OKA'S HEFTUNGSLEMMA AND THE LEVI PROBLEM FOR COMPLEX SPACES

BY

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Introduction. This paper has its origin in an analysis of K. Oka's solution of the Levi problem for unramified domains over C^n [11]. It consists of two parts. The first part contains a new proof of the following theorem [9].

THEOREM 1. *Let X be a complex space and p a continuous strongly pseudoconvex function on X such that for any $\alpha > 0$ the set*

$$\{x \in X \mid p(x) < \alpha\}$$

is relatively compact in X. Then X is a Stein space.

The proof given here is based on a refined version of the Heftungslemma of Oka [11] which we give in §2, and avoids the consideration of completely continuous mappings between Fréchet spaces [4]. It also avoids a detour through the following result, which was required in the older proof.

A holomorph-convex K-complete, space is a Stein space.

In the second part, we consider the question of the extent to which the assumption of *strong* convexity can be dropped. Theorem 1 becomes false if we merely drop the assumption of *strong* convexity as shown by an example of H. Grauert (see [10a] where this example is given). In this direction we prove.

THEOREM 2. *On a K-complete space, any relatively compact open set which is pseudoconvex with a globally defined boundary is a Stein space.*

We next turn to Stein spaces and prove (Theorem 4, Corollary 1) that *on a Stein space with isolated singularities, an open set which is locally a Stein space is also globally a Stein space.*

This we obtain as a special case of a theorem on Stein spaces whose singularities are not necessarily isolated. The particular case when X has no singularities has already been treated by Docquier-Grauert. We have used in the proof the following result (Theorem 5).

A complex space all of whose irreducible components are Stein spaces is itself a Stein space.

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This is a special case of a theorem [10] which asserts that *a complex space whose normalisation is a Stein space is itself a Stein space.* However, we have given a direct proof for the case which we need.

In the final section (§6) we have collected together some observations. We show that for complex manifolds, the statement that a holomorph-convex, K-complete manifold is a Stein manifold can be deduced from a "vanishing theorem" on noncompact kähler manifolds [2] proved by Andreotti and Vesentini. Finally, we apply Theorem 1 to the proof of the following two facts [14; 15].

An unramified covering manifold of a Stein open set in C^n is a Stein manifold.

A holomorphic vector bundle over a Stein manifold is itself a Stein manifold.

1. **Preliminaries.** All complex spaces considered are supposed countable at infinity.

Let X be a complex space. We say X is *holomorph-convex* if for any infinite discrete set $E \subset X$, there is a holomorphic function f on X for which $f(E)$ is unbounded. A holomorph-convex space X for which the following two properties hold is called a *Stein space*: (i) if $x \neq y$ are points of X, there is a holomorphic function f on X with $f(x) \neq f(y)$; (ii) for any $x \in X$, there exist $f_1, \cdots, f_k$ holomorphic on X which map a neighborhood of x isomorphically onto an analytic set in an open set in C^k.

A complex space X is called *K-complete* if for every $x_0 \in X$ there is a holomorphic map f of X into C^p, $p = p(x_0)$, such that x_0 is an isolated point of $f^{-1}f(x_0)$.

It is known [Grauert, Math. Ann. **129** (1955), 233–259] that a pure dimensional space X of dimension n is K-complete if and only if X can be realised as a ramified domain over C^n, but we will not require this theorem.

We refer to [9, §§1, 2] for the definition and elementary properties of Runge domains on Stein spaces and pseudoconvex and strongly pseudoconvex functions on complex spaces (called there convex and strongly convex functions respectively). We will use frequently the following result without explicit mention (see [15]).

(1.1) *The union of an increasing sequence of Stein spaces, each of which is a Runge open subset of the succeeding one, is itself a Stein space.*

The following theorem is proved in [9, p. 211].

(1.2) *Let X be a Stein space and p a continuous pseudoconvex function on X. Then the set*

$$X_0 = \{x \in X \mid p(x) < 0\}$$

is Runge in X. (In particular X_0 is a Stein space.)

We remark that the proof of (1.2) was based on an argument, which gives the following result without any change whatsoever.

(1.3) *Let X be a Stein space and K a compact subset of X. Let $\hat{K}$ be the set $\{x \in X \mid |f(x)| \leqq \sup_{y \in K} |f(y)|$ for all f holomorphic on $X\}$. Let $x_0 \in \hat{K} - K$. Let U be a neighborhood of x_0 in X and $f(x,t)$ a continuous function for $x \in U, 0 \leqq t \leqq 1$ which is holomorphic in U for any t. Set $\sigma_t = \{x \in U \mid f(x,t) = 0\}$, $C = \bigcup_{0 \leqq t \leqq 1} \sigma_t$.*

Suppose that there exists a neighborhood V of $\hat{K}$ such that $C \cap V$ is closed in V while $K \cap C = \emptyset$ and $x_0 \in \sigma_0$. Then there exist arbitrarily small $t > 0$ such that $\sigma_t \cap \hat{K} \neq \emptyset$(1).

From this result we deduce the following one.

(1.4) *Let D be an open subset of the Stein space X. Suppose that ∂D $(= \bar{D} - D)$ has a neighborhood U in which there exists a holomorphic function g such that $C = \{u \in U \mid \operatorname{Re} g(u) = 0\}$ is a closed subset of X such that $\partial D \subset C$, $C \cap D = \emptyset$. Then D is a Runge domain in X (in particular, D is a Stein space).*

Proof of (1.4). Let K be a compact subset of D and $\hat{K}$ its envelope in X (i.e., the set of $x \in X$ where $|f(x)| \leqq \sup_{y \in K} |f(y)|$ for all f holomorphic on X). We assert that $\hat{K} \cap C = \emptyset$. Suppose that, on the contrary, $C_1 = \hat{K} \cap C \neq \emptyset$ and let $s = \sup_{x \in C_1} \operatorname{Im} g(x)$. Put $f(x,t) = g(x) - is - it$ in U, $0 \leqq t \leqq 1$, $\sigma_t = \{x \in U \mid f(x,t) = 0\}$. Then $\sigma_t \subset C$, so that $\bigcup \sigma_t$ is a closed subset of C (hence of X) which does not meet $K \subset D$. Moreover, $\sigma_t \cap \hat{K} = \emptyset$ for $t > 0$. Hence, by (1.3), we have $\sigma_0 \cap \hat{K} = \emptyset$ which is false.

Thus $\hat{K} \cap \partial D = \emptyset$. Since every connected component of $\hat{K}$ meets K [8, p. 919], it follows that $\hat{K} \subset D$. q.e.d.

We will need the following result (see proof of statement (2) on p. 334 in Grauert [5]).

(1.5) Let p be a C^∞ strongly pseudoconvex function in an open set $G \subset C^n$, $K \subset G$ a compact set. Then there exist numbers r, $\delta > 0$ such that, for $z \in K$, the ball H of radius $2r$ about z lies in G, while there exists in H a holomorphic function f such that

$$\{x \in H \mid \operatorname{Re} f(x) = 0\} \cap \{y \in G \mid p(y) \leqq p(z)\} = \{z\}$$

and, if $x \in H$ satisfies $\operatorname{Re} f(x) = 0$, $|x - z| = r$, then we have $p(x) \geqq p(z) + \delta$.

We may suppose also that $\operatorname{Re} f(y) < 0'$ if $p(y) < p(z)$.

2. "Heftungslemmata."

(2.1) *First Heftungslemma.* Let X be a complex space, let $h_1, \cdots, h_k$ be holomorphic functions on X and g a holomorphic function on X which is nowhere zero. We consider the sets:

$$Y = \{x \in X \mid \ |h_i(x)| < 1 \text{ for } 1 \leqq i \leqq k\}$$

and, for every $\delta > 0$,

(1) Essentially (1.3) is proved by G. Stolzenberg [16] for compact sets in C^n. Stolzenberg also remarked that the proof of (1.3) becomes much simpler if $f(x,t)$ is supposed, in addition, to be analytic in x and t. This case is sufficient for the results that we need in this paper.

$$Y_1(\delta) = \{x \in Y \mid |g(x)| < e^{\delta}\},$$

$$Y_2(\delta) = \{x \in Y \mid |g(x)| > e^{-\delta}\}.$$

We have

$$Y = Y_1(\delta) \cup Y_2(\delta).$$

We put

$$Y_3(\delta) = Y_1(\delta) \cap Y_2(\delta).$$

LEMMA 1. *Let us assume that for a suitable* $\delta > 0$

(i) $Y_3(\delta)$ *is relatively compact in* X;

(ii) *the functions* $h_1, \cdots, h_k, g$ *give local coordinates and separate points of* $Y_3(\delta)$.

Then for every holomorphic function f *on* $Y_3(\delta)$ *there exist holomorphic functions* f_1 *on* $Y_1(\delta)$ *and* f_2 *on* $Y_2(\delta)$ *such that on* $Y_3(\delta)$ *we have*:

$$f = f_1 - f_2.$$

Proof. Consider the map $\phi : X \to C^{k+1}$ given by

$$x \to (g(x), h_1(x), \cdots, h_k(x)).$$

Consider the polyhedron in $C^{k+1} = C \times C^k$:

$$P = \{(z, w) \in C \times C^k \mid e^{-\delta} < |z| < e^{\delta}, \|w\| < 1\}.$$

Then ϕ maps $Y_3(\delta)$ on a subset A of P. By conditions (i), (ii) the map $\phi|_{Y_3(\delta)}$ is proper. Hence A is an analytic subset of P isomorphic to $Y_3(\delta)$.

The function f defines therefore a holomorphic function $\tilde{f}$ on A. Since P is a product of $k + 1$ open sets in the complex plane, it is a domain of holomorphy. Therefore we can find a holomorphic function F on P such that $F|_A = \tilde{f}$.

