one makes the usual identifications of $\wedge^n R^n$ with R and of $\wedge^p(A \oplus R)$ with $\wedge^p A \oplus \wedge^{p-1} A$.

COROLLARY 2.2. With f and ρ as above, we obtain the exact sequence

$$\cdots \rightarrow H_q(K(f; 1, n)) \rightarrow H_q(K(f + \rho; 1, n)) \rightarrow H_{q-1}(K(f; n, 1)) \xrightarrow{\delta} H_{q-1}(K(f; 1, n)) \rightarrow \cdots$$

where δ is the induced map in homology of the chain map induced by $\xi(\rho)$.

DEFINITION. If E is any R -module, and $K(f: 1, n)$ is a Koszul complex, denote by $E(f)$ the complex $K(f: 1, n) \otimes E$, and by $E(\wedge f)$ the complex $K(f: n, 1) \otimes E$.

In the case when $n = 1$, the question never arises as to whether the acyclicity of $E(f)$ from a certain point on implies the acyclicity of $E(\wedge f)$ from that same point on. However, when $n > 1$, this question does arise, and the next few lemmas are devoted to answering it.

LEMMA 2.3. Let $\alpha \in \wedge^2 A \otimes E$ be such that $\omega(\xi(i)) \otimes 1[\alpha] = 0$ for $i = 1, \dots, n$. Then for any $\eta \in A^*$ $f \otimes 1(\omega(\eta) \otimes 1[\alpha]) = 0$.

Proof. We may write $\alpha = \sum \alpha'(i) \wedge \alpha''(i) \otimes e(i)$ with $\alpha'(i), \alpha''(i)$ in $A, e(i)$ in E . We will also, in the future, omit $- \otimes 1$ when talking about maps. Now

$$\begin{aligned} f\omega(\eta)[\sum \alpha'(i) \wedge \alpha''(i) \otimes e(i)] \\ &= \sum \eta(\alpha'(i))f(\alpha''(i)) - \eta(\alpha''(i))f(\alpha'(i)) \otimes e(i) \\ &= \sum_i \eta(\sum_j \xi(jf((\alpha''(i))\alpha'(i) - \xi(j)f(\alpha'(i)\alpha''(i))x(j))) \otimes e(i) \\ &= - \sum_j x(j) \otimes \eta \sum_i \omega(\xi(j))[\alpha'(i) \wedge \alpha''(i) \otimes e(i)] \\ &= - \sum_j x(j) \otimes \eta \omega(\xi(j))[\sum_i \alpha'(i) \wedge \alpha''(i) \otimes e(i)] \\ &= 0. \end{aligned}$$

LEMMA 2.4. Let $\alpha \in \wedge^{n-p} A \otimes E$ be such that

$$\omega(\xi(1) \wedge \dots \wedge \partial \xi \sigma(1) \wedge \dots \wedge \partial \xi \sigma(p+1) \wedge \dots \wedge \xi(n))[\alpha] = 0$$

for all $\sigma \in \mathcal{F}(p+1; n)$. Then if $H_1(E(f)) = 0$, and $\eta \in A^*$, we have

$$\gamma = \omega(\eta)[\alpha] - \sum_{\tau} \omega(\xi \tau(1) \wedge \dots \wedge \xi \tau(p+2))[\alpha \tau(1) \wedge \dots \wedge \alpha \tau(p+2)],$$

with

$$\omega(\xi(1) \wedge \dots \wedge \partial \xi \tau(1) \wedge \dots \wedge \partial \xi \tau(p+2) \wedge \dots \wedge \xi(n))[\gamma] = 0$$

for all $\tau \in \mathcal{F}(p+2; n)$. Furthermore, if $z \in \wedge^{p+1} A$, then

$$\omega(\xi(1) \wedge \dots \wedge \partial \xi(i) \wedge \dots \wedge \xi(n))[z \wedge \gamma] = 0 \text{ for } i = 1, \dots, n.$$

(By $z \wedge \gamma$ we mean that we multiply the elements in $\wedge^{n-p-1} A$ by z .)

