THE STRUCTURE OF IDEALS AND POINT DERIVATIONS IN BANACH ALGEBRAS OF LIPSCHITZ FUNCTIONS[1]

BY

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Introduction. The main object of study in this paper is the space of all bounded complex-valued functions defined on a metric space (X, d) which satisfy a Lipschitz condition with respect to the metric d. With a suitable norm this collection of functions becomes a Banach algebra. This Banach algebra, which we shall call a Lipschitz algebra, will be denoted by $\mathrm{Lip}(X, d)$. It is the structure of $\mathrm{Lip}(X, d)$ with which we are primarily concerned here.

Although the notion of Lipschitz continuity is very old and Lipschitz functions have been studied for many years, interest in the Banach space and Banach algebra theory of Lipschitz functions has not developed until quite recently. Very little is known about the Banach space properties of $\mathrm{Lip}\,(X, d)$. The following papers constitute all the work known to this writer which has been done in this area. Mirkil [9] and de Leeuw [4] have considered the space of periodic Lipschitz functions on the real line (or in other words, Lipschitz functions defined on the circle group). Mirkil was interested in the relation between the Lipschitz condition and the translation properties of functions, the fact that the functions were defined on a group being vital. de Leeuw was more directly concerned with the Banach space structure, his paper involving the study of certain dual spaces, extreme points, and isometries. In the long paper of Glaeser [6], a short chapter, somewhat off the main theme of his work, is devoted to the space of n-times continuously differentiable functions defined on a subset of E^m whose nth derivatives satisfy a Lipschitz condition. It is shown as a side result in the paper of Arens and Eells [1] that $\mathrm{Lip}(X, d)$ is always the dual space of some normed linear space.

The only work known to us which treats $\mathrm{Lip}(X, d)$ as a Banach *algebra* was done by S. B. Myers [11]. His paper [11] is a summary of results, the proofs never published because of his untimely death in 1955. His interest in Lipschitz algebras was connected with the process of differentiation, and this interest we pursue in Chapter III of the present study. We supply proofs of many of the unproved statements of Myers, and in some cases extend his results to more general settings.

Chapter I, as its title implies, consists of preliminary material. In it we set some of the notations and establish some of the basic facts concerning Lipschitz functions needed in the subsequent chapters.

In Chapter II we investigate the structure of the closed ideals in Lipschitz algebras. We obtain a concrete description of the important class of ideals $J(K)$ introduced by Šilov for regular Banach algebras. The structure of these ideals is determined by the local behavior of the functions in the algebra, and for Lipschitz algebras we obtain a "smoothness" condition which characterizes these ideals. We also relate these ideals to maximal ideals and this leads to the identification of all the primary ideals in $\mathrm{Lip}(X, d)$ when (X, d) is compact. We are unable to settle the ideal structure of Lipschitz algebras completely. For compact (X, d) we conjecture that every closed ideal in $\mathrm{Lip}(X, d)$ is the intersection of closed primary ideals. We verify this for the special class of ideals $J(K)$, but for general ideals the problem is open. These matters are discussed more precisely and at greater length in §3, the introductory section to the chapter. In §6, the last section of Chapter II, we consider the quotient algebras $\mathrm{Lip}(X, d)/J(x)$. We identify the quotient norm as a certain limit and show that for the case $X = [0, 1]$ these quotient algebras are nonseparable.

Chapter III is devoted to the study of point derivations. A point derivation, which is a generalization of differentiation at a point, is defined to be a linear functional on an algebra of functions which satisfies the product rule. They were first studied on Banach algebras by Singer and Wermer [16]. In §7 we look at the purely algebraic aspects of point derivations, and in §8 the hypothesis of continuity is added and point derivations on Banach algebras are considered. Using heavily the ideal theory developed in Chapter II, we present in §9 three characterizations of the structure of point derivations on Lipschitz algebras. As an application of this, we characterize weak sequential convergence in the Banach space $\mathrm{Lip}(X, d)$ where (X, d) is compact. We conclude the chapter with a restatement of some of the ideal results of Chapter II in terms of point derivations, and show how the process of differentiation in the form of point derivations is connected with the structure of primary ideals in a manner analogous to the structure in the algebra $C^n(0, 1)$.

I. Preliminaries

1. **Some preliminary remarks concerning Lipschitz functions.** Let (X, d) be a metric space. A complex-valued function f defined on X is said to be a *Lipschitz function* if there exists a constant K such that

$$|f(x) - f(y)| \leq Kd(x, y), \quad x, y \in X.$$

The smallest such constant K is called the *Lipschitz norm* of f and will be denoted by $\|f\|_d$. Evidently

$$\|f\|_d = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : \; x, y \in X, \, x \neq y \right\}.$$

For a complex-valued function f defined on X which is bounded on X, the *sup norm* $\|f\|_\infty$ of f is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

The collection of all bounded Lipschitz functions on (X, d) will be denoted by $\mathrm{Lip}(X, d)$. It is well known that $\mathrm{Lip}(X, d)$ is a Banach space with the norm $\| \cdot \|$ defined by

(1.) $$\|f\| = \|f\|_\infty + \|f\|_d, \quad f \in \mathrm{Lip}(X, d).$$

Also, the product of two bounded Lipschitz functions is again a bounded Lipschitz function; for if $f, g \in \mathrm{Lip}(X, d)$, then

(1.2) $$\frac{|(fg)(x) - (fg)(y)|}{d(x, y)} \leqq |f(x)| \frac{|g(x) - g(y)|}{d(x, y)} + |g(y)| \frac{|f(x) - f(y)|}{d(x, y)}.$$

Now forming the supremum in (1.2) over all $x, y \in X$ with $x \neq y$, we see that

$$\|fg\|_d \leqq \|f\|_\infty \|g\|_d + \|g\|_\infty \|f\|_d.$$

From this it is easily seen that the norm $\| \cdot \|$ given by (1.1) is a Banach algebra norm; that is,

$$\|fg\| \leqq \|f\| \, \|g\|, \quad f, g \in \mathrm{Lip}(X, d).$$

The Banach algebras $\mathrm{Lip}(X, d)$ with this norm will be called *Lipschitz algebras*. The norm used on $\mathrm{Lip}(X, d)$ will always be that given by (1.1); the notation $\| \cdot \|$ will always refer to $\| \cdot \|_\infty + \| \cdot \|_d$.

Throughout this paper the metric space (X, d) will be assumed complete. It will be convenient and there is no loss of generality in doing so. For suppose (X, d) were not complete and let (X^*, d^*) denote its completion. Since an element of $\mathrm{Lip}(X, d)$ is uniformly continuous on (X, d), it extends uniquely and in a norm preserving way to an element of $\mathrm{Lip}(X^*, d^*)$. Thus as Banach algebras, $\mathrm{Lip}(X, d)$ and $\mathrm{Lip}(X^*, d^*)$ are isometrically isomorphic.

An important subset of $\mathrm{Lip}(X, d)$ consists of all those functions f in $\mathrm{Lip}(X, d)$ which have the property that

$$|f(x) - f(y)| / d(x, y) \to 0 \text{ as } d(x, y) \to 0.$$

This is more precisely stated as follows: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| f(x) - f(y) \right| / d(x,y) < \varepsilon$$

whenever $d(x,y) < \delta$. This set of functions will be denoted by $\mathrm{lip}(X,d)$.

PROPOSITION 1.1. $\mathrm{lip}(X,d)$ *is a closed subalgebra of* $\mathrm{Lip}(X,d)$.

This was established by Mirkil [9] for the case $X = [0, 2\pi]$ and $d(x,y) = |x - y|^{\alpha}$, $0 < \alpha < 1$, but his proof is valid in general. It was further shown in [9] that for the particular space mentioned, $\mathrm{lip}(X,d)$ is the closed linear subspace of $\mathrm{Lip}(X,d)$ generated by the trigonometric polynomials. Also, a general result of Glaeser [6] implies for this case that the continuously differentiable functions are dense in $\mathrm{Lip}(X,d)$.

$\mathrm{Lip}(X,d)$ is a complex Banach algebra. But we note that if $f = u_1 + iu_2$ belongs to $\mathrm{Lip}(X,d)$, where u_1 and u_2 are the real and imaginary parts of f respectively, then u_1 and u_2 also belong to $\mathrm{Lip}(X,d)$. In fact, $\| f \| \geqq \| u_k \|, k = 1, 2$. Thus we are always able to consider separately the real and imaginary parts of any function in $\mathrm{Lip}(X,d)$. This will technically simplify many of the proofs that follow. The same situation holds in $\mathrm{lip}(X,d)$.

Let f be a real-valued function defined on the metric space (X,d) and let k be a positive number. The *truncation* $T_k f$ *of* f at k is the function on X defined by

$$(1.3) \qquad (T_k f)(x) = \begin{cases} k & \text{if } f(x) > k, \\ f(x) & \text{if } -k \leqq f(x) \leqq k, \\ -k & \text{if } f(x) < -k, \end{cases} \qquad x \in X.$$

LEMMA 1.2.   *For each real-valued function* f *on* (X,d) *the truncated function* $T_k f$ *defined on* (X,d) *by* (1.3) *satisfies*

$$(1.4) \qquad \left| (T_k f)(x) - (T_k f)(y) \right| \leqq \left| f(x) - f(y) \right|$$

for all x, y *in* X.

The validity of the inequality (1.4) can be seen by comparing the graphs of f and $T_k f$. Or a straightforward proof can be given by checking each of the possible cases for a given x and y.

The next proposition is an immediate consequence of Lemma 1.2.

PROPOSITION 1.3.   *Let the real-valued function* f *defined on* (X,d) *satisfy the Lipschitz condition*

$$\left| f(x) - f(y) \right| \leqq K d(x,y), \qquad x, y \in X.$$

Then the truncation $T_k f$ *of* f *belongs to* $\mathrm{Lip}(X,d)$ *and*

$$\| T_k f \|_d \leqq K.$$

The following useful proposition on the extension of Lipschitz functions is

probably well known. The only reference known to this writer, however, is the book by Botts and McShane [2, p. 97] where it appears in slightly different form as a problem.

PROPOSITION 1.4.   *Let Y be a nonempty subset of the metric space (X,d). Let f be a real-valued function defined on Y which is bounded on Y by M and which satisfies the Lipschitz condition*

$$\left| f(x) - f(y) \right| \leqq Kd(x,y), \qquad x, y \in Y,$$

on Y. Then there exists an extension F of f defined on all of X having the same bound M and satisfying

$$\left| F(x) - F(y) \right| \leqq Kd(x,y), \qquad x, y \in X.$$

Proof.   Define f_0 on X by

$$f_0(x) = \sup\{f(y) - Kd(x,y) : y \in Y\}, \qquad x \in X.$$

If $x \in Y$ then the supremum defining f_0 is attained at $y = x$ so that f_0 agrees with f on Y. Thus f_0 is an extension of f to X. From the triangle inequality we obtain for any x_1, x_2 in X and y in Y that

$$f(y) - Kd(y, x_2) - Kd(x_1, x_2)$$
$$\leqq f(y) - Kd(y, x_1)$$
$$\leqq f(y) - Kd(y, x_2) + Kd(x_1, x_2).$$

Hence,

$$\left| f_0(x_1) - f_0(x_2) \right| \leqq Kd(x_1, x_2)$$

for all x_1, x_2 in X. Thus $\| f_0 \|_d \leqq K$.

Then $F = T_M f_0$, the truncation of f_0 at M, is the required extension of f.   Q.E.D.

An important class of functions defined on a metric space consists of those functions defined by the metric d. The next proposition is a collection of facts about such functions which will be used frequently. The proof is straightforward and will be omitted.

PROPOSITION 1.5.   *Let (X,d) be a metric space. The function f defined on X by*

$$f(x) = d(x,t), \qquad x \in X,$$

where $t \in X$ is fixed has Lipschitz norm $\| f \|_d = 1$. If the diameter d' of (X,d) is finite, then $f \in \mathrm{Lip}(X,d)$ and $\| f \| \leqq 1 + d'$. If d' is infinite, then the function g defined by truncating f at 1,

$$(1.5) \qquad g(x) = (T_1 f)(x) = \min[d(x,t), 1], \qquad x \in X,$$

belongs to $\mathrm{Lip}(X,d)$ and $\| g \| \leqq 2$. The family of functions defined by (1.5) as t runs over X separates the points of X.