Let $(z, w) \in P$. Then for every $0 < \varepsilon < \varepsilon(z, w)$ (sufficiently small) we can write

$$F(z,w) = \frac{1}{(2\pi i)^{k+1}} \int_{|\zeta| = e^{\delta} - \varepsilon} \frac{d\zeta}{\zeta - z} \int_{|\eta_1| = 1 - \varepsilon} \frac{d\eta_1}{\eta_1 - w_1} \cdots \int_{|\eta_k| = 1 - \varepsilon} \frac{F(\zeta, \eta)}{\eta_k - w_k} d\eta_k$$

$$- \frac{1}{(2\pi i)^{k+1}} \int_{|\zeta| = e^{-\delta} + \varepsilon} \frac{d\zeta}{\zeta - z} \int_{|\eta_1| = 1 - \varepsilon} \frac{d\eta_1}{\eta_1 - w_1} \cdots \int_{|\eta_k| = 1 - \varepsilon} \frac{F(\zeta, \eta)}{\eta_k - w_k} d\eta_k.$$

The first of these integrals defines a holomorphic function $F_1(z, w, \varepsilon)$ on the set $|z| < e^{\delta} - \varepsilon$, $\|w\| < 1 - \varepsilon$. If $\varepsilon' < \varepsilon''$ the corresponding functions $F_1(z, w, \varepsilon')$ and $F_1(z, w, \varepsilon'')$ coincide in the common region of definition. Therefore we obtain a holomorphic function $F_1(z, w)$ defined on the set

$$P_1 = \{(z, w) \in C \times C^k \mid |z| < e^{\delta}, \|w\| < 1\}.$$

Analogously from the second of the above integrals, we obtain a holomorphic function $F_2(z, w)$ on the set

$$P_2 = \{(z,w) \in C \times C^k \mid e^{-\delta} < |z|, \ \|w\| < 1\}$$

and by the above expression of F we have on $P = P_1 \cap P_2$

$$F = F_1 - F_2.$$

Since $Y_1(\delta) = \phi^{-1}(P_1)$, $Y_2(\delta) = \phi^{-1}(P_2)$ it is enough to take $f_1 = F_1 \circ \phi$, $f_2 = F_2 \circ \phi$ on $Y_1(\delta)$, $Y_2(\delta)$ respectively to get the result.

COROLLARY. *If $Y_1(\delta)$ and $Y_2(\delta)$ are Stein spaces we then have: $H^1(Y, \mathcal{O}) = 0$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X, and, for any coherent sheaf $\mathcal{F}$ on Y, $H^i(Y, \mathcal{F}) = 0$ if $i \geqq 2$.*

(2.2) *Second Heftungslemma.* This is our version of the central idea of Oka [11]. The essential point is to replace the assumption that g, h_i are globally defined in Lemma 1 above by more geometric conditions.

Let g be a *continuous* function nowhere zero on the complex space X. We suppose that g is *holomorphic* in the set

$$X_3(\delta) = \{x \in X \mid e^{-\delta} < |g(x)| < e^{+\delta}\}$$

for some $\delta > 0$. For any $\rho < \delta$, we set

$$X_1(\rho) = \{x \in X \mid |g(x)| < e^{+\rho}\}, \quad X_2(\rho) = \{x \in X \mid |g(x)| > e^{-\rho}\}$$

We suppose that

(i) *$X_1(\delta)$, $X_2(\delta)$ are Stein spaces such that $X_3(\delta)$ is Runge in each of them.*

Let $h_1, \cdots, h_k$ be holomorphic in $X_3(\delta')$ (where $\delta' > \delta$) such that

(ii) *there is $\varepsilon_0 > 0$ such that $|h_i(x)| < 1 - \varepsilon_0$ if $|g(x)| = e^{\pm\delta}$.*

For $0 \leqq \varepsilon \leqq \varepsilon_0$, $0 < \rho < \delta$, we set

$$Y_1(\varepsilon, \rho) = [X_1(\delta) - X_3(\delta)] \cup \{x \in X_3(\delta) \mid |h_i(x)| < 1 - \varepsilon, |g(x)| < e^{+\rho}\}$$

and similarly, replacing $X_1(\delta)$ by $X_2(\delta)$ and $\{|g| < e^{+\rho}\}$ by $\{|g| > e^{-\rho}\}$, we define the set $Y_2(\varepsilon, \rho)$. Clearly, because of (ii), $Y_j(\varepsilon, \rho)$ is an open set for $j = 1,2$. Set $Y_3(\varepsilon, \rho) = Y_1(\varepsilon, \rho) \cap Y_2(\varepsilon, \rho)$ and $Y(\varepsilon) = Y_1(\varepsilon, \rho) \cup Y_2(\varepsilon, \rho)$. We suppose now that

(iii) *for some ρ_0, $0 < \rho_0 < \delta$, the set $Y_3(0, \rho_0)$ is relatively compact in X and the functions $g, h_1, \cdots, h_k$ separate points and give local coordinates on it.*

LEMMA 2. *Under the assumptions* (i), (ii), (iii) *made above, if f is a holomorphic function in $X_3(\delta)$, then for any $\rho < \rho_0$, and $0 < \varepsilon < \varepsilon_0$, there exist holomorphic functions f_r in $Y_r(\varepsilon, \rho)$ $(r = 1,2)$ such that*

$$f_1 - f_2 = f \text{ in } Y_3(\varepsilon, \rho).$$

Moreover, for any compact sets $K_r \subset Y_r(\varepsilon, \rho)$ $(r = 1,2)$, there exist constants $M_r > 0$, independent of f such that

$$\|f_r\|_{K_r} \leqq M_r \|f\|_{Y_3(\varepsilon,\rho)}.$$

Proof. Let $\phi : X_3(\delta) \to C^{k+1}$ be defined by $\phi(x) = (g(x), h_1(x), \cdots, h_k(x))$. Because of (iii), ϕ maps $Y_3(0, \rho_0)$ isomorphically onto an analytic set A of the polyhedron P_{0,ρ_0}, where, for $0 \leqq \varepsilon < \varepsilon_0$, $0 < \rho \leqq \delta$, we set

$$P_{\varepsilon,\rho} = \{(z,w) \in C^{k+1} \,|\, e^{-\rho} < |z| < e^{\rho},\ |w_\nu| < 1 - \varepsilon \text{ for } 1 \leqq \nu \leqq k\}.$$

By assumption (iii), the set $Y_3(\varepsilon, \rho)$, $\rho \leqq \rho_0$, is relatively compact in $X_3(\delta)$, so that

$$\| f \|_{Y_3(\varepsilon,\rho)} = \sup |f(y)| \text{ for } y \in Y_3(\varepsilon, \rho)$$

is finite.

Since $P_{\varepsilon,\rho}$ is a domain of holomorphy, for $\varepsilon < \varepsilon'$, $\rho' < \rho$ there exists [6, §8] a constant $M = M(\varepsilon', \rho') > 0$ independent of f, and a holomorphic function F in $P_{\varepsilon,\rho}$, such that $F \circ \phi = f$ on $Y_3(\varepsilon, \rho)$, with

$$\| F \|_{P_{\varepsilon',\rho'}} \leqq M \| f \|_{Y_3(\varepsilon,\rho)}. \tag{1}$$

Let $(z, w) \in P_{\varepsilon',\rho'}$. Then we have

$$F(z,w) = \frac{1}{(2\pi i)^{k+1}} \int_{|\eta_\nu| = 1-\varepsilon'} \cdots \int \left[\int_{|\zeta| = e^{\rho'}} - \int_{|\zeta| = e^{-\rho'}} \right] \frac{F(\zeta,\eta)}{(\zeta - z)\prod(\eta_\nu - w_\nu)} d\zeta d\eta_1 \cdots d\eta_k.$$

Hence if we set

$$\chi(x,\zeta,\eta) = (2\pi i)^{-k-1}(\zeta - g(x))^{-1} \prod (\eta_\nu - h_\nu(x))^{-1},$$
$$\Gamma_1 = \{(\zeta,\eta) \in C^{k+1} |\ |\zeta| = e^{\rho'},\ |\eta_\nu| = 1 - \varepsilon' \text{ for } 1 \leqq \nu \leqq k\},$$
$$\Gamma_2 = \{(\zeta,\eta) \in C^{k+1} |\ |\zeta| = e^{-\rho'},\ |\eta_\nu| = 1 - \varepsilon', \text{ for } 1 \leqq \nu \leqq k\},$$

we have $f(x) = (\int_{\Gamma_1} - \int_{\Gamma_2}) \chi(x,\zeta,\eta) F(\zeta,\eta) d\zeta d\eta$, for $x \in Y_3(\varepsilon', \rho')$.