Proof. Consider

$$\omega(\xi(1) \wedge \dots \wedge \partial \xi \tau(1) \wedge \dots \wedge \partial \xi \tau(p+2) \wedge \dots \wedge \xi(n))[\alpha] \in \wedge^2 A \otimes E.$$

Then by our hypotheses on α , we have $\omega(\xi(k))\omega(\partial \xi \tau)[\alpha] = 0$ for $k = 1, \dots, n$ (we are here using the obvious device of denoting

$$\omega(\xi(1) \wedge \cdots \wedge \partial \xi \tau(1) \wedge \cdots \wedge \partial \xi \tau(p+2) \wedge \cdots \wedge \xi(n))$$

by $\omega(\partial \xi \tau)$, and shall continue to do so throughout). Thus by 2.3, $\omega(\partial \xi \tau)\omega(\eta)[\alpha]$ is in the kernel of f . Since we are assuming that $H_1(E(f)) = 0$, there is an $\alpha(\tau) \in \wedge^{n+1} A \otimes E$ such that

$$\begin{aligned} \omega(\partial \xi \tau)\omega(\eta)[\alpha] &= \omega(\xi(1) \wedge \cdots \wedge \partial \xi \tau(1) \wedge \cdots \wedge \partial \xi \tau(p+1) \wedge \cdots \\ &\quad \wedge \xi(n)) \omega(\xi \tau(1) \wedge \cdots \wedge \xi \tau(p+2))[\alpha(\tau)]. \end{aligned}$$

If we let $\gamma = \omega(\eta)[\alpha] - \sum_i \omega(\xi \tau(1) \wedge \cdots \wedge \xi \tau(p+2))[\alpha(\tau)]$, it is clear that γ behaves as our lemma says it should.

LEMMA 2.5. *If α is in $\wedge^{n+1} A \otimes E$, $\lambda(1), \dots, \lambda(s)$ are in R^{n*} , and $a(1), \dots, a(s-1)$ are in A , then $a(1) \wedge \cdots \wedge a(s-1) \wedge \omega(\lambda(1) \wedge \cdots \wedge \lambda(s))[\alpha]$ is a boundary in $\wedge^n A \otimes E$ (in the complex $E(\wedge f)$).*

Proof. This is clear for $s = 1$. Hence we proceed by induction on s .

$$\begin{aligned} &a(s-1) \wedge \omega(\lambda(1) \wedge \cdots \wedge \lambda(s))[\alpha] \\ &= (-1)^s \sum (-1)^i \omega(\lambda(1) \wedge \cdots \wedge \partial \lambda(s-i) \wedge \cdots \wedge \lambda(s))[\lambda(s-i)f(a(s-1))\alpha]. \end{aligned}$$

Therefore $a(1) \wedge \cdots \wedge a(s-2) \wedge (a(s-1) \wedge \omega(\lambda(1) \wedge \cdots \wedge \lambda(s))[\alpha])$ is a boundary by the induction hypothesis.

LEMMA 2.6. *Assume that $H_2(E(f)) = 0$. Let α in $\wedge^n A \otimes E$ be such that $\omega(\partial \xi i)[\alpha] = 0$ for $i = 1, \dots, n$ (here i denotes the function in $\mathcal{F}(1; n)$ taking the value i). Let y be in A , and η in A^* . Then $\omega(\eta)[y \wedge \alpha]$ is a boundary in $E(\wedge f)$.*

Proof. It is easy to see that $\omega(\xi(1) \wedge \cdots \wedge \xi(n))[y \wedge \alpha] = 0$. Thus $\xi(1) \wedge \cdots \wedge \xi(n) \otimes y \wedge \alpha$ is a two-cycle in $E(f)$ and hence a boundary, since $H_2(E(f)) = 0$. We thus see that $y \wedge \alpha = \sum_\sigma \sum_\sigma \omega(\xi \sigma)[\beta(\sigma)]$ with the summation running over $\sigma \in \mathcal{F}(s; n)$, $\beta(\sigma)$ in $\wedge^{n+s+1} A \otimes E$, and $\omega(\xi \sigma)$ meaning

$$\omega(\xi \sigma(1) \wedge \cdots \wedge \xi \sigma(s)).$$

Thus

$$\omega(\eta)[y \wedge \alpha] = \sum \sum (-1)^s \omega(\xi \sigma)\omega(\eta)[\beta(\sigma)]$$

so that in $E(\wedge f)$, $\omega(\eta)[y \wedge \alpha]$ is the image of $(-1)^s \sum \sum \xi \sigma \otimes \omega(\eta)[\beta(\sigma)]$, where $\xi \sigma = \xi \sigma(1) \wedge \cdots \wedge \xi \sigma(s)$.