Observe that the function $g \in \text{Lip}(X, d)$ defined by (1.5) does not belong to $\text{lip}(X, d)$ since

$$\lim_{x \to t} \frac{g(x)}{d(x, t)} = 1.$$

Although $\text{lip}(X, d)$ is always a closed subalgebra of $\text{Lip}(X, d)$ that contains the constant functions, it need not in general contain anything but constant functions. For example, if $X = [0, 1]$ and $d(x, y) = |x - y|$, then $f \in \text{lip}(X, d)$ implies that $f'(x) = 0$ for all x, so that f is constant.

To be assured of an abundance of nonconstant functions we shall concern ourselves with the algebras $\text{lip}(X, d^{\alpha}), 0 < \alpha < 1$, where d^{α} is the metric on X defined by $d^{\alpha}(x, y) = [d(x, y)]^{\alpha}$, x, $y \in X$. Although the metrics d and d^{α} are uniformly equivalent on X, the sets of corresponding Lipschitz functions are, of course, much different since the structure of Lipschitz algebras depends on the precise form of the metric and is not invariant under uniform homeomorphisms.

Now if $f \in \text{Lip}(X, d)$, then

$$\frac{|f(x) - f(y)|}{d^{\alpha}(x, y)} = \frac{|f(x) - f(y)|}{d(x, y)} \cdot d^{1 - \alpha}(x, y)$$

$$\leqq \|f\|_d \, d^{1 - \alpha}(x, y)$$

which tends to 0 as $d^{\alpha}(x, y) \to 0$. Hence,

$$\text{lip}(X, d^{\alpha}) \supset \text{Lip}(X, d).$$

By Proposition 1.5, $\text{Lip}(X, d)$ contains sufficiently many functions to separate the points of X. Consequently, for each α, $0 < \alpha < 1$, the algebra $\text{lip}(X, d^{\alpha})$ separates the points of X. We summarize in a proposition the basic facts which will be needed later.

PROPOSITION 1.6. *For each α, $0 < \alpha < 1$, $\text{lip}(X, d^{\alpha})$ with the norm $\| \cdot \|$ $= \| \cdot \|_{\infty} + \| \cdot \|_{d^{\alpha}}$ is a Banach algebra with identity which contains sufficiently many functions to separate the points of X.*

An algebra of functions on a set X is called *inverse-closed* if $f \in A$ and $|f(x)| \geqq \varepsilon > 0$, all $x \in X$, imply that $f^{-1} \in A$.

PROPOSITION 1.7. *$\text{Lip}(X, d)$ and $\text{lip}(X, d)$ are inverse closed algebras.*

Proof. If $|f(x)| \geqq \varepsilon > 0$ for all $x \in X$, then

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| \leqq \frac{1}{\varepsilon^2} |f(x) - f(y)|, \qquad x, y \in X.$$

It follows that if f belongs to $\text{Lip}(X, d)$ or $\text{lip}(X, d^{\alpha})$, then so does f^{-1}.   Q.E.D.

2. **Some Banach algebra preliminaries.** In this section we discuss briefly some of the basic Banach algebra properties of $\mathrm{Lip}(X, d)$ and $\mathrm{lip}(X, d^\alpha)$. We mention only those properties relevant to the work that follows. While some knowledge of the Gelfand theory of commutative Banach algebras would help in appreciating what follows, it is not essential for the most part in understanding the main results and their proofs. The Banach algebra aspects of Lipschitz algebras will be treated more extensively in another paper. The fundamental facts used here concerning Banach algebras can be found in either the book of Loomis [7] or Rickart [12].

The notation Σ will be used for the space of nonzero multiplicative linear functionals of a Banach algebra and we shall refer to Σ as the *carrier space* of the algebra. $C(\Sigma)$ will denote the space of complex-valued functions on Σ that are continuous in the Gelfand topology. The norm on $C(\Sigma)$ will always be the sup norm.

The class of Banach algebras with which we are primarily concerned is the following. The Banach algebra A is called *regular* (in the sense of Šilov) if for every proper subset K of Σ closed in the Gelfand topology and point $\phi \in \Sigma - K$, there exists an f in A such that $\hat{f}(\phi) = 1$ and $\hat{f}(K) = 0$. See [7; 10; 14] for a discussion of regular algebras.

Now let A stand for either $\mathrm{Lip}(X, d)$ or $\mathrm{lip}(X, d^\alpha)$. To each $x \in X$ there corresponds the evaluation functional $\phi_x \in \Sigma$ defined by $\phi_x(f) = f(x)$, $f \in A$. Since A separates the points of X by Propositions 1.5 and 1.6, the injection mapping $x \to \phi_x$ is one-one. Therefore we may identify X as a subset of Σ.

It was shown in §1 that A is a point-separating, inverse-closed algebra of bounded functions on X. Also, A is obviously self-adjoint (closed under complex conjugation). Hence it follows from [7, p. 55] that X is a dense subset of Σ in the Gelfand topology, and if (X, d) is compact, that $X = \Sigma$ and the d-topology coincides with the Gelfand topology. If (X, d) is not compact, then X is a proper subset of Σ. But the inherited Gelfand topology of X is still equivalent to the d-topology since functions of the form (1.5) belong to A; for then spheres $S(x_0, \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$ which are open in the stronger metric topology are also open in the Gelfand topology. Thus regardless of whether (X, d) is compact or not, the relative Gelfand and d-topology of X are the same.

If (X, d) is compact, then it is easy to see by using functions of the form $d(x, K) = \inf\{d(x, t) : t \in K\}$ that A is a regular Banach algebra. It has been shown by Lindberg (unpublished) that A is regular for arbitrary metric spaces (X, d). We collect this and the above results together for easy reference.

PROPOSITION 2.1. *Let A denote either $\mathrm{Lip}(X, d)$ or $\mathrm{lip}(X, d^\alpha)$ and let Σ denote the carrier space of A. Then X is dense in Σ in the Gelfand topology, and the relative Gelfand topology of X coincides with the d-topology of X. Furthermore, A is always regular.*

II. The structure of closed ideals

3. **Introduction.** In this chapter we are concerned with the ideal structure of $\text{Lip}(X, d)$ and $\text{lip}(X, d^\alpha)$. We use several results from the general theory of ideals for regular algebras which we describe in this section. The classes of ideals under consideration are defined and the results of the chapter are briefly discussed.

Let A be a regular Banach algebra with identity and with norm $\|\cdot\|_A$. By means of the Gelfand representation we can regard the elements of A as functions on its carrier space Σ. The regularity of A guarantees the existence of sufficiently many functions to separate points and closed subsets of Σ.

Let I be an ideal in A. By the *hull* of I we mean the point set $\{x \in \Sigma : f(x) = 0,$ all $f \in I\}$ in Σ. Since the hull of I is the intersection of all the null sets $\{x \in \Sigma : f(x) = 0\}$ for $f \in I$, and since each f is continuous, the hull of an ideal is a closed subset of Σ.

One important class of ideals in A is the following. Let K be a closed subset of Σ and define $M(K) = \{f \in A : f = 0 \text{ on } K\}$. Evidently $M(K)$ is an ideal in A. Since A is regular, the hull of $M(K)$ is exactly K. Also, $M(K) = \bigcap_{x \in K} M(x)$, where $M(x) = \{f \in A : f(x) = 0\}$ is the maximal ideal in A corresponding to the point x. $M(K)$ has the property that it is the largest ideal in A having the set K as hull.

Another class of ideals which is of interest is the following. Let K be a closed set in Σ. Define $J(K)$ to be the closure in A of the set $\{f \in A : \text{there exists a neighborhood } U \text{ of } K \text{ such that } f = 0 \text{ in } U\}$. Clearly, $J(K)$ is a closed ideal in A and K is contained in the hull of $J(K)$. The following lemma is a characterization of the functions in $J(K)$ and will be used several times in this chapter. It was first given by Šilov in [14].

Lemma 3.1. *An element $f \in A$ belongs to $J(K)$ if and only if there is a sequence $\{f_n\}$ in A satisfying*
(a) $f_n = f$ *in a neighborhood U_n of K,*
(b) $\|f_n\|_A \to 0$ *as $n \to \infty$.*

Proof. If $f \in J(K)$ then since functions vanishing in a neighborhood of K are dense in $J(K)$, there must exist a sequence $\{g_n\}$ such that $g_n = 0$ in a neighborhood U_n of K and $\|f - g_n\|_A \to 0$. Setting $f_n = f - g_n$ we obtain the desired sequence satisfying (a) and (b). For sufficiency, note that $f - f_n$ vanishes in a neighborhood of K so that $(f - f_n) \in J(K)$ for each n; and $\|f - (f - f_n)\|_A = \|f_n\|_A \to 0$ puts f in $J(K)$ since $J(K)$ is closed.   Q.E.D.

The importance of the ideals $J(K)$ is shown by the following theorem. For a proof see [7], [12], or [14].

Theorem 3.2 (Šilov). *$J(K)$ has K as hull and is the smallest closed ideal in A having K as hull.*

Thus for any closed ideal I in A with K as hull we have $J(K) \subset I \subset M(K)$.

Let us now turn to ideals in Banach algebras of Lipschitz functions. In §4, we obtain a complete identification of the closed ideals in $\mathrm{lip}(X, d^\alpha)$ where (X, d^α) is compact; they are all of the form $M(K)$ for some closed subset K.

In §5 we investigate the ideal structure in $\mathrm{Lip}(X, d)$. The situation there is much more complicated than for $\mathrm{lip}(X, d^\alpha)$. We first consider the ideals $J(K)$. Theorem 5.1 is a concrete characterization of those functions in $\mathrm{Lip}(X, d)$ that belong to $J(K)$. When (X, d) is compact the carrier space of $\mathrm{Lip}(X, d)$ is just X, the Gelfand topology is the same as the d-topology, and the theorem describes $J(K)$ for every closed subset K of (X, d). When (X, d) is not compact the carrier space Σ is larger than X and we restrict our attention to subsets K of X. But not every closed subset K of (X, d) is closed in Σ in the Gelfand topology. For example, the set X is closed in (X, d), but since X is dense in Σ in the Gelfand topology by Proposition 2.1, and since $X \neq \Sigma$, the set X cannot be closed in Σ in the Gelfand topology. However, since the relative Gelfand topology of X is equivalent to the d-topology of X by Proposition 2.1, compact subsets of (X, d) are also compact, hence closed, in Σ in the Gelfand topology. Thus we are interested in $J(K)$ for compact subsets K of (X, d). Theorem 5.1 is stated for this case.

We make one more observation about the ideals $J(K)$ when (X, d) is not compact and where K is compact in (X, d). $J(K)$ has been defined as the closure of the set of functions vanishing in neighborhoods of K where the term "neighborhood" was understood to refer to the Gelfand topology. Because of the equivalence of the relative Gelfand topology and the d-topology of X, we can take $J(K)$ to be the closure in $\mathrm{Lip}(X, d)$ of the set of functions vanishing in d-neighborhoods of K.

For an ideal I the notation I^2 signifies the collection of all finite linear combinations $\sum a_i f_i g_i$ where f_i and g_i belong to I and the a_i are complex numbers. I^2 is again an ideal as is its closure $\bar{I}^2$. Theorem 5.2 states that for a compact subset K of (X, d) the ideals $J(K)$ and $\overline{M(K)^2}$ in $\mathrm{Lip}(X, d)$ are equal. As we shall see in §9, this result applied to the case $K = \{x\}$ relates $J(x)$ to point derivations on $\mathrm{Lip}(X, d)$. The results in §9 are based almost entirely on this relation and the description of $J(x)$ given by Theorem 5.1.

An ideal I in a Banach algebra is called *primary* if it is contained in exactly one maximal ideal. Equivalently, I is primary if the hull of I consists of precisely one point. If the hull of the primary ideal I is $\{x\}$ we say that I is a *primary ideal at* x. For regular algebras, a closed ideal I is primary at x if and only if I lies between $J(x)$ and $M(x)$; that is, $J(x) \subset I \subset M(x)$. $J(x)$ is the smallest and $M(x)$ the largest closed primary ideal at x in regular algebras.

The closed primary ideals in $\mathrm{Lip}(X, d)$ at $x \in X$ are identified in Theorem 5.4; they are just the closed linear subspaces between $J(x)$ and $M(x)$. This identification follows from Theorem 5.2 and the fact that $\mathrm{Lip}(X, d)$ is regular.

If I is a closed ideal with hull K in a regular algebra A, then for each $x \in K$ the *primary component of I at x* is the smallest closed primary ideal at x which contains I. This will be denoted by $P(I, x)$.

Let

$$I^* = \bigcap_{x \in K} P(I, x).$$

Clearly, I^* is a closed ideal with hull K and $I \subset I^*$. If it is the case that $I = I^*$ for every closed ideal I in A we say that A has the *ideal intersection property* [10, p. 29]. That is, A is said to have the ideal intersection property if every closed ideal is equal to the intersection of the closed primary ideals containing it.