Let V_1 be the open set in X defined by

$$V_1 = \{x \in X \,|\, e^{+\delta} - \alpha < |g(x)| < e^{+\delta} + \alpha,\ |h_\nu(x)| < 1 - \varepsilon_0 \text{ for } 1 \leqq \nu \leqq k\}.$$

Analogously, replacing $e^{+\delta}$ by $e^{-\delta}$ we define an open set V_2. We set $V = V_1 \cup V_2$. For $x \in V$ and (ζ, η) satisfying $e^{\pm\rho'} - \alpha < |\zeta| < e^{\pm\rho'} + \alpha, 1 - \varepsilon' - \alpha < |\eta_\nu| 1 - \varepsilon' + \alpha$, if α is sufficiently small the function $\chi(x, \zeta, \eta)$ is holomorphic because of assumption (ii). *Thus there exist Stein neighborhoods N_j of Γ_j such that χ is holomorphic in $V \times (N_1 \cup N_2)$.*

Consider the meromorphic function χ in $X_3(\delta) \times N_j$, 0 in $T_j \times N_j$ where T_j is the open set $X_j(\delta) - \overline{(X_3(\delta) - V)}$ $(j = 1,2)$. By the above remark, χ is holomorphic in $(X_3(\delta) \cap T_j) \times N_j$. Hence, since $X_j(\delta) \times N_j$ is a Stein space, there exists a meromorphic function χ_j in $X_j(\delta) \times N_j$ holomorphic in $T_j \times N_j$, such that $\chi_j - \chi$ is holomorphic in $X_3(\delta) \times N_j$. Now, by assumption (i), for any $\beta > 0$, there is a holomorphic function ϕ_j on $X_j(\delta) \times N_j$ such that

(2) $$|\phi_j - (\chi - \chi_j)| < \beta \text{ on } Y_3(0,\rho_0) \times \Gamma_j.$$

Let $\omega_j = \chi - \chi_j - \phi_j$. Then,

$$\int_{\Gamma_j} \chi F d\zeta d\eta = \int_{\Gamma_j} (\chi_j + \phi_j) F d\zeta d\eta + \int_{\Gamma_j} \omega_j F d\zeta d\eta.$$

Since $\chi_j + \phi_j$ is holomorphic on $Y_j(\varepsilon',\rho')$ for any $(\zeta,\eta) \in \Gamma_j$, we see that the first integral defines a holomorphic function $f_j^{(1)}$ on $Y_j(\varepsilon',\rho')$. Moreover, if $\psi_j' = \int_{\Gamma_j} \omega_j F d\zeta d\eta$, ψ_j' is holomorphic on $X_3(\delta)$ and

(3) $$\text{for } x \in Y_3(0,\rho_0),\ |\psi_j'(x)| \leqq \beta M (2\pi)^{k+1} e^{\rho'} (1-\varepsilon')^k \|f\|_{Y_3(\varepsilon,\rho)}.$$

If $\theta' = \beta M (2\pi)^{k+1} e^{\rho'} (1-\varepsilon')^k$, we may suppose, by suitable choice of β, that $\theta' < \frac{1}{2}$. Now, we have, for $x \in Y_3(\varepsilon',\rho')$,

$$f(x) = f_1^{(1)}(x) - f_2^{(1)}(x) + f^{(1)}(x)$$

where $f^{(1)}(x) = \psi_1'(x) - \psi_2'(x)$. Moreover,

(4) $$\|f^{(1)}\|_{Y_3(0,\rho_0)} \leqq \theta \|f\|_{Y_3(\varepsilon,\rho)}, \quad \theta = 2\theta' < 1.$$

We now proceed with $f^{(1)}$ as we did with f to construct $F^{(1)}$ on $P_{\varepsilon,\rho}$ such that $F^{(1)} \circ \phi = f^{(1)}$ on $Y_3(\varepsilon,\rho)$, while $\|F^{(1)}\|_{P_{\varepsilon',\rho'}} \leqq M \|f^{(1)}\|_{Y_3(\varepsilon,\rho)}$. Then, if

$$f_j^{(2)} = \int_{\Gamma_j} (\chi_j + \phi_j) F^{(1)} d\zeta\, d\eta$$

(this being holomorphic on $Y_j(\varepsilon',\rho')$), we have

$$f^{(1)} = f_1^{(2)} - f_2^{(2)} + f^{(2)}$$

where $f^{(2)}$ is holomorphic in $X_3(\delta)$ and

$$\|f^{(2)}\|_{Y_3(0,\rho_0)} \leqq \theta \|f^{(1)}\|_{Y_3(0,\rho_0)} \leqq \theta^2 \|f\|_{Y_3(\varepsilon,\rho)}.$$

Continuing this process, we obtain sequences of holomorphic functions $f^{(p)}$ in $X_3(\delta)$ and $F^{(p)}$ in $P_{\varepsilon,\rho}$, such that

$$\|f^{(p)}\|_{Y_3(0,\rho_0)} \leqq \theta^p \|f\|_{Y_3(\varepsilon,\rho)},$$
$$\|F^{(p)}\|_{P_{\varepsilon',\rho'}} \leqq M\, \theta^p \|f\|_{Y_3(\varepsilon,\rho)}$$

while if $f_j^{(p)} = \int_{\Gamma_j} (\chi_j + \phi_j) F^{(p)} d\zeta d\eta$, we have, in $Y_3(\varepsilon',\rho')$,

$$f = \sum_{q=1}^{p} (f_1^{(q)} - f_2^{(q)}) + f^{(p)}.$$

As $p \to \infty$, $f^{(p)} \to 0$ in $Y_3(0,\rho_0)$, and if

$$f_j = \int_{\Gamma_j} (\chi_j + \phi_j) \sum_{p=1}^{\infty} F^{(p)} d\zeta\, d\eta,$$

we have the following properties:

$$f = f_1 - f_2, \text{ in } Y_3(\varepsilon', \rho').$$

f_j is holomorphic in $Y_j(\varepsilon', \rho')$. For any compact set $K_j \subset Y_j(\varepsilon', \rho')$, we have $\|f_j\|_{K_j} \leqq \sup_{K_j} |\chi_j + \phi_j| \cdot (M\theta/(1-\theta)) \|f\|_{Y_3(\varepsilon,\rho)} \leqq M_j \|f\|_{Y_3(\varepsilon,\rho)}$ say. This completes the proof of Lemma 2.

(2.3) *Third Heftungslemma.* Let X be a complex space and g a holomorphic function on X which is nowhere zero. For any $\delta > 0$ we define

$$X_1(\delta) = \{x \in X \mid |g(x)| < e^{\delta}\},$$

$$X_2(\delta) = \{x \in X \mid |g(x)| > e^{-\delta}\}$$

so that

$$X = X_1(\delta) \cup X_2(\delta).$$

LEMMA 3. *Suppose that for $\delta = \delta_0$ there exists a C^∞ strongly pseudoconvex function p_i on any relatively compact set $Y_i \subset\subset X_i(\delta_0)$, $i = 1, 2$. Then on any relatively compact set $Y \subset\subset X$ there exists a C^∞ strongly pseudoconvex function on Y.*

Proof. Let U be an open subset of X such that

$$Y \subset\subset U \subset\subset X.$$

Since $U \cap X_i(c\delta_0) \subset\subset X_i(\delta_0)$ there exists a C^∞ strongly pseudoconvex function p_i on $U \cap X_i(c\delta_0)$, $1/2 < c < 1$. Consider the function

$$q_1 = |g|^{-1} - e^{-\delta_0/2} + \varepsilon p_1$$

on the set $U \cap X_1(c\delta_0)$. Since g^{-1} is holomorphic on X, q_1 is a C^∞ strongly pseudoconvex function for every $\varepsilon > 0$. If ε is sufficiently small, the set

$$W_1 = \{x \in U \cap X_1(c\delta_0) \mid q_1(x) > 0\}$$

contains the set $Y \cap X_1(\delta_0/3)$, since on this set

$$|g|^{-1} - e^{-\delta_0/2} > e^{-\delta_0/3} - e^{-\delta_0/2} \text{ and } Y \cap X_1(\delta_0/3) \subset\subset U \cap X_1(\delta_0/2).$$

Moreover on the boundary of $W_1 \cap Y$ in Y, q_1 is defined and equal to zero, since, if ε is sufficiently small, q_1 is negative on the set $\{x \in X \mid |g| = e^{(c-\varepsilon)\delta_0}\} \cap U$.

Consider for any constant $\lambda > 0$ the function ϕ_1 on Y defined by

$$\phi_1(x) = \begin{cases} e^{-\lambda/q_1(x)} & \text{for } x \in W_1 \cap Y, \\ 0 & \text{for } x \in Y - (W_1 \cap Y). \end{cases}$$

This is a C^∞ function on Y which is pseudoconvex and also strongly pseudoconvex on $Y \cap X_1(\delta_0/3)$, if $\lambda > 2 \sup\{q_1(x), x \in Y \cap X_1(\delta_0/3)\}$.

Using the functions g, p_2 instead of g^{-1}, p_1 we construct similarly a C^∞ function $\phi_2(x)$ on Y which is pseudoconvex and also strongly pseudoconvex on $Y \cap X_2(\delta_0/3)$. Then the function $\phi_1 + \phi_2$ is C^∞ and strongly pseudoconvex on Y.

3. **Levi problem.** The object of this section is a proof of the following theorem [9].

THEOREM 1. *Let X be a complex space and p a continuous strongly pseudoconvex function on X such that for any $\alpha > 0$ the set*

$$X_\alpha = \{x \in X \mid p(x) < \alpha\}$$

is relatively compact in X. Then X is a Stein space.

Because of [9, Lemma 5], we may suppose that p is locally the maximum of finitely many C^∞ strongly pseudoconvex functions. We assume this in what follows. The proof then consists of two parts.

(3.1) *If X_α is a Stein space for some α, then there is $\alpha' > \alpha$ such that $X_{\alpha'}$ is a Stein space.*

(3.2) *If $\alpha_\nu < \alpha$, $\alpha_\nu \to \alpha$ as $\nu \to \infty$ and X_{α_ν} is a Stein space for each ν, then X_α is also a Stein space.*

For the proof of (3.1) we need the following lemma.

LEMMA 4. *Let X be a complex space and p a function with the properties given above. Suppose in addition there is a continuous complex-valued function f on X such that for some $\rho > 0$ the sets*

$$X_1(\rho) = \{x \in X \mid \operatorname{Re} f < +\rho\}, \quad X_2(\rho) = \{x \in X \mid \operatorname{Re} f > -\rho\}$$

are Stein spaces, while f is holomorphic in $X_3(\rho) = X_1(\rho) \cap X_2(\rho)$. Then X itself is a Stein space.