LEMMA 2.7. *Assume that $A = R^m$ and that $H_1(E(f)) = H_2(E(f)) = 0$. If α is in $\wedge^{n-p} A \otimes E$ and is such that*

- (i) $\omega(\partial \xi \sigma)[\alpha] = 0$ for all $\sigma \in \mathcal{F}(p+1; n)$;
- (ii) $\omega(\partial \xi i)[y(p) \wedge \cdots \wedge y(1) \wedge \alpha] = 0$ for all $i = 1, \dots, n$;

then $y(p) \wedge \cdots \wedge y(1) \wedge \alpha$ is a boundary in $E(\wedge f)$. Here we are denoting by $y(1), \dots, y(m)$ a basis for R^m , and by $\eta(1), \dots, \eta(m)$ the dual basis for R^{m*} .

Proof. For $p = n - 1$, this is trivial. We then suppose it true for $p = n - q$, and prove it true for $p = n - (q + 1)$.

Since $\omega(\partial \xi i)[y(p) \wedge \cdots \wedge y(1) \wedge \alpha] = 0$, we have by 2.6 that $\omega(\eta(p + 1))[y(p + 1) \wedge y(p) \wedge \cdots \wedge y(1) \wedge \alpha]$ is a boundary in $E(\wedge f)$. But

$$\begin{aligned} \omega(\eta(p + 1))[y(p + 1) \wedge \cdots \wedge y(1) \wedge \alpha] &= y(p) \wedge \cdots \wedge y(1) \wedge \alpha \\ &\quad - y(p + 1) \wedge \cdots \wedge y(1) \wedge \omega(\eta(p + 1))[\alpha]. \end{aligned}$$

By 2.4, $\omega(\eta(p + 1))[\alpha] = \alpha' + \sum_{\tau} \omega(\xi \tau)[\alpha(\tau)]$ where α' satisfies the induction hypothesis, and $\tau \in \mathcal{F}(p + 2; n)$. Thus we have the result.

LEMMA 2.8. *If α in $\wedge^n A \otimes E$ is a one-cycle in $E(\wedge f)$, and if $H_1(E(f)) = 0$, then α is homologous to an α' such that $\omega(\partial \xi i)[\alpha'] = 0$ for all $i = 1, \dots, n$.*

Proof. Since α is a one-cycle, we have $\wedge^n f(\alpha) = 0$. Thus $f\omega(\partial \xi i)[\alpha] = \omega(\partial \xi i)[\wedge^n f(\alpha)] = 0$ for all $i = 1, \dots, n$. Thus $\omega(\partial \xi i)[\alpha] = \omega(\partial \xi i)\omega(\xi(i))[\alpha(i)]$ with $\alpha(i)$ in $\wedge^{n+1} A \otimes E$. If we let $\alpha' = \alpha - \sum \omega(\xi(i))[\alpha(i)]$, this α' will serve our purpose.

The above lemmas show us that if $f: R^m \rightarrow R^n$ is a map, and E is an R -module, then $H_1(E(f)) = H_2(E(f)) = 0$ implies that $H_1(E(\wedge f)) = 0$. In order to go beyond this point, it is convenient to introduce another complex whose relationship to the ones we have been considering will soon be made clear.

Let $g: B \rightarrow A$ be a map of modules. Then g induces maps $\wedge^p g: \wedge^p B \rightarrow \wedge^p A$ for every positive integer p . If β is in $\wedge^p B$ and α is in $\wedge^q A$, we will write $\beta \wedge \alpha$ instead of $\wedge^p g(\beta) \wedge \alpha$. Thus $\wedge A$ becomes a $\wedge B$ -module.

Letting $\wedge = \wedge B = \sum \wedge^p B$ ($p \geq 0$), and $\bar{\wedge} = \sum \wedge^p B$ ($p \geq 1$), we consider the familiar complex

$$\cdots \rightarrow \bigotimes_1^p \bar{\wedge} \otimes \wedge \xrightarrow{d(p)} \cdots \rightarrow \bar{\wedge} \otimes \bar{\wedge} \otimes \wedge \xrightarrow{d(2)} \bar{\wedge} \otimes_R \wedge \xrightarrow{d(1)} \wedge \xrightarrow{\varepsilon} R \rightarrow 0$$

where $\bigotimes_1^p \bar{\wedge}$ means the p -fold tensor product of $\bar{\wedge}$. The map ε is the obvious augmentation map (sending everything of positive degree into zero), and $d(p)$ is defined by:

$$\begin{aligned} d(p)(\lambda(1) \otimes \cdots \otimes \lambda(p) \otimes u) \\ &= \lambda(1) \otimes \cdots \otimes \lambda(p - 1) \otimes \lambda(p) \wedge u \\ &\quad + (-1)^{p+1} \sum_{i=1}^{p-1} (-1)^{i+1} \lambda(1) \otimes \cdots \otimes \lambda(i) \wedge \lambda(i + 1) \otimes \cdots \otimes \lambda(p) \otimes u. \end{aligned}$$