The two most striking examples of Banach algebras which possess the ideal intersection property are the following [7, p. 182]: the algebra of n-times continuously differentiable functions on a domain in m-dimensional Euclidean space and the algebra of bounded Baire measures under convolution on the real line having compact support. The first was proved by Whitney [17] and the second by Schwartz [13]. In both cases the primary ideals are connected with differentiation and the role of differentiation is vital in establishing the ideal intersection property.

We conjecture that for compact (X, d) the algebra $\mathrm{Lip}(X, d)$ possesses the ideal intersection property. For the ideals $I = J(K)$ we are able to verify that $I = I^*$. But for general ideals the problem is open. A proof of the conjecture would be of interest, especially in view of the close relation between primary ideals and the theory of point derivations on $\mathrm{Lip}(X, d)$ given in §9. We shall return to the conjecture at that time and restate it in terms of point derivations.

The remainder of this chapter is devoted to the quotient algebras $Q_x = \mathrm{Lip}(X, d)/J(x)$ for compact (X, d). In Theorem 6.1 we describe the quotient norm in Q_x in terms of a certain lim sup. In Theorem 6.2 we show that for the case $X = [0, 1]$ and $d(x, y) = |x - y|^\alpha$, $0 < \alpha < 1$, the quotients Q_x are non-separable (with respect to the natural quotient norm). We shall discuss these matters in §6 and postpone further discussion until then.

4. Ideals in $\mathrm{lip}(X, d^\alpha)$. This section is devoted to proving that every closed ideal in $\mathrm{lip}(X, d^\alpha)$, where (X, d) is compact, is of the form $M(K)$ for some closed subset K of (X, d). Equivalently, every closed ideal in $\mathrm{lip}(X, d^\alpha)$ is the intersection of the maximal ideals containing it.

It is interesting to note that although $\mathrm{lip}(X, d^\alpha)$ is like $C(X)$ as far as ideal structure goes (i.e., every closed ideal is an $M(K)$), this is definitely not so with respect to subalgebras. For the case $X = [0, 1]$ and $d(x, y) = |x - y|$, Y. Katznelson (unpublished) has constructed a counter-example to the Stone-Weierstrass Theorem. That is, there exists a point-separating self-adjoint subalgebra of $\mathrm{lip}(X, d^\alpha)$ which is not dense in $\mathrm{lip}(X, d^\alpha)$.

We now proceed with the proof of the result on ideals in $\mathrm{lip}(X, d^\alpha)$ stated above. First we have a lemma.

LEMMA 4.1. *Let (X, d) be a metric space (not necessarily compact). For each $f \in \mathrm{lip}(X, d)$ with nonvoid null set $K = \{x \in X : f(x) = 0\}$ there exists a sequence $\{f_n\}$ in $\mathrm{lip}(X, d)$ satisfying*
  (i) *$f_n = f$ in a neighborhood of K,*
  (ii) *$\|f_n\| \to 0$ as $n \to \infty$.*

Proof. We first assume that f is real-valued. Let f_n denote the truncation of f at $1/n$. Since

(4.1) $$|f_n(x) - f_n(y)| \leq |f(x) - f(y)|, \qquad x, y \in X,$$

by Lemma 1.2, it follows that $f_n \in \mathrm{lip}(X, d)$ for each n. Let $U_n = \{x : |f(x)| < 1/n\}$. Then U_n is an open set containing K and $f_n = f$ in U_n. Hence (i) is satisfied.

It is clear that $\|f_n\|_\infty \to 0$ as $n \to \infty$. It remains to show that $\|f_n\|_d \to 0$ as $n \to \infty$ to establish (ii). Let $\varepsilon > 0$ be given. Since $f \in \mathrm{lip}(X, d)$ there is a $\delta > 0$ so that

$$|f(x) - f(y)| / d(x, y) < \varepsilon/2$$

whenever $d(x, y) < \delta$. Fixing δ we have

(4.2)
$$\|f_n\|_d \leq \sup \left\{ \frac{|f_n(x) - f_n(y)|}{d(x, y)} : x, y \in X, x \neq y, d(x, y) < \delta \right\}$$
$$+ \sup \left\{ \frac{|f_n(x) - f_n(y)|}{d(x, y)} : x, y \in X, \ d(x, y) \geq \delta \right\}.$$

It follows from (4.1) and the choice of δ that the first term on the right-hand side of (4.2) is at most $\varepsilon/2$. Also, the second term on the right-hand side of (4.2) is at most $2\|f_n\|_\infty / \delta \leq 2/n\delta$. Thus for each n

$$\|f_n\|_d \leq \varepsilon/2 + 2/n\delta.$$

Choosing $N > 4/\varepsilon\delta$, we have

$$\|f_n\|_d \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n \geq N$. Hence, $\|f_n\|_d \to 0$ as $n \to \infty$. This establishes the lemma for real-valued f. The extension to complex-valued f is clear.   Q.E.D.

THEOREM 4.2. *If I is a closed ideal of $\mathrm{lip}(X, d^\alpha)$ with hull $K \subset X$, then $I = M(K)$.*

Proof. Since $\mathrm{lip}(X, d^\alpha)$ is a regular algebra, $J(K)$ is the smallest and $M(K)$ the largest closed ideal having hull K. But from Lemmas 4.1 and 3.1 we have $M(K) \subset J(K)$, so that $J(K) = M(K)$. The theorem follows.   Q.E.D.

COROLLARY 4.3. *If (X, d) is compact, then every closed ideal in $\mathrm{lip}(X, d^\alpha)$ is of the form $M(K)$ for some closed subset K of (X, d).*

Note that Theorem 4.2 is valid for $\text{lip}(X,d)$ whenever (X,d) is a metric space such that $\text{lip}(X,d)$ is a regular algebra. The sole purpose of introducing (X,d^α) was to insure this condition obtained.

5. Ideals in $\text{Lip}(X, d)$.

THEOREM 5.1. *Let K be a compact subset of the metric space (X,d). A function $f \in \text{Lip}(X,d)$ belongs to $J(K)$ if and only if it satisfies*
(i) $f(x) = 0$, all $x \in K$,
(ii) $|f(x) - f(y)|/d(x,y) \to 0$ as $(x,y) \to K \times K$.

LEMMA. *The set of $f \in \text{Lip}(X,d)$ satisfying* (i) *and* (ii) *of Theorem 5.1 is a closed subset of* $\text{Lip}(X,d)$.

Proof. Let $W = \{(x,y): x,y \in X, x \neq y\}$, the product space $X \times X$ with the diagonal removed. For each $f \in \text{Lip}(X,d)$ define f^* on W by

$$f^*(x,y) = |f(x) - f(y)|/d(x,y), \qquad (x,y) \in W.$$

The mapping $f \to f^*$ takes $\text{Lip}(X,d)$ into $C(W)$, the space of bounded continuous functions on W. Let $\|\cdot\|_W$ denote the sup norm on W. Then

$$\|f^*\|_W = \|f\|_d \leqq \|f\|_\infty + \|f\|_d = \|f\|, \qquad f \in \text{Lip}(X,d).$$

Thus the mapping $f \to f^*$ is norm decreasing, hence continuous.

Condition (ii) can now be stated: $f^*(x,y) \to 0$ as $(x,y) \to K \times K$. The set of functions $h \in C(W)$ such that $h(x,y) \to 0$ as $(x,y) \to K \times K$ is closed in $C(W)$. Therefore, since $f \to f^*$ is continuous, the set of $f \in \text{Lip}(X,d)$ satisfying (i) and (ii) must be closed in $\text{Lip}(X,d)$,   Q.E.D.

Proof of Theorem 5.1. If f vanishes in a neighborhood of K, then f satisfies (i) and (ii) trivially. Since the set of functions satisfying (i) and (ii) is closed by the above lemma we see that all $f \in J(K)$ satisfy the conditions. Thus we have inclusion in one direction.

Now let $f \in \text{Lip}(X,d)$ satisfy conditions (i) and (ii). We must show $f \in J(K)$. It suffices to prove $f \in J(K)$ for $f \geqq 0$. We do this by constructing a sequence $\{f_n\}$ in $\text{Lip}(X,d)$ such that (a) $f_n = f$ in a neighborhood of K for each n, and (b) $\|f_n\| \to 0$ as $n \to \infty$. This shows $f \in J(K)$ by Lemma 3.1.

Let $S_n = \{x: d(x,K) < 1/n\}$ and $E_n = \{x: f(x) < 1/n^3\}$. Then $T_n = E_n \cap S_n$ is an open neighborhood of K in which $0 \leqq f(x) < 1/n^3$. We define f_n in two stages. We first define f_n on $V_n = T_n \cup S'_{n-1}$, where S'_{n-1} denotes the complement of S_{n-1} in X, by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in S'_{n-1}, \\ f(x) & \text{if } x \in T_n. \end{cases}$$

For $x, y \in V_n$ we have

$$\frac{|f_n(x) - f_n(y)|}{d(x,y)} = \begin{cases} 0 & \text{if } x, y \in S'_{n-1}, \\ \dfrac{|f(x) - f(y)|}{d(x,y)} & \text{if } x, y \in T_n, \\ f(x)/d(x,y) & \text{if } x \in T_n, \ y \in S'_{n-1}, \\ f(y)/d(x,y) & \text{if } y \in T_n, \ x \in S'_{n-1}. \end{cases}$$

If $x \in T_n, y \in S'_{n-1}$, then

$$d(x,y) \geqq \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$$

and $f(x) < 1/n^3$. Therefore if $x \in T_n$ and $y \in S'_{n-1}$ we have

$$|f_n(x) - f_n(y)|/d(x,y) < n(n-1)/n^3 \leqq 1/n.$$

The same inequality holds for $y \in T_n$ and $x \in S'_{n-1}$. Thus the Lipschitz bound for f_n on V_n is

$$\sup\left\{ \frac{|f_n(x) - f_n(y)|}{d(x,y)} : x, y \in V_n \right\} \leqq \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in T_n \right\} + \frac{1}{n}.$$

Also

$$\sup\{|f_n(x)| : x \in V_n\} \leqq 1/n^3.$$

We now extend f_n to all of X preserving the Lipschitz bound and without increasing the supremum. This is possible by Proposition 1.4. Then we have a sequence $\{f_n\}$ such that $f_n = f$ in T_n and with the norm bounds

$$\|f_n\|_d \leqq \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in T_n \right\} + \frac{1}{n}$$

and

$$\|f_n\|_\infty \leqq 1/n^3.$$

Now let $\varepsilon > 0$ be given. Since f satisfies (ii) there exists a neighborhood U of K such that

$$|f(x) - f(y)|/d(x,y) < \varepsilon$$

for all x, y in U. Choose N so that we have both $1/N < \varepsilon$ and $T_N \subset U$, this being possible since K is compact. Then for all $n \geqq N$

$$\|f_n\| = \|f_n\|_\infty + \|f_n\|_d$$

$$\leqq \frac{1}{n} + \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in U \right\} + \frac{1}{n}$$

$$< 3\varepsilon.$$

Hence $\|f_n\| \to 0$ as $n \to \infty$.   Q.E.D.

The following theorem will be the key in the identification of primary ideals in $\text{Lip}(X, d)$ given in Theorem 5.3. It will also be of importance in §9 where together with Theorem 5.1 it will lead to a description of the point derivations on $\text{Lip}(X, d)$. Although these applications are for the case $K = \{x\}$, we state it for any compact set K; the proof is no different.

THEOREM 5.2. *For each compact subset K of (X, d), $J(K) = \overline{M(K)^2}$ in* $\text{Lip}(X, d)$.

Proof. Let $g \in M(K)$ and set $f = g^2$; we assume g real-valued. Let f_n denote the truncation of f at $1/n$, and let g_n be the truncation of $|g|$ at $1/\sqrt{n}$. Then $f_n = g_n^2$, and for any $x, y \in X$, $x \neq y$, we have by Lemma 1.2 that

$$\frac{|f_n(x) - f_n(y)|}{d(x, y)} = \frac{|g_n(x)^2 - g_n(y)^2|}{d(x, y)}$$

$$= \frac{|g_n(x)| - |g_n(y)|}{d(x, y)} \cdot [|g_n(x)| + |g_n(y)|]$$

$$\leqq 2\|g\|_d \|g_n\|_\infty.$$

Hence,

$$\lim \|f_n\|_d \leqq 2\|g\|_d \lim \|g_n\|_\infty = 0.$$

It is obvious that the sup norms $\|f_n\|_\infty \to 0$ as $n \to \infty$. Thus $\|f_n\| \to 0$ as $n \to \infty$ and $f_n = f$ in a neighborhood of K. It follows from Lemma 3.1 that $f = g^2 \in J(K)$. Evidently finite linear combinations of squares of functions in $M(K)$ are in $J(K)$ so that for real-valued functions f_1 and f_2 in $M(K)$

$$f_1 f_2 = \frac{1}{4}[(f_1 + f_2)^2 - (f_1 - f_2)^2]$$

is again in $J(K)$. Finally if f_1 and f_2 in $M(K)$ are complex-valued we can consider the real and imaginary parts of the product separately to show that $f_1 f_2 \in J(K)$. Hence we have $M(K)^2 \subset J(K)$. Since $J(K)$ is closed, $\overline{M(K)^2} \subset J(K)$.