Proof. For any $\alpha > 0$, consider the set

$$\{x \in X \mid p(x) < \alpha\} = X_\alpha.$$

It follows from (1.5) that there exists $\beta > \alpha$ such that for any $x_0 \in \partial X_\alpha$ there is an analytic set A_{x_0} in X_β and a holomorphic function f_{x_0} in a neighborhood U_{x_0} of A_{x_0} in X_β satisfying the conditions

$$A_{x_0} \cap \bar{X}_\alpha = \{x_0\}, \quad A_{x_0} = \{x \in U_{x_0} \mid f_{x_0}(x) = 0\}.$$

For $0 < \delta \leqq \rho$, let $\Omega_j(\delta) = X_j(\delta) \cap X_\beta$, and let $\Omega_j^\alpha = \Omega_j(\rho) \cap X_\alpha$. Since $\Omega_3(\rho)$ is Stein, there is, for x_0 in $\Omega_3(c\rho) \cap \partial X_\alpha$, $1/3 < c < 1/2$, a meromorphic function ψ_{x_0} in $\Omega_3(\rho)$, holomorphic in $\Omega_3(\rho) - A_{x_0}$, for which $\psi_{x_0} - 1/f_{x_0}$ is holomorphic in $U_{x_0} \cap \Omega_3(\rho)$. Now, ψ_{x_0} is bounded on the set $\Omega_3^\alpha \cap \{x \in X_3(\rho) \mid |e^f| = e^{\pm \rho/2}\}$ and on any compact subset of Ω_3^α, while $|\psi_{x_0}(x)| \to \infty$ as $x \to x_0, x \in X_\alpha$. It follows that given any compact subset K of X_α, we can construct finitely many holomorphic functions $h_1,\cdots,h_k$ on Ω_3^α having the following properties. Put $g = e^f$ in $X_3(\rho)$. Then

(a) There is an $\varepsilon_0 > 0$ such that

$$|h_\nu(x)| < 1 - \varepsilon_0 \text{ for } x \in \Omega_3^\alpha \cap \{y \in X_3(\rho) \mid |g(y)| = e^{\pm\rho/2}\}.$$

(b) If $0 \leqq \varepsilon < \varepsilon_0$, the set

$$Y(\varepsilon) = (X_\alpha - \Omega_3^\alpha) \cup \{x \in \Omega_3^\alpha \mid |h_\nu(x)| < 1 - \varepsilon,\ \nu = 1,\cdots,k\}$$

contains K.

(c) The set $\Omega_3(\rho/3) \cap Y(0)$ is relatively compact in X and the functions $h_1,\cdots,h_k$ separate points and give local coordinates on it.

We can apply now the second Heftungslemma to the space X_α since by (1.2) and (1.4), the open sets $\Omega_1^\alpha,\Omega_2^\alpha$ are Stein and Ω_3^α is Runge in each of them. Hence,

(*) for any compact sets $C_j \subset \Omega_j^\alpha$, there exists a compact subset C of Ω_3^α such that the following result holds: there exist constants $M > 0$ and $\delta > 0$ such that for any holomorphic function h in Ω_3^α, there exist holomorphic functions h_j $(j = 1,2)$ in $Y(\varepsilon) \cap \Omega_j(\delta)$, such that $h_1 - h_2 = h$ on $Y(\varepsilon) \cap \Omega_3(\delta)$, while $\| h_j \|_{C_j} \leqq M \| h \|_C$.

We assert now that for any $\gamma > 0$, the set X_γ is Stein. To prove this, let $\alpha > \gamma$ be so chosen that for any $x_0 \in \partial X_\gamma$, there exists an analytic set B_{x_0} in X_α, a neighborhood U_{x_0} of B_{x_0} in X_α and a holomorphic function g_{x_0} in U_{x_0} such that $B_{x_0} \cap \bar{X}_\gamma = \{x_0\}$ and $B_{x_0} = \{x \in U_{x_0} \mid g_{x_0}(x) = 0\}$. Let ψ_j be meromorphic in Ω_j^α and such that ψ_j is holomorphic in $\Omega_j^\alpha - B_{x_0}$, $\psi_j - 1/g_{x_0}$ is holomorphic in $U_{x_0} \cap \Omega_j^\alpha$; finally let $h = \psi_1 - \psi_2$ in Ω_3^α. Clearly h is holomorphic in Ω_3^α.

We now place ourselves in the situation of the Heftungslemma in such a way that $Y(\varepsilon) \supset \bar{X}_\gamma$ and find functions h_j in $\Omega_j(\delta) \cap Y(\varepsilon)$ with $h_1 - h_2 = h$. Then the function ψ in $Y(\varepsilon)$ defined to be $\psi_j - h_j$ in $Y(\varepsilon) \cap \Omega_j(\delta)$ is meromorphic in $Y(\varepsilon)$, holomorphic in $Y(\varepsilon) - B_{x_0}$, and $\psi - 1/g_{x_0}$ is holomorphic in $U_{x_0} \cap Y(\varepsilon)$. Clearly the restriction of ψ to X_γ is a holomorphic function and $|\psi(x)| \to \infty$ as $x \to x_0$. Since $x_0 \in \partial X_\gamma$ is arbitrary, *X_γ is holomorph-convex.*

Let $x_1 \neq x_2$ be two points of X_γ and suppose first that both are points of Ω_1^α. Let ϕ be holomorphic in Ω_1^α and $\phi(x_1) \neq \phi(x_2)$. Take $C_1 = \{x_1\} \cup \{x_2\}$ and let $C \subset \Omega_3^\alpha$ be as in (*) above. Let ϕ' be holomorphic in Ω_2^α and, with $h = \phi - \phi'$, let $\| h \|_C < \varepsilon$ (such a ϕ' exists since Ω_3^α is Runge in Ω_2^α). As above, there are holomorphic functions h_j on $Y(\varepsilon) \cap \Omega_j^\alpha$ with $\| h_1 \|_{C_1} < M\varepsilon$, $h_1 - h_2 = h$ in $Y(\varepsilon) \cap \Omega_3(\delta)$. The function ψ defined to be $\phi - h_1$ in $Y(\varepsilon) \cap \Omega_1(\delta)$, $\phi' - h_2$ in $Y(\varepsilon) \cap \Omega_2(\delta)$ is holomorphic in $Y(\varepsilon)$ $(\supset X_\gamma)$ and $\| \psi - \phi \|_{C_1} < M\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we can choose it such that $\psi(x_1) \neq \psi(x_2)$. In the case $x_1 \in \Omega_1^\alpha$, $x_2 \in \Omega_2^\alpha$, we take $C_1 = \{x_1\}$, $C_2 = \{x_2\}$ and the compact set $C \subset \Omega_3^\alpha$ as in (*) above. We then find a holomorphic function ϕ_j in Ω_j^α such that $|\phi_1(x_1) - \phi_2(x_2)|$

$\geqq 1$ and such that $\| \phi_1 - \phi_2 \|_C < \varepsilon$. Then, as above, we can find a holomorphic function ψ on $Y(\varepsilon)$ such that $\| \psi(x_j) - \psi_j(x_j) \| < M\varepsilon$. Clearly if $\varepsilon < 1/M$, $\psi(x_1) \neq \psi(x_2)$. A similar argument applies when $x_1, x_2 \in \Omega_2^\alpha$. Thus we conclude that *holomorphic functions separate points of* X_γ.

The proof that holomorphic functions on X_γ give local coordinates at each point is similar and so is omitted.

This completes the proof that X_γ is a *Stein space for any* $\gamma > 0$.

By (1.1) and (1.2), we conclude that X is a Stein space, which proves Lemma 4.

We now go to the proof of (3.1).

Let α be such that X_α is a Stein space. Now, by (1.5), there is $\beta > \alpha$ such that if $p(x_0) = \alpha$, there exists a Stein neighborhood U_{x_0} and a holomorphic function ϕ in U_{x_0} such that

(i) $B = \{x \in U_{x_0} \mid \operatorname{Re} \phi(x) = 0\}$ is closed in X_β;

(ii) $B \cap Y_\alpha = \{x_0\}$ where $Y_\alpha = \{x \in X \mid p(x) \leqq \alpha\}$.

We may further suppose that if $x \in U_{x_0} \cap Y_\alpha$, then $\operatorname{Re} \phi(x) \leqq 0$. Let now β', $\alpha < \beta' < \beta$ be fixed.

If $c < 0$ is sufficiently near to zero, then we have

(a) $B_c = \{x \in U_{x_0} \mid \operatorname{Re} \phi(x) = c\}$ is closed in $X_{\beta'}$;

(b) $V_c = \{x \in U_{x_0} \mid \operatorname{Re} \phi(x) > c\}$ is a neighborhood of x_0 such that $\operatorname{Re} \phi(x) = c$ if $x \in \partial V_c \cap X_{\beta'}$.

Let $c < d < 0$; the set V_d is also a neighborhood of x_0.

We now find points $x_0, \cdots, x_p \in \{x \in X \mid p(x) = \alpha\}$ and corresponding sets V_{c_ν}, V_{d_ν} as above (the corresponding functions being ϕ_ν) such that the V_{d_ν} form a covering of $\{x \in X \mid p(x) = \alpha\}$. We find C^∞ function l_ν with compact support contained in $V_{d_\nu} \cap X_{\beta'}$, such that $p \pm l$ is strongly convex in $X_{\beta'}$, l being any finite sum of the l_ν, while $\sum_{\nu=0}^p l_\nu(x) > 0$ for any x with $p(x) = \alpha$.