Tensoring this complex with $\wedge A$ over \wedge , we of course obtain $\bar{K}(g)$:

$$\cdots \rightarrow \bar{\wedge} \otimes \bar{\wedge} \otimes \wedge A \xrightarrow{d(2)} \bar{\wedge} \otimes \wedge A \xrightarrow{d(1)} \wedge A$$

where the boundary operators are the same as the above, except that u is now an element of $\wedge A$ instead of $\wedge B$. It is clear that the homology groups of this complex are $\text{Tor}^\wedge(R, \wedge A)$.

Now for every pair of positive integers (p, q) we obtain a subcomplex of this last complex as follows:

$$\bar{K}(g; p, q): \cdots \rightarrow \sum \wedge^{s(1)} B \otimes \wedge^{s(2)} B \otimes \wedge^{q-|s|} A \rightarrow \sum \wedge^s B \otimes \wedge^{-s} A \rightarrow \wedge^q A$$

where $s(1)$, $s \geq p$, $s(2) \geq 1$, and $|s| = s(1) + s(2)$.

Letting $B = R^n$ and $A = R^m$, we may consider the complexes $\bar{K}(g; 1, m-n)$ and $\bar{K}(g; n, m-1)$. If E is any R -module, we denote the complex $\bar{K}(g; 1, m-n) \otimes E$ by $\bar{E}(\wedge g)$ and the complex $\bar{K}(g; n, m-1)$ by $\bar{E}(g)$.

If $f: R^m \rightarrow R^n$ is a map, we obtain the map $f^*: R^{n*} \rightarrow R^{m*}$. Making the canonical identification of R^{n*} with R^n and of R^{m*} with R^m , we obtain the map $g: R^n \rightarrow R^m$. If E is any R -module, we have an isomorphism between the $(p+1)$ -chains of $E(f)$ (of $E(\wedge f)$) and the p -chains of $\bar{E}(g)$ (of $\bar{E}(\wedge g)$) which commutes with boundaries (i.e., a chain isomorphism of degree -1 between $E(f)$ and $\bar{E}(g)$ ($E(\wedge f)$ and $\bar{E}(\wedge g)$)). This isomorphism is obtained by making the usual identification between $\wedge^t R^s$ and $\wedge^{s-t} R^{s*}$ for any integers s and t .

We are now ready to start proving the main result of this section.

THEOREM 2.9. *Let $f: R^m \rightarrow R^n$ be a map, and E an R -module. If $H_i(E(f)) = 0$ for all $i \geq p$ ($p > 0$), then $H_i(E(\wedge f)) = 0$ for all $i \geq p$.*

It is clear that under the given hypotheses, we need only show that $H_p(E(\wedge f)) = 0$. As we observed at the conclusion of 2.8, we have already shown that if $p = 1$, we have $H_1(E(\wedge f)) = 0$. Hence we may assume $p \geq 2$. However, by the preceding discussion, we see that 2.9 then boils down to showing that (with the notations of the foregoing paragraphs) $H_i(\bar{E}(g)) = 0$ for all $i \geq p$ ($p \geq 1$) implies $H_p(\bar{E}(\wedge g)) = 0$. Since the module E plays a dummy role in all of the discussion, we will omit it throughout the rest of this section.

Let $g: R^n \rightarrow R^m$ be a map, let $x(1), \dots, x(n)$ be a basis for R^n , $y(1), \dots, y(m)$ a basis for R^m , and $\eta(1), \dots, \eta(m)$ the dual basis for R^{m*} . We are going to show that if $H_p(\bar{K}(g; n, q+n-1)) = H_{p+1}(\bar{K}(g; n, q+n-1)) = 0$ for any positive integer q , then $H_p(\bar{K}(q; 1, q)) = 0$, where $p > 0$.

First of all, let us observe that $\bar{K}(g; n, q+n-1)$ is nothing but $\bar{K}(g; 1, q-1)$ augmented by $\wedge^{q-1} A \rightarrow \wedge^{q+n-1} A$ where this map is multiplication by $x(1) \wedge \cdots \wedge x(n)$. (For the sake of notational convenience, we are setting $B = R^n$, and $A = R^m$.)