To show the reverse inclusion let $f \in \text{Lip}(X, d)$ vanish in a neighborhood U of K. Since K is compact, there exists a function $g \in \text{Lip}(X, d)$ satisfying $g = 0$ on K and $g = 1$ on $X - U$. (We may either use Proposition 1.4 on the extension of Lipschitz functions or construct g using the metric d.) Then $g^2 \in M(K)^2$ and $g^2 = 1$ on $X - U$. Since $M(K)^2$ is an ideal, $f = fg^2$ belongs to $M(K)^2$. It follows that $J(K) \subset \overline{M(K)^2}$.   Q.E.D.

Recall that a closed ideal I is primary at x if $J(x) \subset I \subset M(x)$. The next theorem identifies the closed primary ideals of $\text{Lip}(X, d)$ at $x \in X$.

THEOREM 5.3. *The closed primary ideals of $\text{Lip}(X, d)$ at $x \in X$ are precisely the closed linear subspaces of $\text{Lip}(X, d)$ between $J(x)$ and $M(x)$.*

Proof. Let $x \in X$. Suppose I is a closed linear subspace lying between $J(x)$ and $M(x)$. Let $g \in I$, $f \in \mathrm{Lip}\,(X, d)$. Then $g(f - f(x))$ belongs to $M(x)^2$, hence by Theorem 5.2 to $J(x)$. Since $J(x) \subset I$, $(gf - f(x)g) \in I$. Since I is a subspace and $g \in I$ we see that $gf \in I$. Hence I is an ideal. The hull of I is $\{x\}$ since $I \supset J(x)$. Thus I is a closed primary ideal at x. The converse statement follows from the fact that $\mathrm{Lip}\,(X, d)$ is regular and the definition of primary ideal.   Q.E.D.

Now let I be any closed proper ideal in $\mathrm{Lip}\,(X, d)$. Let K denote the hull of I. The primary components $P(I, x)$ of I for $x \in K$ were discussed in §3. We now state formally the conjecture mentioned there.

Conjecture. If I is a closed ideal in $\mathrm{Lip}\,(X, d)$ where (X, d) is compact and I has hull K, then

$$I = \bigcap_{x \in K} P(I, x).$$

We can verify the conjecture for the special case $I = J(K)$. Evidently the primary component of $J(K)$ at $x \in K$ is just $J(x)$.

PROPOSITION 5.4. *If K is a closed subset of the compact space (X, d), then*

$$J(K) = \bigcap_{x \in K} J(x).$$

Proof. Since $J(K) \subset \bigcap_{x \in K} J(x)$ trivially, it remains to show the reverse inclusion. Suppose $f \notin J(K)$. Then there exists an $\varepsilon > 0$ and a sequence $\{(x_n, y_n)\}$ such that $(x_n, y_n) \to K \times K$ as $n \to \infty$, and

(5.1) $|f(x_n) - f(y_n)|/d(x_n, y_n) \geqq \varepsilon$

for all n. It follows from the compactness of (X, d) that there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ such that $(x_{n_k}, y_{n_k}) \to (x, y)$ where x and y belong to K. We must have $x = y$; otherwise $d(x_{n_k}, y_{n_k}) \to d(x, y) \neq 0$, which implies that

$$\frac{|f(x_{n_k}) - f(y_{n_k})|}{d(x_{n_k}, y_{n_k})} \to \frac{|f(x) - f(y)|}{d(x, y)} = 0$$

since $f = 0$ on K. But this contradicts (5.1). Hence we have found an $x \in K$ such that $f \notin J(x)$, so that $f \notin \bigcap_{x \in K} J(x)$.   Q.E.D.

We shall discuss the conjecture and its relation to the theory of point derivations in §9.

6. **The quotient algebras $Q_x = A/J(x)$.**   For any regular algebra A the quotient algebras $Q_x = A/J(x)$ are of interest. The structure of Q_x is determined completely by the behavior of the elements of A in a neighborhood of x. The algebra Q_x is thus called the *local algebra at x*. We shall use the notation $f + J(x)$ to signify the element in $Q_x = A/J(x)$ corresponding to $f \in A$. The norm in the Banach algebra Q_x is given by

(6.1) $$\|f + J(x)\| = \inf\{\|f - g\| : g \in J(x)\}, \quad f \in A.$$

Because of the notation for elements in Q_x no confusion can arise from using $\|\cdot\|$ to denote both the norm in A and in Q_x.

Šilov introduced the *norm at the point* x [14, p. 25] which is denoted by $\|\cdot\|_x$ and defined by

(6.2) $$\|f\|_x = \inf\{\|g\| : g \in A,\ g = f \text{ in a neighborhood of } x\}, \qquad f \in A.$$

It is not difficult to see that

$$\|f + J(x)\| = \|f\|_x$$

for all $f \in A$, since functions which vanish in a neighborhood of x are dense in $J(x)$. The norm at the point x is not actually a norm on A, but if we define $\|\cdot\|_C$ on A by

$$\|f\|_C = \sup\{\|f\|_x : x \in \Sigma(A)\}, \qquad f \in A,$$

this is a norm on A. Šilov calls A an algebra of *type C* if $\|\cdot\|_C$ is equivalent to the original norm on A. If the two norms are not equivalent, then we may form the completion of A with respect to $\|\cdot\|_C$. The resulting algebra is called the *C-completion* of A. For a discussion of these notions in general, see [10; 14].

We now discuss the structure of the local algebras Q_x for the case $A = \text{Lip}(X, d)$ where (X, d) is compact. We first give a direct description of the norm in Q_x in terms of the functions themselves. For notational convenience, set

$$F(f, x) = \limsup_{(s,t) \to (x,x)} |f(s) - f(t)| / d(s, t)$$

for each $f \in \text{Lip}(X, d)$ and $x \in X$. It was expected that $\|f\|_x$ and $|f(x)| + F(f, x)$ would be equivalent. What is of interest is that they are identical.

THEOREM 6.1. *For each $f \in \text{Lip}(X, d)$ and $x \in X$,*

$$\|f\|_x = |f(x)| + F(f, x).$$

Proof. If $g \in \text{Lip}(X, d)$ is such that $g = f$ in a neighborhood of x, then

$$\|g\| \geq |g(x)| + F(g, x) = |f(x)| + F(f, x).$$

Hence, by definition of $\|\cdot\|_x$,

$$\|f\|_x \geq |f(x)| + F(f, x).$$

There remains the reverse inequality. The following construction is essentially the same as in the proof of Theorem 5.1, so we shall just give a sketch here and refer to that proof for details.

Let $S_n = \{t : d(t, x) < 1/n\}$, $E_n = \{t : |f(x) - f(t)| < 1/n^3\}$ and $T_n = S_n \cap E_n$. Define g_n on $V_n = T_n \cup (X - S_{n-1})$ by

$$g_n(t) = \begin{cases} f(x) & \text{if } t \in X - S_{n-1}, \\ f(t) & \text{if } t \in T_n. \end{cases}$$

Let

$$K_n = \sup \left\{ \frac{|f(s) - f(t)|}{d(s, t)} : s, \ t \in T_n, \ s \neq t \right\}.$$

Then as in the proof of Theorem 5.1, g_n has a Lipschitz bound of $K_n + 1/n$ on V_n and $|g_n(t) - f(x)| < 1/n^3$ for $t \in V_n$. Extending g_n to all of X while preserving these bounds (Proposition 1.4), we obtain a sequence $\{g_n\}$ such that $g_n = f$ in T_n and

$$\| g_n \|_d \leqq K_n + 1/n,$$

$$\| g_n - f(x) \|_\infty \leqq 1/n^3.$$

It is clear that

$$\| f \|_x \leqq \lim_{n \to \infty} \| g_n \|.$$

(In fact, it is not difficult to check that $\{g_n\}$ is actually a minimizing sequence for $\| f \|_x$; i. e., $\| f \|_x = \lim \| g_n \|$.) Obviously, $\lim_{n \to \infty} \| g_n \|_\infty = |f(x)|$. Also

$$\lim_{n \to \infty} \| g_n \|_d \leqq \lim_{n \to \infty} (K_n + 1/n) = \lim_{n \to \infty} K_n,$$

the limit existing because $\{K_n\}$ is a decreasing sequence bounded from below. Call this limit K. We must show that $K \leqq F(f, x)$ to finish the proof. To do this, we must show that given any $\varepsilon > 0$ and neighborhood U of x, there exist s, t in U such that

$$|f(s) - f(t)|/d(s, t) \geqq K - \varepsilon.$$

We first choose N so that for all $n > N$ we have $K_n > K - \varepsilon/2$. Now by definition of K_n we can choose points s_n, t_n in T_n such that

$$|f(s_n) - f(t_n)|/d(s_n, t_n) \geqq K_n - \varepsilon/2.$$

Choosing $n > N$ large enough to guarantee $T_n \subset U$, we see that there exist $s = s_n, t = t_n$ in U such that

$$|f(s) - f(t)|/d(s, t) \geqq K_n - \varepsilon/2 \geqq K - \varepsilon.$$

This shows that

$$\| f \|_x \leqq \lim_{n \to \infty} (\| g_n \|_\infty + \| g_n \|_d) \leqq |f(x)| + F(f, x)$$

and the proof is complete.   Q.E.D.

For $f \in \mathrm{lip}(X, d^\alpha)$, $0 < \alpha < 1$, we have

$$\lim_{(s,t) \to (x,x)} |f(s) - f(t)| / d(s, t) = 0,$$

so that from Theorem 6.1 the norm at the point x is simply $\|f\|_x = |f(x)|$. Hence $\|f\|_C = \|f\|_\infty$ for all $f \in \mathrm{lip}(X, d^\alpha)$. This is obviously not equivalent to the norm in $\mathrm{lip}(X, d^\alpha)$ so that $\mathrm{lip}(X, d^\alpha)$ is not of type C. Since $\mathrm{lip}(X, d^\alpha)$ separates the points of X and is closed under complex conjugation, it follows from the Stone-Weierstrass theorem that the C-completion of $\mathrm{lip}(X, d^\alpha)$ is $C(X)$.

The type C structure for $\mathrm{Lip}(X, d)$ seems much more involved and complicated. Obtaining a satisfactory description of the C-completion of $\mathrm{Lip}(X, d)$ appears to be quite difficult.

The algebraic structure of Q_x is not very interesting. From Proposition 5.2 we have $J(K) = \overline{M(K)^2}$. Thus the quotient $M(K)/J(K)$ is algebraically trivial; that is, all products are zero. The quotients Q_x, though not algebraically trivial, are not much more interesting. Products depend only on the linear structure of Q_x and the value of the functions at the point x.

For the remainder of this section we restrict our attention to $X = [0, 1]$ with the metric $d(x, y) = |x - y|^\alpha$, $0 < \alpha < 1$. The Lipschitz algebra for this metric space will be denoted by $\mathrm{Lip}\,\alpha$. We prove by a construction that the quotient algebras $Q_x = \mathrm{Lip}\,\alpha/J(x)$ are nonseparable. The construction can be modified so as to hold in more general metric spaces, but for the sake of clarity and to keep calculational complications to a minimum we consider only Lipschitz functions on $[0, 1]$. The Lipschitz norm in $\mathrm{Lip}\,\alpha$ will be denoted by $\|\cdot\|_\alpha$.

THEOREM 6.2. $Q_x = \mathrm{Lip}\,\alpha/J(x)$ is nonseparable.

Proof. The proof is for $x = 0$; the case $x \neq 0$ is similar.

Let P denote the set of all sequences $p = \{p_1, p_2, \cdots, p_n, \cdots\}$ where each term p_n of p is either 0 or 1 and where $p_n = 1$ for infinitely many n. The set P is well known to be uncountable. For each $p \in P$ we define the function F_p by setting $F_p(0) = 0$, $F_p(1/2^n) = p_n/2^{n\alpha}$, $n = 1, 2, \cdots$, and extending F_p linearly in the intervals $(1/2^{n+1}, 1/2^n)$. A short computation shows that $\|F_p\|_\alpha \leq 1$ so that the uncountable collection of functions $P^* = \{F_p : p \in P\}$ belongs to $\mathrm{Lip}\,\alpha$.