Consider the space

$$D_q = \{x \in X \mid p(x) < \alpha + l_0(x) + \cdots + l_q(x)\},$$

$q = -1, \cdots, p$. *Suppose that* D_q *is a Stein space.* We assert that *then* D_{q+1} *is also a Stein space.* To prove this we proceed as follows. We define the function f_{q+1} in D_{q+1} in the following way: $\operatorname{Re} f_{q+1}(x) = c_{q+1}$ for x in $D_{q+1} - U_{x_{q+1}}$, $\operatorname{Re} f_{q+1}(x) = \max(c_{q+1}, \operatorname{Re} \phi_{q+1}(x))$ for $x \in U_{x_{q+1}}$, while $\operatorname{Im} f_{q+1}$ is any continuous extension to D_{q+1} of the restriction of $\operatorname{Im} \phi_{q+1}$ to $D_{q+1} \cap V_{d_{q+1}}$. Then f_{q+1} is holomorphic on the set $\{x \in D_{q+1} \mid c_{q+1} < \operatorname{Re} f_{q+1} < d_{q+1}\}$. Now, since D_q is a Stein space the set $\{x \in D_{q+1} \mid \operatorname{Re} f_{q+1} < d_{q+1}\} = \{x \in D_q \mid \operatorname{Re} f_{q+1} < d_{q+1}\}$ is a Stein space (by (1.4)) while, since $U_{x_{q+1}}$ is a Stein space, the set $\{x \in D_{q+1} \mid \operatorname{Re} f_{q+1} > c_{q+1}\} = \{x \in U_{x_{q+1}} \mid \operatorname{Re} f_{q+1} > c_{q+1}\}$ is a Stein space (again by (1.4)). Moreover, in D_{q+1}, there is a strongly pseudoconvex function P with $\{x \in D_{q+1} \mid P(x) < r\} \subset\subset D_{q+1}$ for any $r > 0$ (in fact, we may take $P(x) = (\alpha - p(x) + l_0(x) + \cdots + l_{q+1}(x))^{-1}$). Hence, by Lemma 4, D_{q+1} is Stein.

Since by supposition, $D_{-1} = X_\alpha$ is Stein, it follows that D_p is Stein. Since D_p is a neighborhood of Y_α, we may choose $\alpha' > \alpha$ so that $X_{\alpha'} \subset D_p$. Then, by (1.2), $X_{\alpha'}$, is Stein. This completes the proof of (3.1).

Proof of (3.2). This is a simple consequence of the Runge theorem (1.2) and (1.1).

Proof of Theorem 1. Let E be the set of real α for which X_α is Stein (E is nonempty since it contains all α for which $X_\alpha = \emptyset$). Clearly, if $\alpha \in E$ and $\gamma < \alpha$, then $\gamma \in E$. By (3.1), E is open. By (3.2), E is closed. Hence E is the whole real line. Hence, for any α, X_α is Stein. Again by the Runge theorem, X itself is Stein.

4. **Pseudoconvex domains in K-complete spaces.** Let D be a relatively compact open set on the complex space X. D is called *pseudoconvex* if to any $x_0 \in \partial D$, there is a neighborhood U_{x_0} in X and a continuous pseudoconvex function p_{x_0} in U_{x_0} with

$$D \cap U_{x_0} = \{x \in U_{x_0} \mid p_{x_0}(x) < 0\}.$$

It is called a pseudoconvex domain with *globally defined boundary* if there is a neighborhood U of ∂D and a continuous pseudoconvex function p in U such that $D \cap U = \{x \in U \mid p(x) < 0\}$. The object of this section is to prove the following

THEOREM 2. *A relatively compact open set D on the K-complete complex space X which is pseudoconvex with globally defined boundary is a Stein space.*

Before giving the proof of Theorem 2, we prove the following lemma, which is of independent interest.

LEMMA 5. *Let X be a K-complete space and $D \subset\subset X$ an open subset. Then, there exists a C^∞ strongly pseudoconvex function in D.*

The proof is based on

LEMMA 6. *Let X, Y be two locally compact spaces and $\phi : X \to Y$ a continuous map with discrete fibres. Let K be a compact set in X and $W_1, \cdots, W_k$ a finite open covering of K in X. Then there exists a finite open covering $V_1, \cdots, V_p$ of $\phi(K)$ in Y such that every connected component of $\phi^{-1}(V_\nu)$ which meets K is contained in some W_μ.*

Proof. It is enough to prove that for any $y \in \phi(K)$, there is an open neighborhood V_y of y such that any connected component of $\phi^{-1}(V_y)$ which meets K is contained in some W_μ. Let $x_1, \cdots, x_q$ be the points of K with $\phi(x_j) = y$ and let K' be a compact neighborhood of K containing no other point of $\phi^{-1}(y)$. Let U_j be an open neighborhood of x_j with $U_j \subset W_\mu \cap K'$ for some μ, $U_j \cap U_{j'} = \emptyset$ if $j \neq j'$. Now $C = \phi(\partial K') \cup \phi(K' - \bigcup_{j \leq q} U_j)$ is a compact subset of Y not containing y. Let V_y be an open neighborhood of y not meeting C. Then

$\phi^{-1}(V_y) \cap K' \subset \bigcup_{j \leqq q} U_j$, and $\phi^{-1}(V_y) \cap \partial K' = \varnothing$. Hence any connected component of $\phi^{-1}(V_y)$ meeting K (or K') is contained in $\bigcup U_j$, and since the U_j are disjoint, in one of them. Since each U_j is contained in some W_μ, Lemma 6 is proved.

Proof of Lemma 5. We may suppose that there is a holomorphic map $\phi : X \to C^N$ with discrete fibres. Let $\{W_1, \cdots, W_k\}$ be a finite covering of $\bar{D}$ by open sets in each of which there exists a C^∞ strongly convex function. Let $V_1, \cdots, V_p$ be an open covering of $\phi(\bar{D})$ in C^N such that each connected component of $\phi^{-1}(V_\nu) \subset W_\mu$ for some μ. We divide C^N into closed cubes of side $\varepsilon > 0$: if $z_j = x_j + ix_{N+j}$, consider the cube (1) $K_{n,\varepsilon} = \{\varepsilon n_j \leqq x_j \leqq \varepsilon(n_j + 1)\}$, $n = (n_1, \cdots, n_{2N}), n_j$ integers. For sufficiently small ε, if $K_{n,\varepsilon} \cap \phi(\bar{D}) \neq \varnothing$, then $K_{n,\varepsilon} \subset V_r$ for some r.

Let U be a neighborhood of $\bar{D}$ contained in $\bigcup W_\mu$. Let $K_{n'}$ be the union of the $K_{n,\varepsilon}$ with fixed $n' = (n_2, \cdots, n_{2N})$, which meet $\phi(\bar{D})$. Then, by successive applications of the Heftungslemma Lemma 3, we see that there is a neighborhood $V_{n'}$ of $K_{n'}$ such that any relatively compact subset of $\phi^{-1}(V_{n'}) \cap U$ carries a C^∞ strongly convex function. If $K_{n''}$ is the union of the sets $K_{n'}$ with fixed $n'' = (n_3, \cdots, n_{2N})$, we see, in the same way that there is a neighborhood $V_{n''}$ of $K_{n''}$ such that any relatively compact subset of $\phi^{-1}(V_{n''}) \cap U$ carries a C^∞ strongly convex function. Repeating this process, there is a neighborhood V of $\phi(\bar{D})$ for which any relatively compact subset of $\phi^{-1}(V) \cap U$ (in particular D) carries a C^∞ strongly pseudoconvex function, which gives Lemma 5.

It is now simple to give the proof of Theorem 2.

Proof of Theorem 2. Let X be K complete, $D \subset\subset X$ and U a neighborhood of ∂D in which there exists a continuous pseudoconvex function p with $U \cap D = \{x \in U \mid p(x) < 0\}$. We assert that there is a neighborhood V of $\bar{D}$ and a continuous convex function p' in V with $D = \{x \in V \mid p'(x) < 0\}$. In fact let $D' \subset\subset D$ and $\partial D' \subset U$; let $-\delta = \operatorname{Sup} p(x)$ for $x \in \partial D' \cap U$. Then clearly $-\delta < 0$. Let $p'(x) = -\delta$ in D', $p'(x) = \max(p(x), -\delta)$ for $x \in U - D'$. We may take $V = D \cup U$. Because of Lemma 5, there is a strongly pseudoconvex function q in D. Let

$$P(x) = -\frac{1}{p'(x)} + e^{q(x)} \text{ for } x \in D.$$

Then $P(x)$ is strongly convex and for any r, the set $\{x \in D \mid P(x) < r\} \subset\subset D$. By Theorem 1, D is Stein.

From Theorem 2 and the Runge theorem (1.2) we deduce

THEOREM 3. *Let X be a K-complete complex space. If there exists a continuous pseudoconvex function p on X such that for any $\alpha > 0$ the set $\{x \in X \mid p(x) < \alpha\}$ is relatively compact, then X is Stein.*

5. **Pseudoconvex domains in Stein spaces.** In Theorem 1, we had to sup-

pose that the boundary of the domain D in X was globally defined. For domains spread over C^n, the method of Oka provides a method to reduce the case of arbitrary pseudoconvex domains to this. Since we shall require his result for our next theorem, we state it as a separate lemma. The proof is contained in [11]; see also [3].

LEMMA 7. *Let X be a complex space of pure dimension n and $\phi : X \to C^n$ a holomorphic map. Let U be the set of points of X where ϕ is a local isomorphism, and let $D \subset\subset X$, be an open set contained in U. Suppose that every point of ∂D has a neighborhood W in X such that $W \cap D$ is Stein. Then, if for $x \in D$, $d(x)$ denotes the "radius" of the largest connected open subset of D which is isomorphic, by ϕ, to a polycylinder about $\phi(x)$, then $-\log d(x)$ is a pseudoconvex function in D.*

(Here "polycylinder" means a set of the form $|z_i - z_i^{(0)}| < r$, r not depending on i.)