If $\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha$ is an element of $\bigwedge^{s(1)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^r A$, and if λ is any element of A^* , then $\omega(\lambda)[\beta(i)]$ and $\omega(\lambda)[\alpha]$ are defined and we define $\omega(\lambda)[\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha]$ as follows:

$$\begin{aligned} & \omega(\lambda)[\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha] \\ &= \omega(\lambda)[\beta(1)] \otimes \beta(2) \otimes \cdots \otimes \beta(p) \otimes \alpha \\ &+ \sum_{i=1}^{p-1} (-1)^{t(i)} \beta(1) \otimes \cdots \otimes \omega(\lambda)[\beta(i+1)] \otimes \cdots \otimes \beta(p) \otimes \alpha \\ &+ (-1)^{t(p)} \beta(1) \otimes \cdots \otimes \beta(p) \otimes \omega(\lambda)[\alpha] \end{aligned}$$

where $t(i) = s(1) + \cdots + s(i)$. We make the convention in this definition that $\beta(1) \otimes \cdots \otimes \omega(\lambda)[\beta(i)] \otimes \cdots \otimes \beta(p) \otimes \alpha$ is taken to be zero if $\beta(i)$ is of degree one, i.e., if $\beta(i) \in \bigwedge^1 B$.

If d denotes the boundary map in the complex $\bar{K}(g)$, then $\omega(\lambda)d = d\omega(\lambda)$, if the leading term is of degree greater than 1.

If y is an element of A , then we define $(\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha) \wedge y$ to be $\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha \wedge y$. It is then clear that

$$\begin{aligned} & \omega(\lambda)[(\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha) \wedge y] \\ &= (\omega(\lambda)[\beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha]) \wedge y \pm \lambda(y) \beta(1) \otimes \cdots \otimes \beta(p) \otimes \alpha. \end{aligned}$$

With these auxiliary operations, we are able to get to the proof of the theorem.

LEMMA 2.10. *With $g: R^n \rightarrow R^m$ as above, assume that $H_p(\bar{K}(g; n, q+n-1)) = H_{p+1}(\bar{K}(g; n, q+n-1)) = 0$. For each $\tau \in \mathcal{F}(r+2; n)$, let $\gamma(\tau)$ be an element of $\sum \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^{q-2-|s|} A$ ($s(i) \geq 1$, $|s| = \sum s(i)$) such that $d(\sum_{\tau} x\tau \otimes \gamma(\tau)) = 0$ where d is the boundary operator in $\bar{K}(g)$ and $x\tau$ stands for $x(\tau(1)) \wedge \cdots \wedge x(\tau(r+2))$. Then $\omega(\eta(1) \wedge \cdots \wedge \eta(r))[\sum_{\tau} x\tau \otimes \gamma(\tau)]$ is a p -cycle of $\bar{K}(g; 1, q)$ which is actually a p -boundary in $\bar{K}(g; 1, q)$. (We of course define $\omega(\eta(1) \wedge \cdots \wedge \eta(r))$ to be $(-1)^{r+1} \omega(\eta(1)) \wedge \cdots \wedge \omega(\eta(r))$.)*

Proof. Since the computations involved in this proof are unfortunately quite tedious, we will just outline here the basic steps required. We first make the rather spurious remark that for $r > n-2$, the theorem is obviously true, so that we may proceed by downward induction on r .

That the element in question is a p -cycle somewhere follows from the fact that the boundary map commutes with $\omega(\eta)$. Using the purely formal fact that for α in $\bigwedge^q B$ and β in $\sum \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^t A$, $\omega(\lambda(1) \wedge \cdots \wedge \lambda(r))[\alpha \otimes \beta] = \sum_{s=0}^r (-1)^{sq} \sum_{\sigma} (-1)^{|\sigma|} \omega(\partial \lambda \sigma)[\alpha] \otimes \omega(\lambda \sigma)[\beta]$ for any $\lambda(1), \dots, \lambda(r)$ in A^* , where $\sigma \in \mathcal{F}(s; r)$, it is not difficult to show that the element we are considering is actually a p -chain in $\bar{K}(g; 1, q)$.