We now show that P^* is a discrete subset of $\mathrm{Lip}\,\alpha$. For let $p, q \in P$ with $p \neq q$. Then for some n, $p_n \neq q_n$, say $p_n = 1$ and $q_n = 0$. Then

$$\|F_p - Fq\|_\alpha \geq |F_p(1/2^n) - F_q(1/2^n)|/(1/2^n)^\alpha = 1.$$

Hence, P^* is discrete.

Since $F_p(0) = 0$, all $p \in P$, we have $P^* \subset M(0)$. Furthermore $P^* \cap J(0) = \varnothing$; to see this let $p \in P$ be given. Now $p = \{p_n\}$ has infinitely many terms which have value 1. Let $\{p_{n_1}, p_{n_2}, \cdots, p_{n_k}, \cdots\}$ be the collection of those p_n which are 1. Define the sequence $\{x_k\}$ by setting $x_k = 1/2^{n_k}$. Then $x_k \to 0$ as $k \to \infty$, and $F_p(x_k) = x_k^\alpha$ for all k. Hence

$$\limsup_{x \to 0} \frac{F_p(x)}{x^\alpha} \geqq 1$$

so that $F_p \notin J(0)$.

We now form the quotient space Q_0. Unfortunately, the discrete set $P^\#$ does not remain discrete when projected into Q_0. To rectify this we call p and q in P equivalent if $p_n = q_n$ except for finitely many n. Let P_0 denote the set of equivalence classes of P. Then P_0 is still uncountable and we now assert that $P_0^\#$ remains discrete when projected into Q_0. For suppose p and q are in P_0 and $p \neq q$. Then $p_n \neq q_n$ for infinitely many n. Now

$$\| F_p - F_q + J(0) \| = \inf \{ \| F_p - F_q + f \| : f \in \mathrm{Lip}\, \alpha, f \in J(0) \}.$$

The infimum can be taken over all f which vanish in a neighborhood of 0 since these functions are dense in $J(0)$. But for any such f we can choose N so large that $f(1/2^n) = 0$ for all $n \geqq N$. By construction of F_p and F_q there exists $n_0 \geqq N$ so that $F_p(1/2^{n_0}) \neq F_q(1/2^{n_0})$. Then,

$$\| F_p - F_q + f \|_\alpha \geqq \frac{| F_p(1/2^{n_0}) - F_q(1/2^{n_0}) |}{(1/2^{n_0})^\alpha} \geqq 1.$$

Thus the infimum taken over all f that vanish in a neighborhood of 0 will remain greater than or equal to 1. Hence, the set $P_0^\# = \{ F_p : p \in P_0 \}$ is an uncountable set whose image in Q_0 is discrete. Therefore Q_0 is nonseparable.   Q.E.D.

III. Point derivations

7. **Some algebraic properties of point derivations.** The notion of point derivation is a generalization of the process of evaluating the derivative of a function at a point. Since the product rule requires products of functions and evaluation of functions at a point to be defined, the most general setting for the notion of point derivation is an algebra of functions defined on a set. Of course, the algebraic operations will be the usual pointwise addition and multiplication of functions.

Throughout this section A will denote an algebra of complex-valued functions defined on a set X. To avoid uninteresting complications we assume that A contains sufficiently many functions to separate the points of X and that A contains the constant functions. A linear functional D on A is called a *point derivation at x* if

$$Dfg = f(x)Dg + g(x)Df, \qquad f, g \in A.$$

In this initial section we consider the purely algebraic properties of point derivations adding topological considerations in the next section.

For each $x \in X$ let $\mathfrak{D}_x$ denote the set of all point derivations on A at the point x. That $\mathfrak{D}_x$ is a linear space is easily seen.

The first lemma is a much used formula which is easily proved by induction.

LEMMA 7.1. *If* $D \in \mathfrak{D}_x$ *and* $f \in A$, *then* $Df^n = nf(x)^{n-1} Df$.

Since $D1 = D1^2 = 2D1$, we have $D1 = 0$. Thus an immediate consequence of Lemma 7.1 is that derivations annihilate the constant functions.

The next lemma is somewhat more complicated, but it is a straightforward calculation using only the product rule and Lemma 7.1.

LEMMA 7.2. *Let P be a polynomial in n variables and let P_k, $k = 1, 2, \cdots, n$, denote the partial derivatives of P with respect to the kth variable. Then for $D \in \mathfrak{D}_x$ and $f_1, \cdots, f_n$ in A,*

$$DP(f_1, \cdots, f_n) = \sum_{k=1}^{n} P_k(f_1(x), \cdots, f_n(x))Df_k.$$

LEMMA 7.3. *Suppose $f \in A$ satisfies $f^2 = f$. Then $Df = 0$ for every point derivation D on A.*

Proof. Let $x \in X$ and $D \in \mathfrak{D}_x$. By Lemma 7.1 we have

(7.1) $Df = Df^2 = 2f(x)Df$.

Since $f = f^2 = f^4$, we also have

(7.2) $Df = Df^4 = 4f(x)^3 Df$.

But $f^3 = f^2 \cdot f = f \cdot f = f^2 = f$. Therefore, equation (7.2) can be written

(7.3) $Df = 4f(x)Df$.

Thus, from equations (7.1) and (7.3) we obtain $2Df = Df$, so that $Df = 0$. Q.E.D.

PROPOSITION 7.4. *If $f \in A$ satisfies a polynomial identity $P(f) = 0$ where P is a polynomial in one variable, then $Df = 0$ for all point derivations D on A.*

Proof. If $P(f) = 0$, then the range of f is a finite set $\{\lambda_1, \cdots, \lambda_n\}$. That is, $f = \sum_{i=1}^{n} \lambda_i \chi_{E_i}$, where the sets $E_i \subset X$ are disjoint and $\bigcup_{i=1}^{n} E_i = X$ (χ_E denotes the characteristic function of the set E). Now given the points $\lambda_1, \cdots, \lambda_n$ in the complex plane, there exist polynomials $Q_1, \cdots, Q_n$ such that $Q_i(\lambda_j) = \delta_{ij}$, $i, j = 1, 2, \cdots, n$. Then $Q_i(f) \in A$ and $Q_i(f) = \chi_{E_i}$ for each i. Thus $\chi_{E_i} \in A$, $i = 1, 2, \cdots, n$. Since $\chi_{E_i}^2 = \chi_{E_i}$, from Lemma 7.3 we have $D\chi_{E_i} = 0$. Hence

$$Df = \sum_{n=1}^{n} \lambda_i D\chi_{E_i} = 0. \qquad \text{Q.E.D.}$$

The next propositions use the fact that A separates the points of X.

PROPOSITION 7.5. *If D is a point derivation on A and $D \not\equiv 0$, then there is a unique $x \in X$ such that $D \in \mathfrak{D}_x$.*

Proof. Assume $D \in \mathfrak{D}_x \cap \mathfrak{D}_y$ and $x \neq y$. Since D is not the zero functional, there exists $f \in A$ with $Df \neq 0$. We may assume that $f(x) = 0$, for otherwise f is replaced by $f - f(x)$. Since $D \in \mathfrak{D}_x \cap \mathfrak{D}_y$,

$$Df^2 = 2f(x)Df = 2f(y)Df.$$

Since $f(x) = 0$ and $Df \neq 0$, we have $f(y) = 0$.

We now choose $g \in A$ such that $g(x) = 0, g(y) \neq 0$, which is possible since A separates the points of X. Then the relation $g(x)Dg = g(y)Dg$ implies that $Dg = 0$. But then $D \in \mathfrak{D}_x$ yields

$$Dfg = f(x)Dg + g(x)Df = 0 \cdot Dg + 0 \cdot Df = 0,$$

while $D \in \mathfrak{D}_y$ implies

$$Dfg = f(y)Dg + g(y)Df = 0 + g(y)Df \neq 0,$$

a contradiction.   Q.E.D.

LEMMA 7.6. *Let* $D \in \mathfrak{D}_x$, $D \not\equiv 0$. *Then given any finite number of points* $x_1, \cdots, x_n$ *in* X, *it is possible to choose* $f \in A$ *so that* $Df = 1$ *and* $f(x_1) = \cdots = f(x_n) = 0$.

The proof of this lemma is a straightforward induction. It is used in proving the following.

PROPOSITION 7.7. *Let* $D_1 \in \mathfrak{D}_x$ *and* $D_2 \in \mathfrak{D}_y$ *be nonzero point derivations on* A *with* $x \neq y$. *Then a linear combination* $aD_1 + bD_2$ *of* D_1 *and* D_2 *is a point derivation on* A *if and only if* $a = 0$ *or* $b = 0$.

Proof. Sufficiency is clear. For necessity we assume $b \neq 0$ and prove $a = 0$. We may assume $b = 1$. Set $D = aD_1 + D_2$ and assume $a \neq 0$. For all f, g in A we have

(7.4) $Dfg = af(x)D_1g + ag(x)D_1f + f(y)D_2g + g(y)D_2f.$

D cannot be the zero derivation; for if $D \equiv 0$, we choose $f \in A$ with $D_1f = 1$ and $f(x) = f(y) = 0$ so that $g(x) = g(y)$ for all $g \in A$, which contradicts the point-separating property of A.

By hypothesis, D is a point derivation on A. Since $D \not\equiv 0$ there is by Proposition 7.5 a unique $u \in X$ such that $D \in \mathfrak{D}_u$. Then

(7.5) $Dfg = f(u)Dg + g(u)Df.$

We now choose $f \in A$ such that $Df = 1$ and $f(u) = f(x) = f(y) = 0$, which is possible by Lemma 7.6. Then equating (7.4) and (7.5) we have

(7.6) $g(u) = ag(x)D_1f + g(y)D_2f,$ $g \in A.$

If either $D_1f = 0$ or $D_2f = 0$, we obtain a contradiction to the fact that A sepa-

rates the points of X. If both $D_1 f$ and $D_2 f$ are nonzero, we can choose $g \in A$ so that $g(x) = 1$, $g(u) = g(y) = 0$. Then (7.6) becomes $0 = a D_1 f$ so that $a = 0$. Thus $a \neq 0$ is impossible and the proof is complete.   Q.E.D.

Proposition 7.5 and a weak form of Proposition 7.7 were first stated by S. B. Myers in [11] for regular Banach algebras A.

8. Point derivations on Banach algebras.   In this section A will denote a semi-simple commutative Banach algebra with identity. The carrier space of A will be denoted by X and we shall regard the elements of A (via the Gelfand representation) as functions on X. Together with the usual algebraic properties we now require point derivations to be continuous. The space $\mathfrak{D}_x$ of point derivations on A at x is thus a subspace of the dual space A^* of A.

It should be noted that many of the results of this section are valid in more general topological algebras, but since our main interest is in Lipschitz algebras we consider only the case of Banach algebras.

A subset B of a Banach algebra A is said to *generate A* if the smallest closed subalgebra of A containing B and the identity 1 of A is all of A. Equivalently, B generates A if the set of all polynomials (with constant term) in the elements of B are dense in A. The elements of B are called *generators* of A. The next proposition follows from Lemma 7.2 and continuity.

PROPOSITION 8.1. *If A is generated by the set B and if a point derivation D vanishes on B, then D vanishes on all of A.*

PROPOSITION 8.2. *For each $x \in X$, $\mathfrak{D}_x$ is a linear subspace of A^* closed in the weak* topology.*

Proof. To see that the subspace $\mathfrak{D}_x$ of A^* is closed in the weak* topology, let D be a point of closure of $\mathfrak{D}_x$ and let $\{D_\nu\}$ be a net from $\mathfrak{D}_x$ converging to D. Then for any f and g in A,

$$Dfg = \lim D_\nu fg = \lim [f(x)D_\nu g + g(x)D_\nu f] = f(x)Dg + g(x)Df.$$

Thus $D \in \mathfrak{D}_x$.   Q.E.D.

PROPOSITION 8.3. *If A has n generators, then for each $x \in X$ the dimension of $\mathfrak{D}_x$ is at most n.*

Proof. Let $f_1, \cdots, f_n$ generate A. Suppose there exist $n + 1$ point derivations $D_1, \cdots, D_{n+1}$ at x. Set $D_i f_k = a_{ik}$. For any polynomial P in n variables we have from Lemma 7.2

$$D_i P(f_1, \cdots, f_n) = \sum_{k=1}^{n} a_{ik} P_k(f_1(x), \cdots, f_n(x)),$$

where P_k denotes the first partial derivative of P with respect to the kth variable.