THEOREM 4. *Let X be a Stein space, S the set of its singular points. Let D be an open subset of X such that every point of ∂D has a neighborhood W in X for which $W \cap D$ is Stein. Moreover, suppose that there is a neighborhood U of $S \cap \partial D$ such that $D \cap U$ is a Stein space.*

Then D itself is a Stein space.

Proof. We will first prove the theorem under the additional assumption that X is *irreducible*; suppose that $\dim_C X = n$.

We may suppose D to be relatively compact in X. In fact if Δ is any Runge domain in X, $\Delta \subset\subset X$, it is enough to prove that $D' = \Delta \cap D$ is Stein. But D' satisfies the same conditions as D.

We assert that for any U' with $D \subset U' \subset\subset X$, there are finitely many holomorphic maps

$$f^{(\nu)} : X \to C^n, \qquad \nu = 1, \cdots, k,$$

and holomorphic functions g_ν on X such that if we set $A_\nu = \{x \in X \mid g_\nu(x) = 0\}$, the following conditions are fulfilled: (i) each $f^{(\nu)}$ has discrete fibres in X, (ii) $f^{(\nu)} : U' - A_\nu \to C^n$ is a local isomorphism and (iii) $\bigcap_{\nu=1}^k A_\nu \cap U' = S \cap U'$.

Since each analytic set in X is the set of common zeros of finitely many holomorphic functions, it is sufficient to find holomorphic mappings $f^{(\nu)}$ and analytic sets A_ν satisfying (i), (ii), (iii).

To do this, we note first the following:

Given finitely many points $x_1, \cdots, x_p \in X - S$, there is a holomorphic map $f : X \to C^n$ with discrete fibres in X such that the jacobian determinant j_f of f at x_ν is $\neq 0$, $\nu = 1, \cdots, p$. This is an immediate consequence of the fact that the holomorphic maps of X into C^n with discrete fibres in X are dense in the space of all holomorphic maps of X into C^n

To complete the construction of $f^{(\nu)}, A_\nu$, suppose that these are already constructed for $\nu \leqq l-1$ and choose one point $x_r, r = 1, \cdots, p$, from each irreducible component of $A_1 \cap \cdots \cap A_{l-1}$ which meets $U' - S$; and construct $f^{(l)}: X \to C^n$ so that it has discrete fibres, while $j_{f^{(l)}}(x_r) \neq 0$; and define A_l to be the closure in X of the set of points $x \in X - S$ where $j_{f^{(l)}}(x) = 0$. A_l is an analytic set by [10, Satz 16], and since

$$\dim(A_1 \cap \cdots \cap A_{l-1} \cap (U' - S)) \leqq \dim(A_1 \cap \cdots \cap A_l \cap (U' - S)) - 1,$$

the mappings $f^{(\nu)}$, $\nu \leqq n+1$, satisfy our requirements.

We define $U_\nu = \{x \in U' \mid g_\nu(x) \neq 0\}$, $D_\nu = D \cap U_\nu$. Now, the pair $(U_\nu, f^{(\nu)})$ is an unramified domain over C^n, and we denote by $d_\nu(x)$ the polycylindrical distance of x from the boundary of U_ν (see statement of Lemma 7). Let $\delta_\nu(x)$, $x \in D_\nu$, denote the distance of x from the boundary of D_ν. Then we have

(a) $\delta_\nu(x) \leqq d_\nu(x)$ for $x \in D_\nu$,

(b) $-\log \delta_\nu(x)$ is pseudoconvex in D_ν (Lemma 7).

We now assert that the following two results hold.

(5.1) *There is a neighborhood V_ν of $A_\nu \cap D$ such that for $x \in V_\nu \cap D_\nu$, we have*

$$\delta_\nu(x) = d_\nu(x).$$

(5.2) *There is an integer $h > 0$ and a constant $C > 0$ such that for any $x \in D_\nu$, we have*

$$|g_\nu(x)| \leqq C \, \{d_\nu(x)\}^{1/h}.$$

Proof of (5.1). We have only to show that for any $x_0 \in A_\nu \cap D$, there is a neighborhood W such that for $x \in W - A_\nu$, we have $\delta_\nu(x) = d_\nu(x)$. Now choose a connected neighborhood $W_1 \subset D$ which is a ramified covering, by means of $f^{(\nu)}$, of a polycylinder of radius r about $z_0 = f^{(\nu)}(x_0)$ and which contains no other point x_1 with $f^{(\nu)}(x_1) = z_0$. Let W be the inverse image in W_1 of the polycylinder of radius $\frac{1}{2}r$ about z_0. If $x \in W - A_\nu$, the "maximal polycylinder about x in U_ν" cannot contain x_0, so has radius $< \frac{1}{2}r$ and so is contained in $W_1 \subset D$. Hence, for $x \in W - A_\nu$, we have $d_\nu(x) = \delta_\nu(x)$.

Proof of (5.2). For $x_0 \in A_\nu \cap \bar{D}$, let $W = W(x_0)$ be a connected, relatively compact open set, containing no other point of x, with $f^{(\nu)}(x_1) = f^{(\nu)}(x_0)$ and such that $f^{(\nu)} | W$ is a ramified covering of a polycylinder Z of radius r in C^n about $z_0 = f^{(\nu)}(x_0)$. For the sake of simplicity, we drop the indices ν in the rest of the proof of (5.2). We denote by Z_1 the polycylinder of radius $\frac{2}{3}r$ about z_0.

Let W' be the inverse image in W of the polycylinder Z' about z_0 of radius $\frac{1}{3}r$, and let $S(x)$, for $x \in W' - A$, be the maximal polycylinder (of radius $d(x)$) about x in $X - A$. Then $\partial S(x) \cap A \neq \emptyset$, and $f(S(x))$ is contained in the polycylinder Z_1.

For $x \in W$ and $z = f(x)$, $w = g(x)$ $(= g_\nu(x))$ satisfies an equation

$$w^p + a_1(z) w^{p-1} + \cdots + a_p(z) = 0, \tag{1}$$

where the a_i are bounded holomorphic functions on Z. Moreover for $z',z'' \in Z_1$, we have an inequality

$$|a_i(z') - a_i(z'')| \leqq C_1 |z' - z''|.$$

Suppose now that $x \in W' - A$ and $\xi \in S(x), f(x) = z, f(\xi) = \zeta$. Then $z, \zeta \in Z_1$, and from the equation (1) satisfied by g in $S(x)$ it follows that (2)

$$|g(x) - g(\xi)| \leqq C_2 \sup_{0 \leqq t \leqq 1, i} |a_i(z) - a_i(tz + (1-t)\zeta)|^{1/p}$$

$$\leqq C_3 |z - \zeta|^{1/p} \leqq C_4 [d(x)]^{1/p}.$$

If we let ξ tend to a point of $\partial S(x) \cap A$, then $g(\xi)$ tends to zero and we obtain, for $x \in W' - A$,

$$|g(x)| \leqq C_4 [d(x)]^{1/p}.$$

The proof of (5.2) follows immediately from this.

Consider on D_ν the function

$$p'_\nu(x) = -\log \delta_\nu(x) + h \log |g_\nu(x)| - \log C,$$

where C, h satisfy (5.2). Now, because of remark (b) before (5.1) $p'_\nu(x)$ is pseudoconvex in D_ν. By (5.1) and (5.2), there is a neighborhood V_ν of $A_\nu \cap D$ such that if $x \in V_\nu \cap D_\nu$, we have $p'_\nu(x) \leqq 0$. Hence the function

$$p_\nu(x) = \begin{cases} \max(p'_\nu(x), 0) & \text{for } x \in D_\nu, \\ 0 & \text{for } x \in V_\nu \end{cases}$$

is a *pseudoconvex function in* D, and if $x_0 \in \partial D - A_\nu$, then $p_\nu(x) \to \infty$ as $x \in D, x \to x_0$. Hence, if $p(x) = \max p_\nu(x)$, then p is pseudoconvex in D and $p(x) \to \infty$ if $x \to x_0 \notin \bigcap A_\nu \cap \partial D = S \cap \partial D$. Also, if $x \in S \cap D$, we have $p(x) = 0$. To complete the proof of Theorem 4 in the special case, we have only to show that for any $\alpha > 0$, the set

$$D_\alpha = \{x \in D \mid p(x) < \alpha\}$$

is a Stein space.

Suppose that U is a neighborhood of $\partial D \cap S$ such that $V = U \cap D$ is Stein. Let $s(x)$ be a pseudoconvex function in V such that for any $r > 0$, the set $\{x \in V | s(x) < r\} \subset\subset V$. Let $\Omega \subset\subset U$ be a neighborhood of $\partial D \cap S$. Now $\bar{D}_\alpha \cap \partial D \subset \partial D \cap S$ since $p(x) \to \infty$ if $x \to x_0 \in \partial D$, $x_0 \notin S$. Hence $s(x)$ is bounded for $x \in \partial\Omega \cap D_\alpha$, say $s(x) \leqq M$. Consider the function s' on D_α defined to be

(2) We have, in fact the following lemma (see, e.g., Lokasiewicz [7]). Let B be a connected open set in R^m and f a continuous function on B satisfying $f^p(x) + \sum_{1 \leqq i \leqq p} a_i(x) f^{p-i}(x) = 0$ where the $a_i(x)$ are bounded continuous functions on B, $|a_i(x)| \leqq T$. Then, there is a constant $M = M(T)$ such that if $x, x' \in B$ and γ is a curve joining x and x' in B, we have

$$|f(x) - f(x')| \leqq M(T) \sup_{y \in \gamma, i = 1, \cdots, p} |a_i(y) - a_i(x')|^{1/p}.$$

$s'(x) = M + 1$ if $x \in D_\alpha - \Omega$, $s'(x) = \max(s(x), M + 1)$ if $x \in D_\alpha \cap \Omega$. Then, s' is pseudoconvex in D_α and, for any $r > 0$, the set $\{x \in D_\alpha | s'(x) < r\}$ has a closure in D_α which does not meet $\partial D \cap S$. Hence, the function $P(x) = (\alpha - p(x))^{-1} + s'(x)$ is pseudoconvex in D_α and, for any $r > 0$, we have $\{x \in D_\alpha | P(x) < r\} \subset\subset D_\alpha$. Hence, by Theorem 3, D_α is Stein, which proves Theorem 4 in the special case when X is irreducible.