If we denote by γ the element $\sum_{\tau} x\tau \otimes \gamma(\tau)$ ($\tau \in \mathcal{F}(r+2; n)$), and by γ' the element $\omega(\eta(1) \wedge \cdots \wedge \eta(r))[\gamma]$, it is again clear that $\omega(\eta(r+1))[\gamma']$ is a

$(p+1)$ -cycle in $\bar{K}'(g; 1, q-1)$, where $\bar{K}'(g; 1, q-1)$ denotes the complex $\bar{K}(g; 1, q-1)$ augmented by the map $\bigwedge^{q-1} A \rightarrow \bigwedge^{q+n-1} A$ (defined by multiplication by $x(1) \wedge \cdots \wedge x(n)$). Hence $\omega(\eta(r+1))[\gamma']$ is a boundary by our assumption (since $H_{p+1}(\bar{K}'(g; 1, q-1)) = H_{p+1}(\bar{K}(g; n, q+n-1))$). Thus it is easy to show that $\omega(\eta(r+1))[\gamma'] \wedge y(r+1)$ is a p -boundary in $\bar{K}(g; 1, q)$. But $\omega(\eta(r+1))[\gamma' \wedge y(r+1)] = \omega(\eta(r+1))[\gamma'] \wedge y(r+1) \pm \gamma'$. Therefore it remains to show that $\omega(\eta)[\gamma' \wedge y]$ is a p -boundary in $\bar{K}(g; 1, q)$.

Now since $\gamma(\tau) \in \sum \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^{q-2-|s|} A$, we have that $\gamma(\tau) \wedge y$ is a p -chain in $\bar{K}'(g; 1, q-1)$. Also, since $d(\gamma) = \sum x\tau \otimes d\gamma(\tau) + (\text{other terms involving leading terms of higher degree}) = 0$, we have $d\gamma(\tau) = 0$, so that $d(\gamma(\tau) \wedge y) = 0$. Thus $\gamma(\tau) \wedge y$ is a p -cycle in $\bar{K}'(g; 1, q-1)$ and hence a p -boundary. We may therefore write $\gamma(\tau) \wedge y = d(\sum_k x(k) \otimes \gamma(k, \tau))$ where $\gamma(k, \tau)$ is also in $\sum \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^{q-2-|s|} A$. Letting

$$\gamma'(\sigma) = \sum (-1)^{r+i+1} \gamma(k(i), \partial\sigma(i))$$

for $\sigma \in \mathcal{F}(r+3; n)$, where $\partial\sigma(i) \in \mathcal{F}(r+2; n)$ is defined by omitting the i th vertex from σ , we see that

$$d(\sum_{\sigma} x\sigma \otimes \gamma'(\sigma)) = 0.$$

Now consider the element $\beta = \sum x\tau \otimes \sum x(k) \otimes \gamma(k, \tau)$. Then $\omega(\eta)\omega(\eta(1) \wedge \cdots \wedge \eta(r))[\beta]$ is a $(p+1)$ -chain in $\bar{K}(g; 1, q)$. Now with the observation that $\omega(\eta)[\gamma' \wedge y] = \omega(\eta)\omega(\eta(1) \wedge \cdots \wedge \eta(r))[\gamma \wedge y]$, it is not too difficult to see that subtracting the boundary of $\omega(\eta)\omega(\eta(1) \wedge \cdots \wedge \eta(r))[\beta]$ from $\omega(\eta)[\gamma' \wedge y]$ leaves $\omega(\eta(1) \wedge \cdots \wedge \eta(r+1))(\sum x\sigma \otimes \gamma'(\sigma))$. But by our induction hypothesis, this is a boundary so the proof of 2.10 is complete.

LEMMA 2.11. *Let $g: R^n \rightarrow R^m$ be as in 2.10, and suppose that $H_p(\bar{K}(g; n, q+n-1)) = 0$. If α is any p -cycle of $\bar{K}(g; 1, q)$, then α is homologous to a cycle of the form $\sum_{\rho} x\rho \otimes \alpha(\rho)$ with $\rho \in \mathcal{F}(2; n)$, and*

$$\alpha(\rho) \in \sum \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^{q-2-|s|} A.$$

Proof. It is very clear from the nature of the boundary map that α is homologous to a cycle of the form $\sum x(i) \otimes \alpha(i)$ with $\alpha(i)$ in $\sum \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^{q-1-|s|} A$. Therefore we may assume that α itself is of this form. Since $\sum x(i) \otimes \alpha(i)$ is a cycle, we have $d(\alpha(i)) = 0$. But this means that $\alpha(i)$ is a p -cycle in $\bar{K}'(g; 1, q-1)$; thus $\alpha(i)$ is a boundary and we may write

$$\alpha(i) = d(\sum x(k) \otimes \beta(k, i)) \text{ with } \beta(k, i) \in \bigwedge^{s(2)} B \otimes \cdots \otimes \bigwedge^{s(p)} B \otimes \bigwedge^{q-2-|s|} A.$$

Then the element $\sum x(i) \otimes \sum x(k) \otimes \beta(k, i)$ is a $(p+1)$ -chain in $\bar{K}(g; 1, q)$ whose boundary is $\sum x(i) \otimes \alpha(i) + \sum x\rho \otimes \alpha(\rho)$ where $\alpha(\rho) = \beta(\rho(2), \rho(1)) - \beta(\rho(1), \rho(2))$. Thus α is homologous to an element of the prescribed form.