To show that $D_1, \cdots, D_{n+1}$ are linearly dependent we must find constants $c_1, \cdots, c_{n+1}$, not all zero, such that

$$\sum_{i=1}^{n+1} c_i D_i P(f_1, \cdots, f_n) = 0,$$

that is,

$$\sum_{k=1}^{n} P_k(f_1(x), \cdots, f_n(x)) \sum_{i=1}^{n+1} c_i a_{ik} = 0,$$

for all polynomials P. But this amounts to solving the system of equations

$$\sum_{i=1}^{n+1} c_i a_{ik} = 0, \qquad k = 1, 2, \cdots, n,$$

for the c_i. Since there are more unknowns than equations, this is trivial. Q.E.D.

The next proposition is due to Singer and Wermer [16].

PROPOSITION 8.4. $D \in A^*$ is a point derivation at x if and only if D vanishes on $\overline{M(x)^2}$ and the identity 1. $\mathfrak{D}_x$ is nontrivial if and only if $M(x) \neq \overline{M(x)^2}$.

The last proposition states that $\mathfrak{D}_x$ is the orthogonal complement of the subspace $\overline{M(x)^2} \oplus C$ of A; in symbols,

$$\overline{(M(x)^2 \oplus C)}^{\perp} = \mathfrak{D}_x.$$

Thus by a familiar Banach space theorem [5, p. 72], the spaces $\mathfrak{D}_x$ and $[A/(\overline{M(x)^2} \oplus C)]^*$ are isometrically isomorphic.

We now look at a few examples of Banach algebras and their point derivations.

(a) For $C(X)$, X a compact Hausdorff space, Singer and Wermer [16] showed that $M(x) = M(x)^2$ for every $x \in X$. Thus there are no (not even unbounded) nonzero point derivations on $C(X)$.

(b) $C^n(0, 1)$ has one generator [12, p. 300]; therefore by Proposition 8.3 the dimension of $\mathfrak{D}_x$, $0 \leqq x \leqq 1$, is at most 1. For each constant k and each point x, the mapping $f \to kf'(x)$, $f \in C^n(0, 1)$, is a nontrivial point derivation at x. Thus $\mathfrak{D}_x$ is 1-dimensional at each x and the only point derivations are the natural ones.

(c) Let $A(\Delta)$ denote the Banach algebra of functions which are analytic on the unit disc $\Delta = \{z : |z| < 1\}$ and continuous on the closure. Then $\Sigma = \overline{\Delta}$ and $A(\Delta)$ has one generator so that the dimension of $\mathfrak{D}_z$, $z \in \Delta$, is at most 1. For each $z \in \Delta$ we can exhibit the nontrivial point derivation $f \to kf'(z)$, $f \in A(\Delta)$, where k is a constant. Thus $\mathfrak{D}_z$ is 1-dimensional for each $z \in \Delta$. For z on the boundary of Δ, we have the following.

LEMMA. If $|z| = 1$, then $\mathfrak{D}_z = \{0\}$.

Proof. If $|z| = 1$, $M(z)$ is the closure of the principal ideal generated by the

function $g \in A(\Delta)$ defined by $g(\zeta) = z - \zeta$, $\zeta \in \bar{\Delta}$. See [3, p. 178]. That is, every $f \in M(z)$ is the limit of a sequence $\{gf_n\}$, $f_n \in A(\Delta)$. Since $|z| = 1$, the function $\sqrt{g}$ is also in $A(\Delta)$. Thus

$$f = \lim gf_n = \lim (\sqrt{g})(\sqrt{g}f_n)$$

where both $\sqrt{g}$ and $\sqrt{g}f_n$ are in $M(z)$. Hence, $f \in \overline{M(z)^2}$ so that $\overline{M(z)^2} = M(z)$. Hence, $\mathfrak{D}_z = \{0\}$.   Q.E.D.

(d) For $\text{lip}(X, d^\alpha)$, where (X, d) is compact and $0 < \alpha < 1$, it has been shown in §4 that every closed ideal is the intersection of the maximal ideals containing it. In particular, $\overline{M(x)^2} = M(x)$ for every $x \in X$. Hence, there are no nontrivial point derivations of $\text{lip}(X, d^\alpha)$.

(e) For $A = \text{Lip}\,\alpha$ (i.e., $X = [0, 1]$ with $d(x, y) = |x - y|^\alpha$), it has been shown in §6 that $A/J(x)$ is nonseparable. Manifestly, $A/(J(x) \oplus C)$ is also nonseparable. By Theorem 5.2, $J(x) = \overline{M(x)^2}$. Hence, $\mathfrak{D}_x = [A/(J(x) \oplus C)]^*$ is a nonseparable subspace of A^*. This shows that point derivations occur in great abundance on Lipschitz algebras. The structure of $\mathfrak{D}_x$ for $\text{Lip}(X, d)$ will be studied in the next section.

PROPOSITION 8.5. *Suppose $x \in X$ is an isolated point. Then $\mathfrak{D}_x = \{0\}$.*

Proof. Let $E = X - \{x\}$. Since x is an isolated point, the characteristic function χ_E belongs to A [15]. For any $f \in M(x)$ we then have $f = f\chi_E$ belonging to $M(x)^2$. Hence, $M(x) = M(x)^2$ so that there are no (not even unbounded) nontrivial point derivations on A at x.   Q.E.D.

The next two propositions were stated by Myers in [11]. We supply proofs here. It seems likely that the regularity hypothesis in Proposition 8.6 can be removed, but it remains an open question.

PROPOSITION 8.6. *If A is a regular algebra and if $f \in A$ vanishes in an open set $U \subset X$, then $Df = 0$ for all $D \in \mathfrak{D}_x$, all $x \in U$.*

Proof. Since A is regular there exists $g \in A$ such that $g = 0$ on $X - U$ and $g(x) = 1$. Then $fg \equiv 0$. Hence,

$$0 = Dfg = f(x)Dg + g(x)Df = Df.$$ Q.E.D.

PROPOSITION 8.7. *$\mathfrak{D} = \bigcup_{x \in X} \mathfrak{D}_x$ is closed in the weak* topology of A^*.*

Proof. Let D be a point of weak* closure for $\mathfrak{D}$. Then there is a net $\{D_\alpha\}$ in $\mathfrak{D}$ such that $D_\alpha \to D$ in the weak* topology. Each D_α is a point derivation at some $x_\alpha \in X$. $\{x_\alpha\}$ is then a net in X and since X is compact there exists a cluster point x_0 for $\{x_\alpha\}$. Choose a subnet $\{x_\beta\}$ such that $x_\beta \to x_0$ in X. Since each $f \in A$ is continuous on X, $f(x_\beta) \to f(x_0)$. The subnet $\{D_\beta\}$ of $\{D_\alpha\}$ still converges to D in the weak* topology. Hence given f, $g \in A$

$$Dfg = \lim D_\beta(fg) = \lim \left[f(x_\beta)D_\beta g + g(x_\beta)D_\beta f \right]$$

$$= f(x_0)Dg + g(x_0)Df.$$

Thus $D \in \mathfrak{D}_{x_0}$, so $D \in \mathfrak{D}$.   Q.E.D.

9. Point derivations on $\mathrm{Lip}\,(X, d)$. We now investigate the structure of the space of point derivations $\mathfrak{D}_x$ on $\mathrm{Lip}\,(X, d)$ for points $x \in X$. We give three characterizations of $\mathfrak{D}_x$, the first in terms of cluster points of certain sequences of functionals, the second in terms of Banach limits, and the third in terms of the Stone-Čech compactification of a certain space. The first two treatments hold in any metric space (X, d), but for the third we require (X, d) to be compact. S. B. Myers stated the first two results, Theorems 9.3 and 9.5, for compact (X, d) in [11]. The three descriptions of $\mathfrak{D}_x$ are similar in nature and all depend on the same basic facts; namely, (a) $\mathfrak{D}_x$ is the orthogonal complement of $\overline{M(x)^2} \oplus C$, (b) for $\mathrm{Lip}\,(X, d)$ the ideals $\overline{M(x)^2}$ and $J(x)$ are the same, and (c) by Theorem 5.1, the functions $f \in J(x)$ are those which satisfy $f(x) = 0$ and

$$\left| f(s) - f(t) \right| / d(s, t) \to 0 \text{ as } (s, t) \to (x, x).$$

Fix the point $x \in X$. We assume x is not isolated; otherwise $\mathfrak{D}_x = \{0\}$ by Proposition 8.5. Let W denote the product space $X \times X$ with the diagonal removed; that is, $W = \{(s, t) : s, t \in X, s \neq t\}$. Let W_x denote the set of all sequences $\{(x_n, y_n)\}$ from W which converge to (x, x). In other words, W_x is the set of all sequences $\{(x_n, y_n)\}$ of pairs of points from X such that $x_n \neq y_n$, all n, and $x_n \to x$, $y_n \to x$ in (X, d).

To each element $\{(x_n, y_n)\}$ of W_x we associate the sequence $\{\phi_n\}$ of bounded linear functionals on $\mathrm{Lip}\,(X, d)$ where ϕ_n is defined by

(9.1) $\phi_n(f) = [f(x_n) - f(y_n)]/d(x_n, y_n), \qquad f \in \mathrm{Lip}\,(X, d).$

Clearly $\left| \phi_n(f) \right| \leqq \|f\|_d$. Thus if $\| \cdot \|^*$ denotes the norm in the dual space $\mathrm{Lip}\,(X, d)^*$, we have

$$\| \phi_n \|^* = \sup \{ |\phi_n(f)| : \|f\| \leqq 1 \} \leqq 1.$$

That is, the sequence $\{\phi_n\}$ lies in the unit sphere S^* of $\mathrm{Lip}\,(X, d)^*$.

Let Φ_x denote the collection of all weak* cluster points of the set of all sequences $\{\phi_n\}$ defined by (9.1) as $\{(x_n, y_n)\}$ varies over W_x. Clearly $\Phi_x \subset S^*$. And since S^* is compact in the weak* topology, the set Φ_x is nonempty.

LEMMA 9.1. $\Phi_x \subset \mathfrak{D}_x$.

Proof. Suppose $D \in \Phi_x$. Then D is a weak* cluster point of a sequence $\{\phi_n\}$ given by (9.1) in terms of an element $\{(x_n, y_n)\}$ of W_x. Let $\{\phi_{n_\gamma}\}$ be a subnet of $\{\phi_n\}$ which converges to D in the weak* topology. Let $\{(x_{n_\gamma}, y_{n_\gamma})\}$ be the corresponding subnet of $\{(x_n, y_n)\}$. Then for any f and g in $\mathrm{Lip}\,(X, d)$,

$$Dfg = \lim \phi_{n_\gamma}(fg) = \lim \left[g(x_{n_\gamma})\phi_{n_\gamma}(f) + f(y_{n_\gamma})\phi_{n_r}(g) \right]$$

$$= g(x) \lim \phi_{n_\gamma}(f) + f(x) \lim \phi_{n_\gamma}(g)$$

$$= g(x)Df + f(x)Dg.$$

Hence $D \in \mathfrak{D}_x$.   Q.E.D.

PROPOSITION 9.2. *There exists a nontrivial point derivation on* $\mathrm{Lip}\,(X, d)$ *at each nonisolated point of* (X, d).

Proof. If x is not isolated, let $x_n \to x$ and $x_n \neq x$, all n. Then $\{(x_n, y_n)\} = \{(x_n, x)\}$ is an element of W_x and the sequence $\{\phi_n\}$ is given by (9.1) with $y_n = x$, all n. Define g on X by $g(t) = d(t, x), t \in X$, truncating if necessary, so that $g \in \mathrm{Lip}\,(X, d)$. Then $\phi_n(g) = 1$ for all sufficiently large n. If D is any weak* cluster point of $\{\phi_n\}$, then $Dg = 1$. Hence D is a nontrivial point derivation at x.   Q.E.D.

We now form the weak* closure of the linear subspace of $\mathrm{Lip}\,(X, d)^*$ spanned by Φ_x; call this $\mathrm{sp}(\Phi_x)$. Since $\mathfrak{D}_x$ is a weak* closed linear subspace of $\mathrm{Lip}\,(X, d)^*$ by Proposition 8.2, and since $\Phi_x \subset \mathfrak{D}_x$ by Lemma 9.1, it follows that $\mathrm{sp}(\Phi_x) \subset \mathfrak{D}_x$. We now show that the reverse inclusion holds.