To complete the proof of Theorem 4 in the general case, we have only to use the following theorem.

THEOREM 5. *A complex space, all of whose irreducible components are Stein spaces is itself a Stein space.*

This theorem is a special case of the theorem in [10], but since this is simpler to prove directly, we give a proof.

Proof of Theorem 5. Theorem 5 follows from the following result:

(5.3) *Suppose that a complex space X is the union of two closed analytic subsets X_1, X_2. If X_1, X_2 are Stein spaces, so is X.*

In fact, suppose this proved and let X be a complex space with the irreducible components $\{Y_\nu\}$, if $V_k = \bigcup_{\nu \leqq k} Y_\nu$, then V_k is a Stein space. Let f be a holomorphic function on V_k and E a discrete subset of V_{k+1}, $E \cap V_k = \varnothing$, and let ϕ be a holomorphic function on E (i.e., to every point $x \in E$ is assigned a complex number $\phi(x)$). Since V_{k+1} is Stein and $V_k \cup E$ an analytic set in V_{k+1}, there is a holomorphic function g in V_{k+1} such that $g | V_k = f$, $g | E = \phi$. Theorem 5 follows from this remark.

To prove (3.3), it is enough to show that for any coherent sheaf $\mathscr{T}$ of ideals on X, we have $H^1(X, \mathscr{T}) = 0$. Let $\mathscr{O}$ be the sheaf of germs of holomorphic functions on X and $\mathfrak{a}_\nu \subset \mathscr{O}$ the subsheaf of germs of functions which vanish on X_ν. Let $\mathscr{O}_\nu = \mathscr{O}/\mathfrak{a}_\nu$ and $\mathscr{T}_\nu = \mathscr{T} \otimes \mathscr{O}_\nu$, the tensor product being over $\mathscr{O}$. If e_ν is the image in $\mathscr{O}_\nu$ of the section 1 of $\mathscr{O}$, then any element of $(\mathscr{T}_\nu)_x$ can be represented in the form $\alpha \otimes (e_\nu)_x$ where $\alpha \in \mathscr{T}_x$. Let $\eta : \mathscr{T} \to \mathscr{T}_1 \oplus \mathscr{T}_2$ be the homomorphism defined by $\eta(\alpha) = \alpha \otimes e_1 \oplus \alpha \otimes e_2$. Then η is injective and if $\mathscr{H}$ is the cokernel of η, we have the exact sequence

$$(*) \qquad 0 \to \mathscr{T} \xrightarrow{\eta} \mathscr{T}_1 \oplus \mathscr{T}_2 \xrightarrow{\rho} \mathscr{H} \to 0.$$

We assert that $\mathscr{H}$ has support contained in $Y = X_1 \cap X_2$ and that $\mathscr{H}$ is a coherent $\mathscr{O}_\nu$ sheaf ($\nu = 1,2$). To prove the first statement, if $x \notin X_\nu$, then $(\mathscr{O}_\nu)_x = 0$ so that η is surjective at x, which shows that $\mathscr{H}_x = 0$. To prove the second statement, it suffices to prove that $\mathfrak{a}_\nu$ annihilates $\mathscr{H}$ for $\nu = 1, 2$, i.e., that $\mathfrak{a}_\nu \cdot (\mathscr{T}_1 \oplus \mathscr{T}_2) \subset \eta(\mathscr{T})$. Now, $\mathfrak{a}_\nu \cdot \mathscr{T}_\nu = 0$ and $\mathfrak{a}_\nu \cdot \mathscr{T}_\mu$, $\nu \neq \mu$, considered as a subsheaf of $\mathscr{T}_1 \oplus \mathscr{T}_2$, is contained in $\eta(\mathscr{T})$, which proves that $\mathfrak{a}_\nu$ annihilates $\mathscr{H}$.

Consider now the mapping

$$\rho^* : H^0(X, \mathscr{T}_1) \oplus H^0(X, \mathscr{T}_2) \to H^0(X, \mathscr{H})$$

induced by ρ. Now, $H^0(X, \mathscr{T}_\nu) = H^0(X_\nu, \mathscr{T}_\nu)$, and $H^0(X, \mathscr{H}) = H^0(X_\nu, \mathscr{H})$. Moreover the composition ρ' of the injection of $\mathscr{T}_\nu$ into $\mathscr{T}_1 \oplus \mathscr{T}_2$ with ρ is a surjective mapping of $\mathscr{T}_\nu$ onto $\mathscr{H}$, so that, *since X_ν is Stein*, ρ' induces a surjective mapping of $H^0(X_\nu, \mathscr{T}_\nu)$ onto $H^0(X_\nu, \mathscr{H})$. Hence *the map ρ^* is surjective.*

Consider now the exact cohomology sequence associated to (*). We have the exact sequence

$$\cdots \to H^0(X, \mathscr{T}_1) \oplus H^0(X, \mathscr{T}_2) \xrightarrow{\rho^*} H^0(X, \mathscr{H}) \to H^1(X, \mathscr{T}) \to 0$$

since, the X_ν being Stein, we have $H^1(X, \mathscr{T}_\nu) = H^1(X_\nu, \mathscr{T}_\nu) = 0$. But since ρ^* is surjective, we deduce that $H^1(X, \mathscr{T}) = 0$, which proves (5.3).

Corollary 1 to Theorem 4. *If X is a Stein space with isolated singularities and D an open set on X which is locally Stein (i.e., any point of ∂D has a neighborhood in X whose intersection with D is Stein), then D itself is a Stein space.*

Corollary 2. *If X is any Stein space and D is a locally Stein open subset of X such that D is strongly pseudoconvex at any point of S, the singular locus of X, then D is a Stein space.*

This follows from Theorem 4 since $\partial D \cap S$ has then a strongly convex neighborhood, and any strongly convex domain on a Stein space is Stein (see [9]). The hypotheses of Corollary 2 are fulfilled in particular if $\partial D \cap S = \emptyset$.

Remarks. (1) We can deduce from Corollary 1 that if X is any two (complex) dimensional Stein space and D a locally Stein open subset of X, then D is Stein. In fact, it follows from Corollary 1 that if $\pi : Y \to X$ is the normalization of X, then $\pi^{-1}(D)$ is Stein. But $\pi : \pi^{-1}(D) \to D$ is the normalization of D, so that, by [10, Theorem 1], D is Stein.

(2) We have not used the full force of our supposition that X is a Stein space, and our proof gives for example the following theorem.

Let X be a complex manifold on which global holomorphic functions give local coordinates at any point and suppose that any analytic subset of X of codimension 1 can be defined by global equations. Then if D is a relatively compact open subset of X which is locally Stein, then D is Stein.

6. ***K*-complete manifolds: application of a vanishing theorem.** Let X be a complex *manifold* and F a (holomorphic) line bundle on X. Suppose that F is given by a covering $\{U_i\}$ and transition functions $f_{ij} : U_i \cap U_j \to C^*$. A (hermitian) metric on the fibres of F is given locally by a form

$$h(\xi, \eta) = h_i(x)\xi_i\bar{\eta}_i$$

where ξ_i, η_i are the fibre coordinates of the vectors ξ, η over U_i, $h_i(x) > 0$ being C^∞ in U_i. In $U_i \cap U_j$, we have $h_j = |f_{ij}|^2 h_{ji}$.

F is called *positive*, if $h(\xi, \eta)$ can be so chosen that the hermitian form on X associated to the alternate form

$$\chi = -\frac{1}{2\pi i}\,\partial\cdot\bar{\partial}\log h_i \text{ in } U_i$$

is positive definite. Then χ defines a kähler metric on X. If $h(\xi,\eta)$ can be so chosen that the kähler metric defined by χ is *complete*, we say that F is *positive complete.* (Note that the trivial bundle on X is positive if and only if there exists a C^∞ strongly pseudoconvex function on X.)

The following lemma is merely Proposition 11 of [2] formulated for the trivial bundle F.

LEMMA 8. *Suppose that X is a complex manifold on which there exists a strongly pseudoconvex function. If K is the canonical bundle of X, suppose that K^{-1} is positive complete. Then holomorphic functions on X separate points and give local coordinates at each point of X.*

To apply this, we make use of the following

LEMMA 9. *Let X be a complex manifold and $ds^2 = \sum h_{\mu\nu}dz_\mu d\bar{z}_\nu$ be a hermitian form on X such that the coefficients are "bounded" on any compact set. Suppose that p is a C^∞ strongly pseudoconvex function on X such that the sets $\{x\in X \mid p(x)\leqq \alpha\}$ are compact. Then there is a C^∞ function $\phi: R\to R$ such that if $u=\phi\circ p$, we have*

$$L(u) > ds^2.$$

Here, $L(u)$ denotes the Levi form

$$\sum \frac{\partial^2 u}{\partial z_\mu\,\partial\bar{z}_\nu}\,dz_\mu d\bar{z}_\nu.$$

Proof. We may suppose that $p>0$. Let $c_\nu>0$ be a constant such that $c_\nu L(p)>ds^2$ on the set of x with $\nu\leqq p(x)\leqq \nu+1$. Let $h(t)$ be a C^∞ function on $0\leqq t<\infty$ such that $h(t)>c_\nu$ for $\nu\leqq t\leqq \nu+1$, $h'(t)>0$. Define $\phi(t)=\int_0^t h(\tau)\,d\tau$, and put $u(x)=\phi(p(x))$. We have

$$\begin{aligned} L(u) &= \phi'(p)L(p)+\phi''(p)\,|dp|^2 \\ &= h(p)L(p)+h'(p)\,|dp|^2 \geqq h(p)L(p) > ds^2. \end{aligned}$$

The next lemma is a part of Theorem 1. However, we will use it for a proof of Theorem 5 below which is independent of the Heftungslemma.