Now applying Lemma 2.10 to the case $r = 0$, we have a proof of 2.9.

We state the following proposition without proof. Although a direct proof could now be given, a less computational proof can be obtained from some results in a subsequent paper with D. S. Rim.

PROPOSITION 2.12. *Let $f: R^m \rightarrow R^n$ be a map, and E an R -module. Let $a(1), \dots, a(n)$ be in R^m , and $\beta(1), \dots, \beta(n)$ be in R^{n*} . Then*

$$\omega(\beta(1) \wedge \dots \wedge \beta(n))[a(1) \wedge \dots \wedge a(n)]$$

is an element in R which annihilates $H_p(E(f))$ and $H_p(E(\wedge f))$ for all $p \geq 0$.

PROPOSITION 2.13. *Let $f: R^m \rightarrow R^n$ be a map, and let α be the ideal generated by the determinants of the $n \times n$ minors of the matrix corresponding to f . If $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is an exact sequence of R -modules, we have*

(a) $0 \rightarrow H_s(E'(f)) \rightarrow H_s(E(f)) \rightarrow H_s(E''(f)) \rightarrow H_{s-1}(E'(f)) \rightarrow \dots \rightarrow H_0(E''(f)) \rightarrow 0$ is exact, where $s = m - n + 1$;

(b) $0 \rightarrow H_s(E'(\wedge f)) \rightarrow H_s(E(\wedge f)) \rightarrow H_s(E''(\wedge f)) \rightarrow H_{s-1}(E'(\wedge f)) \rightarrow \dots \rightarrow H_0(E''(\wedge f)) \rightarrow 0$ is exact;

(c) $\alpha H(E(f)) = 0 = \alpha H(E(\wedge f))$.

Proof. (a) and (b) are trivial. The statement (c) is an obvious consequence of 2.12.

3. E -regularity. Throughout this section, the canonical basis of R^n will be denoted by $x(1), \dots, x(n)$, and that for R^m will be denoted by $y(1), \dots, y(m)$. If $p < m$, then R^p will be imbedded in R^m in the canonical way, so that when $y(1), \dots, y(m)$ is the basis for R^m , $y(1), \dots, y(p)$ is the basis for R^p . If $f: R^m \rightarrow R^n$ is a map, we will denote by $f(i)$ the restriction of f to R^{n+i-1} , where $1 \leq i \leq m - n + 1$.

DEFINITION. Let $f: R^m \rightarrow R^n$ be a map, and E an R -module. We will say that f is E -regular if $H_1(E(f(i))) = 0$ for $1 \leq i \leq m - n + 1$. We assume throughout that $m \geq n$.

REMARK. In the next section, we will see that if R is a local ring, the E -regularity of a map does not depend on the choice of the basis.

PROPOSITION 3.1. *Let E be an R -module, and $f: R^m \rightarrow R^n$ an E -regular map. Then $H_q(E(f)) = 0$ for all $q \neq 0$.*

Proof. We proceed by induction on $m - n$. If $m - n = 0$, there is nothing to prove. Suppose, then, that $m - n = s + 1$, and that the proposition is true for $m - n = s$. It is clear that $f(s+1): R^{m-1} \rightarrow R^n$ is an E -regular map. By our induction hypothesis, $H_q(E(f(s+1))) = 0$ for all $q \neq 0$ and by 2.9 we have that $H_q(E(\wedge f(s+1))) = 0$ for all $q \neq 0$. Since $E(f)$ is the mapping cylinder of $E(\wedge f(s+1)) \rightarrow E(f(s+1))$ by 2.1, we have the exact sequence:

$$\begin{aligned} \cdots \rightarrow H_q(E(f(s+1))) &\rightarrow H_q(E(f)) \rightarrow H_{q-1}(E(\wedge f(s+1))) \\ &\rightarrow H_{q-1}(E(f(s+1))) \rightarrow H_{q-1}(E(f)) \rightarrow \cdots \end{aligned}$$

It then follows immediately that $H_q(E(f)) = 0$ for all $q \neq 0$.