THEOREM 9.3. $\mathrm{sp}(\Phi_x) = \mathfrak{D}_x$; *that is, the space of point derivations* $\mathfrak{D}_x$ *on* $\mathrm{Lip}\,(X, d)$ *at the point* $x \in X$ *is the weak* closure of the linear space spanned by* Φ_x.

Proof. The inclusion $\mathrm{sp}\,(\Phi_x) \supset \mathfrak{D}_x$ remains to be proved. Let

$$\mathfrak{D}_x^\perp = \{f \in \mathrm{Lip}\,(X, d) : Df = 0 \text{ all } D \in \mathfrak{D}_x\}$$

be the orthogonal complement of $\mathfrak{D}_x$ in $\mathrm{Lip}\,(X, d)$. Similarly, let $\mathrm{sp}\,(\Phi_x)^\perp$ denote the orthogonal complement of $\mathrm{sp}\,(\Phi_x)$ in $\mathrm{Lip}\,(X, d)$. Then the inclusion $\mathrm{sp}\,(\Phi_x) \supset \mathfrak{D}_x$ is equivalent to the inclusion

$$(9.2) \qquad\qquad \mathrm{sp}(\Phi_x)^\perp \subset \mathfrak{D}_x^\perp.$$

We know $\mathfrak{D}_x^\perp = \overline{M(x)^2} \oplus C = J(x) \oplus C$ from Proposition 9.4 and Theorem 5.2. To prove (9.2) suppose $f \notin \mathfrak{D}_x^\perp$. Then since $f \notin J(x)$ we have from Theorem 5.1 that there exists an $\varepsilon > 0$ such that for each n there exists points x_n and y_n in $S(x, 1/n)$, the sphere of radius $1/n$ centered at x, such that $x_n \neq y_n$ and

$$\left| f(x_n) - f(y_n) \right| / d(x_n, y_n) \geqq \varepsilon.$$

As $n \to \infty$, the sequence $\{(x_n, y_n)\}$ converges to (x, x) and

$$\liminf \frac{\left| f(x_n) - f(y_n) \right|}{d(x_n, y_n)} \geqq \varepsilon.$$

Then any weak* cluster point D of the sequence $\{\phi_n\}$ where $\{\phi_n\}$ is defined in terms of $\{(x_n, y_n)\}$ by (9.1) is in Φ_x and $Df \neq 0$. Hence $f \notin \mathrm{sp}\,(\Phi_x)^\perp$ and the inclusion (9.2) is established.   Q.E.D.

REMARK. We observe here that $\mathfrak{D}_x$ is *not* spanned by the weak* cluster points of sequences $\{\phi_n\}$ where ϕ_n is of the form

$$(9.3) \qquad\qquad \phi_n(f) = \frac{f(x_n) - f(x)}{d(x_n, x)}, \qquad f \in \text{Lip}(X, d),$$

where $x_n \to x$ and $x_n \neq x$, all n. That is, sequences of functionals defined by only a single sequence of points converging to x are inadequate to obtain all of $\mathfrak{D}_x$. Thus strict analogy with the definition of derivative of a function is not good enough to describe all point derivations at x. This can be seen by considering the function $f(x) = x^2 \sin(1/x)$, $x \neq 0$, $f(0) = 0$, in Lip 1. Details are left to the reader.

Let $\mathbf{m}$ denote the Banach space of all bounded sequences $\{x_n\}$ of complex numbers with sup norm. A Banach limit, $\text{LIM}(\cdot)$, is a bounded linear functional on $\mathbf{m}$ satisfying

(1) $\text{LIM}(\alpha x + \beta y) = \alpha \, \text{LIM}(x) + \beta \, \text{LIM}(y)$ for all $x, y \in \mathbf{m}$ and complex numbers α, β.

(2) If $x_n = 1$, all n, then $\text{LIM}(\{x_n\}) = 1$.

(3) If $x_n \geqq 0$, all n, then $\text{LIM}(\{x_n\}) \geqq 0$.

(4) $\text{LIM}(\{x_{n+1}\}) = \text{LIM}(\{x_n\})$, all $\{x_n\} \in \mathbf{m}$.

From these properties it is easy to deduce the further properties

(5) If $\lim x_n = \alpha$, then $\text{LIM}(\{x_n\}) = \alpha$.

(6) If $\lim x_n = \alpha$ and $\{y_n\} \in \mathbf{m}$, then $\text{LIM}(\{x_n y_n\}) = \alpha \, \text{LIM}(\{y_n\})$.

Let x be a nonisolated point of (X, d). Each element $w = \{(x_n, y_n)\}$ of W_x induces a mapping $\bar{w}$ of $\text{Lip}(X, d)$ into $\mathbf{m}$ defined by

$$\bar{w}(f) = \{[f(x_n) - f(y_n)] / d(x_n, y_n)\}, \qquad f \in \text{Lip}(X, d).$$

Clearly, $\| \bar{w}(f) \|_\infty \leqq \| f \|_d$, so that the mapping is norm decreasing, hence continuous. It is also clear that $\bar{w}$ is a linear mapping.

LEMMA 9.4. *Let* $\text{LIM}(\cdot)$ *be a Banach limit. Then for each* $w \in W_x$, *the bounded linear functional* D_w *on* $\text{Lip}(X, d)$ *defined by*

$$(9.4) \qquad\qquad D_w f = \text{LIM}(\bar{w}(f)), \qquad f \in \text{Lip}(X, d)$$

is a point derivation at x.

Proof. It is clear that D_K is bounded and linear. Let $f, g \in \text{Lip}(X, d)$. Then

$$D_w fg = \text{LIM}(\bar{w}(fg))$$

$$= \text{LIM}\left(\left\{ \frac{(fg)(x_n) - (fg)(y_n)}{d(x_n, y_n)} \right\} \right)$$

$$= \text{LIM}\left(\left\{ f(x_n) \frac{g(x_n) - g(y_n)}{d(x_n, y_n)} + g(y_n) \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \right)$$

$$= f(x) \, \text{LIM}(\bar{w}(g)) + g(x) \, \text{LIM}(\bar{w}(f))$$

$$= f(x) \, D_w g + g(x) \, D_w f,$$

by properties (1) and (6) of LIM($\cdot$) and the fact that f and g are continuous. Hence $D_w \in \mathfrak{D}_x$.   Q.E.D.

Let B_x denote the set of all such D_w defined on Lip(X,d) by (9.4) as w varies over W_x and where LIM($\cdot$) is a fixed Banach limit. By Lemma 9.4, $B_x \subset \mathfrak{D}_x$. Let sp$(B_x)$ denote the weak* closure of the linear subspace of Lip$(X,d)^*$ spanned by B_x.

THEOREM 9.5.   sp$(B_x) = \mathfrak{D}_x$.

Proof. Since $\mathfrak{D}_x$ is a weak* closed subspace of Lip$(X,d)^*$ and $B_x \subset \mathfrak{D}_x$, we have sp$(B_x) \subset \mathfrak{D}_x$.

To show the reverse inclusion we can again show sp$(B_x)^\perp \subset \mathfrak{D}_x^\perp$ as in the proof of Theorem 9.3. For if $f \notin \mathfrak{D}_x^\perp$, then it is possible to select a $w \in W_x$ such that $D_w f \neq 0$ (this uses property (2) of LIM($\cdot$)). The details are the same as for the proof of Theorem 9.3 and are omitted.   Q.E.D.

We proceed to the third and last characterization of $\mathfrak{D}_x$. (X,d) is now assumed to be compact. The Stone-Čech compactification of $W = X \times X - \{(x,x) : x \in X\}$ will be denoted by βW. Since (X,d) is compact the compactification of W occurs (crudely speaking) along the removed diagonal. It will be shown that each of the points in $\beta W - W$ determines a point derivation on Lip(X,d), and all point derivations can be obtained from these.

First we need a lemma about βW. It states that each $w \in \beta W - W$ "sits above" some point $x \in X$.

LEMMA 9.6. *Suppose the net* $\{w_\alpha\}$, *where* $w_\alpha = (x_\alpha, y_\alpha) \in W$, *converges to* $w \in \beta W - W$. *Then the nets* $\{x_\alpha\}$ *and* $\{y_\alpha\}$ *converge in* (X,d) *to the same point* $x \in X$.

Proof. The compactness of (X,d) implies the existence of cluster points of the nets $\{x_\alpha\}$ and $\{y_\alpha\}$; choose cluster points x and y respectively. Then there exists a subnet $\{w'_\alpha\}$ of $\{w_\alpha\}$ such that $w'_\alpha, x'_\alpha, y'_\alpha$ converge to w, x, y respectively. And since $w \in \beta W - W$, we must have $x = y$. Finally, the point x must be unique; that is, there can be no other cluster point of the nets $\{x_\alpha\}$ or $\{y_\alpha\}$. To see this, suppose $x'' \neq x$ were a cluster point of $\{x_\alpha\}$. Then arguing as above, we can find another subnet $\{(x''_\alpha, y''_\alpha)\}$ of $\{w_\alpha\}$ such that $x''_\alpha \to x''$ and $y''_\alpha \to x''$ in (X,d). But every bounded continuous function f on W can be extended to a continuous function θf on βW. Then

(9.5) $(\theta f)(w) = \lim f(x''_\alpha, y''_\alpha) = \lim f(x'_\alpha, y'_\alpha)$.

Define f on W by $f(u,v) = d(u,x)$, $(u,v) \in W$. Then f is a bounded continuous function on W and from (9.5) we get $d(x,x) = d(x'',x)$. This contradicts the assumption that $x \neq x''$.

Hence, the nets $\{x_\alpha\}$ and $\{y_\alpha\}$ each have only one cluster point x in (X, d) and therefore must converge to x. Q.E.D.

Let $C(W)$ denote the space of bounded continuous functions on W with sup norm $\|\cdot\|_W$. Each $f \in C(W)$ has a unique extension $\theta f \in C(\beta W)$ and $\|f\|_W = \|\theta f\|_{\beta W}$. Thus $C(W) = C(\beta W)$; that is, the two spaces are isometrically isomorphic. For each $f \in \mathrm{Lip}(X, d)$ we define $f^* \in C(W)$ by

$$f^*(x, y) = [f(x) - f(y)] / d(x, y), \qquad (x, y) \in W.$$

Since $\|f^*\|_W = \|f\|_d \leq \|f\|$, the mapping $f \to f^*$ from $\mathrm{Lip}(X, d)$ into $C(W)$ is continuous.

Now from Lemma 9.6 we obtain a partition of $\beta W - W$. Since W is dense in βW, we can select a net $\{w_\alpha\}$ from W converging to $w \in \beta W - W$. Then $w_\alpha = (x_\alpha, y_\alpha)$ and the nets $\{x_\alpha\}$ and $\{y_\alpha\}$ converge to some x_w in (X, d). This point x_w is independent of the choice of net $\{w_\alpha\}$ from W. For each $x \in X$, let $\Omega_x = \{w \in \beta W - W : x_w = x\}$. Then the collection of Ω_x for $x \in X$ is a partition of $\beta W - W$. Each Ω_x is the set of points in $\beta W - W$ which "sit above" the point $x \in X$.

LEMMA 9.7. *Let $x \in X$ and $w \in \Omega_x$. Define the functional D on $\mathrm{Lip}(X, d)$ by*

(9.6) $$Df = (\theta f^*)(w), \qquad f \in \mathrm{Lip}(X, d).$$

Then $D \in \mathfrak{D}_x$.

Proof. That D is a bounded linear functional is clear. We must show that D satisfies the product rule. Choose a net $\{w_\alpha\}$ from W converging to w in βW. Let $w_\alpha = (x_\alpha, y_\alpha)$. Then $x_\alpha \to x$ and $y_\alpha \to x$ in (X, d). Since each $f \in \mathrm{Lip}(X, d)$ is continuous we have $f(x_\alpha) \to f(x)$ and $f(y_\alpha) \to f(x)$. Hence, for any f, g in $\mathrm{Lip}(X, d)$

$$\begin{aligned}
Dfg &= (\theta(fg)^*)(w) = \lim (\theta(fg)^*)(w_\alpha) \\
&= \lim (fg)^*(w_\alpha) \\
&= \lim [g(x_\alpha)f^*(w_\alpha) + f(y_\alpha)g^*(w_\alpha)] \\
&= g(x) \lim (\theta f^*)(w_\alpha) + f(x) \lim (\theta g^*)(w_\alpha) \\
&= g(x)(\theta f^*)(w) + f(x)(\theta g^*)(w) \\
&= g(x)Df + f(x)Dg.
\end{aligned}$$

Thus $D \in \mathfrak{D}_x$. Q.E.D.