LEMMA 8′. *Let X be a complex manifold and p a C^∞ strongly convex function on X such that for any $\alpha>0$, the set $\{x\in X \mid p(X)<\alpha\}\subset\subset X$. Then, holomorphic functions on X separate points and give local coordinates.*

This follows at once from Lemma 8 and Lemma 9.

THEOREM 5. *A holomorph-convex, K-complete manifold X is a Stein manifold.*

Proof. Since X is holomorph-convex, there is a sequence $\{\Delta_r\}$ of open sets in X, and holomorphic functions $f_{r,j}$ $(j \leqq k_r)$ such that if we put $p_r'(x) = \sum_{j \leqq k} |f_{r,j}|^2$, then $\Delta_r = \{x \in \Delta_{r+1} | p_r'(x) < 1\}$; moreover $\Delta_r \subset \subset \Delta_{r+1}, \bigcup \Delta_r = X$. Let q be a C^∞ strongly pseudoconvex function on Δ_{r+1} (this exists by Lemma 5). Then the function $p_r(x) = 1/(1 - p_r'(x)) + q(x)$ is a C^∞ strongly pseudoconvex function in Δ_r and, for any $\alpha > 0$, we have $\{x \in \Delta_r | p_r(x) < \alpha\} \subset \subset \Delta_r$. By Lemma 8′, holomorphic functions separate points and give local coordinates on Δ_r. Clearly Δ_r is holomorph-convex, so that it is Stein. But now it is seen that Δ_r is Δ_{r+1}-convex, so that holomorphic functions on Δ_r can be approximated by those on Δ_{r+1}. It follows that holomorphic functions on Δ_r can be approximated by those on X. Hence holomorphic functions on X separate points and give local coordinates. Hence X is Stein.

We give two applications of Theorem 1. The proof that we give for the next theorem is very similar to that of Oka's theorem given in [9], but it is more natural in the context of covering manifolds.

THEOREM 6. *Let D be an open subset of C^n which is Stein, and let $\tilde{D}$ be a covering manifold of D. Then $\tilde{D}$ is Stein.*

Proof. Let $\pi : \tilde{D} \to D$ be the projection map; let p be a C^∞ strongly pseudoconvex function on D such that for any $\alpha > 0$, we have

$$D_\alpha = \{x \in D \mid p(x) < \alpha\} \subset\subset D.$$

Let $\tilde{p} = p \circ \pi$ and $\tilde{D} = \pi^{-1}(D_\alpha)$. Since $\tilde{p}$ is strongly convex and $\tilde{D}_\alpha = \{y \in \tilde{D} | \tilde{p}(y) < \alpha\}$, we have only to prove that $\tilde{D}_\alpha$ is Stein.

Let ds^2 be a C^∞ complete riemannian metric on D and $d\tilde{s}^2 = \pi^*(ds^2)$ the induced metric on $\tilde{D}$. Then $d\tilde{s}^2$ is complete in $\tilde{D}$ (see [13, p. 208]). Let $y_0 \in \tilde{D}_\alpha$ and $\rho(y)$ the distance of $y \in \tilde{D}$ from y_0 with respect to $d\tilde{s}^2$. Let $|dz|^2$ be the Euclidean metric on C^n. There is a constant $M > 0$ so that

$$\frac{1}{M}|dz|^2 < ds^2 < M|dz|^2 \text{ on } D_{\alpha+1}.$$

This implies that if $y, y' \in \tilde{D}_\alpha$ belong to an open connected set which is mapped isomorphically onto a convex set in $D_{\alpha+1}$ by π, then

$$|\rho(y) - \rho(y')| \leqq M|\pi(y) - \pi(y')|. \tag{1}$$

Let $\gamma \geqq 0$ be a C^∞ function with compact support in the open ball $B_{\delta/2}$ of radius $\delta/2$ about $0 \in C^n$; we suppose that $\gamma(0) > 0$. If δ is sufficiently small, for $y \in \tilde{D}_\alpha$, we can consider the open set $B_\delta(y)$ which is mapped isomorphically onto the ball of radius δ about $\pi(y)$. Then γ defines a C^∞ function γ_y in $B_\delta(y)$ if we put $\gamma_y(y') = \gamma(\pi(y) - \pi(y'))$. Define

$$\rho_1(y) = \int_{B_\delta(y)} \gamma_y(y')\rho(y')dv_{y'},$$

where $dv_{y'}$ is the volume element induced on $\tilde{D}$ by the Euclidean volume dv. Clearly $|\rho_1(y) - \rho(y)| \leqq M \int_{B_{\delta/2}} \gamma(x)\,dv = C$. Moreover ρ_1 is a C^∞ function in $\tilde{D}_\alpha$ and

$$D_{\mu\nu}\rho_1 = \frac{\partial^2 \rho_1}{\partial y_\mu \partial \bar{y}_\nu} = \int_{B_\delta(y)} \frac{\partial \gamma_y(y')}{\partial y_\mu} \cdot \frac{\partial \rho(y')}{\partial \bar{y}'_\nu} dv_{y'}$$

has absolute value $\leqq M \int_{B_{\delta/2}} |\partial\gamma/\partial x_\mu|\, dv_x$, because of (1). Hence $|D_{\mu\nu}\rho_1|$ is bounded on $\tilde{D}_\alpha$. Hence there is a constant $K > 0$ such that

$$q(y) = \rho_1(y) + K|\pi(y)|^2$$

is strongly pseudoconvex in $\tilde{D}_\alpha$ and $q(y) \geqq \rho(y) - C$. Since $d\tilde{s}^2$ is complete, if $\beta < \alpha$, for any $r > 0$ we have

$$\{y \in \tilde{D}_\beta \mid q(y) < r\} \subset\subset \tilde{D}_\alpha.$$

Hence $u(y) = (\alpha - \tilde{p}(y))^{-1} + q(y)$ is a C^∞ strongly pseudoconvex function on $\tilde{D}_\alpha$ such that for $r > 0$, $\{y \in \tilde{D}_\alpha \mid u(y) < r\} \subset\subset \tilde{D}_\alpha$. By Theorem 1, $\tilde{D}_\alpha$ is Stein.

REMARK. A method of K. Stein [15] enables one to deduce from Theorem 6 the theorem that any covering space of a Stein space is itself Stein. As a final application we prove

THEOREM 7. *A holomorphic vector bundle E over a Stein manifold X is itself a Stein manifold.*

Proof. Let $\pi: E \to X$ be the projection and $g_{ij}: U_i \cap U_j \to \mathrm{GL}(q,C)$ transition functions with respect to a covering $\{U_i\}$ defining E. Let h_i be a C^∞ map of U_i into the space of positive definite hermitian matrices such that, on $U_i \cap U_j$, we have

$$h_j = {}^t\bar{g}_{ij} h_i g_{ij}.$$

Let $\zeta^{(i)}$ be fibre coordinates on $\pi^{-1}(U_i)$. Define the C^∞ function ϕ_h on E by

$$\phi_h(x,\zeta) = {}^t\bar{\zeta}^{(i)} h_i(x) \zeta^{(i)} \quad \text{if} \quad (x,\zeta) \in \pi^{-1}(U_i).$$

Let Θ_i be the hermitian form associated to the alternate form $(1/2\pi i)\bar{\partial}(h_i^{-1}\partial h_i)$. For differentials $dx, d\bar{x}$ at $x \in U_i$, define

$$F_h(dx, d\bar{x}, \zeta) = {}^t\bar{\zeta}^{(i)} h_i(x)\, \Theta_i(dx, d\bar{x})\, \zeta^{(i)}.$$

It is easy to verify (see e.g. [1, p. 257]) that *if this hermitian form in dx, $d\bar{x}$ is positive definite for all $\zeta \neq 0$, then ϕ_h is strongly pseudoconvex outside the zero section of E.*

We now consider a strongly pseudoconvex function p on X such that all sets $\{x \in X \mid p(x) < \alpha\}$ are relatively compact and consider the function

$$\phi = \phi_{h'} \text{ with } h'_i = e^p \cdot h_i \text{ in } U_i.$$

We have

$$F_{h'} = e^p\{L(p)\,{}^t\bar{\zeta}^{(i)} h_i \zeta^{(i)} + {}^t\bar{\zeta}^{(i)} h_i \Theta_i \zeta^{(i)}\}.$$

Now, since h_i is positive definite, there is a constant $c_i > 0$ corresponding to any $U_i' \subset\subset U_i$ such that for ${}^t\bar{\zeta}^{(i)}\zeta^{(i)} = 1$, ${}^t\bar{\zeta}^{(i)}h_i\zeta^{(i)} \geqq c_i$. By Lemma 8, we can choose p such that F_h is positive definite for any $\zeta^{(i)} \neq 0$. The function ϕ is then strongly pseudoconvex for $\zeta \neq 0$ and pseudoconvex everywhere in E. Consider now the function

$$P(x, \zeta) = p \circ \pi + \phi$$

on E. Clearly P is strongly pseudoconvex everywhere on E and, if $r > 0$, we have $\{e \in E \mid P(e) < r\} \subset\subset E$. By Theorem 1, E is Stein.

REMARK. We can prove in the same way that any holomorphic vector bundle on any Stein space is Stein.

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