REMARK. It should be noted that the notion of E -regularity is not only a generalization of the notion of E -sequence, but actually corresponds to a fairly familiar notion. Namely, suppose $E = R$, and that R is a field. Then a map $f: R^m \rightarrow R^n$ being R -regular simply says that f has maximum rank, and that the kernel of f (the null-space) is generated by the usual vectors obtained by Cramer's rule.

DEFINITION. A map $f: R^m \rightarrow R^n$ is called *properly E -regular* if it is E -regular and $H_0(E(f)) \neq 0$.

4. Local rings. Throughout this section, R will denote a local ring (i.e., a noetherian ring with only one maximal ideal) and \mathfrak{m} will denote its maximal ideal. All modules will be assumed finitely generated.

If $f: R^m \rightarrow R^n$ is a map, we use the same conventions as in the preceding section to define the maps $f(i): R^{n+i-1} \rightarrow R^n$ for $1 \leq i \leq m - n + 1$.

PROPOSITION 4.1. *Let E be an R -module, and $f: R^m \rightarrow R^n$ a map such that $f(R^m) \subset \mathfrak{m}R^n$. If for some $p > 0$, we have that $H_p(R(f)) = 0$, then $H_q(E(f(i))) = 0$ for all i such that $0 \leq i \leq m - n + 1$, and all $q \geq p$.*

Proof. Since we are assuming $m \geq n$, we may write $m = n + s$, and proceed by induction on s . We may also assume that $p > 0$, for otherwise we have $E = 0$ (since $E^n \subset \mathfrak{m}E^n$ implies $E = 0$).

If $s = 0$, there is really nothing to prove. Hence we consider the case $s = t + 1$, $t \geq 0$. In this case, we have $f: R^{n+t+1} \rightarrow R^n$, and $f(s): R^{n+t} \rightarrow R^n$, and we recall that $E(f)$ is the mapping cylinder of a map of $E(\wedge f(s))$ into $E(f(s))$. In fact, the map of $E(\wedge f(s))$ into $E(f(s))$ is obtained in the following way.

Using the notation of §3, we have $f(y(m)) = f(y(n+s)) = \sum a(j)x(j)$, with $a(j)$ in \mathfrak{m} . We thus obtain a map $\rho: R \rightarrow R^n$ defined by $\rho(1) = \sum (-1)^{j+1} a(j)x(j)$, and thereby the element $\xi(\rho) = \sum a(j)\xi(1) \wedge \cdots \wedge \partial \xi(j) \wedge \cdots \wedge \xi(n)$ in $\wedge^{n-1} R^{n*}$. Using this element $\xi(\rho)$, we obtain the map of $E(\wedge f(s))$ into $E(f(s))$ as described in §1.

A straightforward computation shows that because $H_p(E(f)) = 0$ and all $a(j)$ are in \mathfrak{m} , we have $Z_p(E(f(s)))$ is contained in $B_p(E(f(s))) + \mathfrak{m}Z_p(E(f(s)))$ (where B and Z stand for boundaries and cycles, respectively). Since everything in sight is finitely generated, we see that $H_p(E(f(s))) = 0$. But $f(s)(R^{n+t}) \subset \mathfrak{m}R^n$, so we may apply the inductive hypothesis to $f(s)$; since the acyclicity of $E(f(s))$ from a certain point on implies the acyclicity of $E(\wedge f(s))$ from that same point on, we obtain the complete result.

PROPOSITION 4.2. *Let E be a nonzero R -module, and $f: R^m \rightarrow R^n$ a map such that $f(R^m) \subset \mathfrak{m}R^n$. Then the following statements are equivalent:*

- (a) *f is properly E -regular.*
- (b) *$H_p(E(f)) = 0$ for all $p > 0$.*
- (c) *$H_1(E(f)) = 0$.*

Proof. Clearly (a) implies (b) and (b) implies (c). That (c) implies (a) follows from 4.1 (the fact that f is properly E -regular follows from the fact that $f(R^m) \subset \mathfrak{m}R^n$ and that $E \neq 0$).

COROLLARY 4.3. *If $f: R^m \rightarrow R^n$ and E are as above, and if f is properly E -regular, then so is $f(\pi)$ where π is any permutation of $\{1, \dots, m\}$ and $f(\pi)$ is defined by: $f(\pi)[y(i)] = f(y(\pi(i)))$.*

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