Now let ψ_x be the set of all point derivations D of the form (9.6) for $w \in \Omega_x$. Let $\mathrm{sp}(\psi_x)$ be the weak* closure of the linear space spanned by ψ_x.

THEOREM 9.8. $\mathfrak{D}_x = \mathrm{sp}(\psi_x)$.

Proof. It must be shown that $\mathrm{sp}(\psi_x) \supset \mathfrak{D}_x$, which is equivalent to showing that the orthogonal complements in $\mathrm{Lip}(X, d)$ satisfy $\mathrm{sp}(\psi_x)^\perp \subset \mathfrak{D}_x^\perp$. As in the

proof of Theorem 9.3 if $f \notin \mathrm{sp}\,(\psi_x)^{\perp}$, then $f \notin J(x)$, so that there exists an $\varepsilon > 0$ and a sequence $\{(x_n, y_n)\}$ in W such that $x_n \to x$, $y_n \to x$, and $|f^*(x_n, y_n)| \geqq \varepsilon$ for all n. The sequence $\{(x_n, y_n)\}$ must have a cluster point w in βW since βW is compact. It is easily seen that the point w must be in $\beta W - W$ and in fact, $w \in \Omega_x$. Then defining D by (9.6) we obtain a $D \in \mathfrak{D}_x$ with $Df \neq 0$. Thus $f \notin \mathfrak{D}_x^{\perp}$ and the proof is complete.   Q.E.D.

As an application of the theory of point derivations, we now give a characterization of weak sequential convergence in $\mathrm{Lip}(X, d)$ for compact (X, d). This is of interest since very little is known about the Banach space properties of $\mathrm{Lip}(X, d)$. The theory of point derivations may play an important role in future work along these lines.

THEOREM 9.9. *Let (X, d) be compact and let f, f_n, $n = 1, 2, \cdots$, be elements of $\mathrm{Lip}(X, d)$. The sequence $\{f_n\}$ converges weakly to f if and only if the set $\{f_n\}$ is bounded in $\mathrm{Lip}(X, d)$ and*

(1) $f_n(x) \to f(x)$, *all $x \in X$,*

(2) $Df_n \to Df$, *all $D \in \mathfrak{D}$,*

where $\mathfrak{D} = \bigcup_{x \in X} \mathfrak{D}_x$.

Proof. The necessity is obvious. For sufficiency, consider the sequence $\{\theta f_n^*\}$ in $C(\beta W)$. It is clearly bounded. If $w \in W$, then $(\theta f_n^*)(w) = f_n^*(w) \to f^*(w) = (\theta f^*)(w)$ from (1). If $w \in \beta W - W$, then by Lemma 9.7, we get $(\theta f_n^*)(w) \to (\theta f^*)(w)$ from (2). Hence, the sequence $\{\theta f_n^*\}$ converges pointwise in $C(\beta W)$ to θf^*. But this implies by [5, p. 265] that $\theta f_n^* \to \theta f^*$ weakly in $C(\beta W)$. Since $C(W) = C(\beta W)$, we have $f_n^* \to f^*$ weakly in $C(W)$.

Now fix a point $x_0 \in X$. Instead of the usual norm in $\mathrm{Lip}(X, d)$, we shall use the norm $\| \cdot \|'$ defined by $\|f\|' = |f(x_0)| + \|f\|_d$. It is easily verified that $\| \cdot \|'$ is a Banach space norm for $\mathrm{Lip}(X, d)$ which is equivalent to the usual norm. With the norm $\| \cdot \|'$, the closed subspace $M(x_0) = \{f \in \mathrm{Lip}(X, d) : f(x_0) = 0\}$ is isometrically isomorphic to a subspace of $C(W)$ under the mapping $f \to f^*$. Hence, the sequence $\{f_n - f(x_0)\}$ converges weakly to $f - f(x_0)$ in $M(x_0)$. But since $\mathrm{Lip}(X, d) = M(x_0) \oplus C$, and since $f_n(x_0) \to f(x_0)$ from (1), we see that $f_n \to f$ weakly in $\mathrm{Lip}(X, d)$.   Q.E.D.

For the remainder of the section the metric space (X, d) will be assumed compact. We now look at the relation between point derivations on $\mathrm{Lip}(X, d)$ and functions in $\mathrm{lip}(X, d)$. Let us first consider the case $X = [0, 1]$ and $d(x, y) = |x - y|$; i.e., Lip 1. If $f \in$ Lip 1 and if $Df = 0$ for all point derivations D, then $f'(x) = 0$ for almost all x. Since functions in Lip 1 are absolutely continuous, this implies that f is constant. More generally, we have the following.

PROPOSITION 9.10. *Let (X, d) be compact and let $f \in \mathrm{Lip}(X, d)$. Then $f \in \mathrm{lip}(X, d)$ if and only if $Df = 0$ for all point derivations D on $\mathrm{Lip}(X, d)$.*

Proof. Let $f \in \mathrm{lip}(X, d)$ and $x \in X$. Let $\{(x_n, y_n)\}$ be any element of W_x; i.e., $x_n \neq y_n$ for all n, and $(x_n, y_n) \to (x, x)$. Then $d(x_n, y_n) \to 0$, so by definition of $\mathrm{lip}(X, d)$, $f(x_n, y_n) \to 0$ as $n \to \infty$. Hence, by Theorem 9.3, $Df = 0$ for all $D \in \mathfrak{D}_x$.

Now suppose $f \notin \mathrm{lip}(X, d)$. Then there exists an $\varepsilon > 0$ and a sequence $\{(x_n, y_n)\}$ such that $d(x_n, y_n) \to 0$, $x_n \neq y_n$ all n, and

$$(9.7) \qquad\qquad\qquad |f^*(x_n, y_n)| > \varepsilon$$

for all n. Since (X, d) is compact there exists an $x \in X$ and a subsequenc $\{(x_{n_k}, y_{n_k})\}$ converging to (x, x). To this subsequence, an element of W_x, we may correspond a point derivation D at x by Lemma 9.1. For this D we have from (9.7) that $Df \neq 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Q.E.D.

For each closed subset K of (X, d) let $\mathfrak{D}_K$ denote the weak* closure of the linear subspace spanned by $\bigcup_{x \in K} \mathfrak{D}_x$. Then the above proposition states that

$$\mathrm{lip}(X, d)^\perp = \mathfrak{D}_X.$$

As a consequence of this it follows from a familiar Banach space theorem [5, p. 72] that

$$[\mathrm{Lip}(X, d) / \mathrm{lip}(X, d)]^* = \mathrm{lip}(X, d)^\perp = \mathfrak{D}_X.$$

We now turn to the ideal theory of $\mathrm{Lip}(X, d)$ and the relation between ideals and point derivations. First we shall use point derivations to construct some simple counter-examples.

It was shown in §5 that every closed subspace of $\mathrm{Lip}(X, d)$ between $J(x)$ and $M(x)$ is a closed primary ideal. We now show that this does not extend beyond primary ideals. The following example shows that if K contains two points, at least one of which is nonisolated, then there exists a closed subspace between $J(K)$ and $M(K)$ which is not an ideal. Let $x, y \in K$, $x \neq y$, and assume x is nonisolated. Choose point derivations $D_1 \in \mathfrak{D}_x$ and $D_2 \in \mathfrak{D}_y$ and assume $D_1 \neq 0$. Set $B = \{f \in \mathrm{Lip}(X, d) : f(K) = 0, \ (D_1 + D_2)f = 0\}$. Then B is clearly a closed linear subspace between $J(K)$ and $M(K)$, but B is not an ideal.

The next example shows that there exist ideals in $\mathrm{Lip}(X, d)$ which are not self-adjoint (i.e., closed under complex conjugation). Let D_1 and D_2 be nonzero elements of $\mathfrak{D}_x$ for some $x \in X$, and assume D_1 not proportional to D_2. Set $E = \{f : (D_1 + iD_2)f = 0, \ f(x) = 0\}$. Then E is a closed primary ideal at x. If $f = u + iv$ belongs to E, then a computation shows that $D_1 u = D_2 v$ and $D_2 u = -D_1 v$. If also $\bar{f} = u - iv$ belongs to E, then $D_1 u = -D_2 v$ and $D_2 u = D_1 v$. This implies that $D_1 u = D_2 u = 0$ and $D_1 v = D_2 v = 0$. But there clearly exists an $f = u + iv$ such that $f \in E$ while $D_1 f \neq 0$, so that either $D_1 u \neq 0$ or $D_1 v \neq 0$. Thus E is not self-adjoint.

We now relate primary ideals and primary components of ideals to the theory of point derivations. Let I be a closed ideal in $\mathrm{Lip}(X, d)$ with K as hull. For each $x \in K$ we associate with I a set of point derivations:

$$\mathfrak{D}_x(I) = \{D \in \mathfrak{D}_x : Df = 0, \text{ all } f \in I\} = I^\perp \cap \mathfrak{D}_x.$$

Plainly $\mathfrak{D}_x(I)$ is a weak* closed linear subspace of $\mathfrak{D}_x$.

Also for each weak* closed linear subspace H of $\mathfrak{D}_x$ we associate the set

$$I(H) = \{f : f(x) = 0, Df = 0 \text{ all } D \in H\} = M(x) \cap H^\perp.$$

Clearly $I(H)$ is a closed linear subspace of $\mathrm{Lip}(X, d)$. Since $H \subset \mathfrak{D}_x$, we have $H^\perp \supset \mathfrak{D}_x^\perp = J(x) \oplus C$. Hence, $I(H)$ is a closed subspace of $\mathrm{Lip}(X, d)$ which is between $J(x)$ and $M(x)$, so that by Theorem 5.3, $I(H)$ is a closed primary ideal at x.

PROPOSITION 9.11.   *Let I be a closed ideal in $\mathrm{Lip}(X, d)$ with hull K. Then for each $x \in K$, the primary component of I at x is $I_x = I(\mathfrak{D}_x(I))$.*

Proof.   We must show that I_x is the smallest closed primary ideal at x containing I. It has been noted above that I_x is a closed primary ideal at x, and clearly $I \subset I_x$. It remains to show that I_x is the smallest such ideal. Suppose I' is a closed primary ideal at x such that $I \subset I' \subset I_x$, and suppose that $I' \neq I_x$. Let $f \in I_x - I'$. Then by the Hahn-Banach theorem, there exists a bounded linear functional D on $\mathrm{Lip}(X, d)$ such that $Df \neq 0$ and $D = 0$ on $I' \oplus C$. Since I' is primary at x, $I' \supset J(x)$. Hence, $D = 0$ on $J(x) \oplus C$, so that $D \in \mathfrak{D}_x(I)$. Since $Df \neq 0$, this contradicts the definition of I_x.   Q.E.D.

It has been conjectured that $\mathrm{Lip}(X, d)$ has the ideal intersection property; that is, for every closed ideal I we have $I = \bigcap_{x \in K} I_x$ where K is the hull of I. This may be expressed in terms of point derivations as follows. Let $\mathfrak{D}(I) = \bigcup_{x \in K} \mathfrak{D}_x(I)$. If $f \in I$, then $Df = 0$ for all $D \in \mathfrak{D}(I)$. The ideal I will equal the intersection of its primary components if the facts that $f = 0$ on K and $Df = 0$ for all $D \in \mathfrak{D}(I)$ imply that $f \in I$. Thus the ideal intersection property can be recast to state that each closed ideal I is characterized by the vanishing of its elements on a subset K of X together with the annihilation of its elements by a certain set of point derivations. Formally, we can state the conjecture as follows.

Conjecture.   Let I be a closed ideal in $\mathrm{Lip}(X, d)$ with hull K. Then $f \in I$ if and only if $f = 0$ on K and $Df = 0$ for all $D \in \mathfrak{D}(I)$.

For closed primary ideals in $\mathrm{Lip}(X, d)$ such a characterization is valid. To each closed primary ideal at x, there corresponds the weak* closed subspace $\mathfrak{D}_x(I)$ of $\mathfrak{D}_x$; and conversely, to each weak* closed subspace H of $\mathfrak{D}_x$ there corresponds the closed primary ideal $I(H)$ at x. That is, there is a one-one correspondence between closed linear subspaces lying between $J(x)$ and $M(x)$ and weak* closed subspaces of $\mathfrak{D}_x$. Thus the functions $f \in I$ are characterized by their vanishing at x together with a certain collection of point derivations at x vanishing at f. In this sense, the ideal structure of $\mathrm{Lip}(X, d)$ is analogous to that of $C^n(0, 1)$ where each closed primary ideal is characterized by the functions in it vanishing at a certain point x together with derivatives at the point vanishing up to a certain order.

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