

DISTINGUISHED RINGS OF LINEAR TRANSFORMATIONS⁽¹⁾

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A ring R of linear transformations of a vector space M over a division ring D is called distinguished iff (1) the lattice J of all R -submodules of M is a distributive sublattice of the lattice L of all subspaces of M , and (2) the set of all linear transformations of M leaving J invariant is R . The study of such rings is motivated by a paper of Wolfson [2] in which J is a chain and by several recent papers of Behrens [4]. Behrens studies Artinian rings R with unity having faithful R -modules M such that the lattice of R -submodules of M is distributive.

Our primary interest is with distinguished rings R for which J is finite as well as distributive. Such a condition does not force the ring R to be Artinian. A basic tool in our study is a lattice theorem (1.1) stating that every element of J is a direct sum of elements of L associated with the irreducible elements of J . It is shown that every finite distributive sublattice of L containing 0 and M is the lattice of submodules of a distinguished ring R . If D has characteristic 0, then a finite sublattice of L must be distributive in order to be the lattice of submodules of a ring of linear transformations of M .

A subspace N of M is called J -distributive iff $N \cap (A \cup B) = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$. It is shown that N is J -distributive iff $N = Me$ for some idempotent $e \in R$. All subspaces of M are J -distributive iff J is a chain. Wolfson proved that R is a Baer ring (i.e., every annihilating right or left ideal of R is generated by an idempotent) if J is a chain. We show that this is almost the only case in which a distinguished ring is a Baer ring.

Every distinguished ring R is a direct sum of subrings of the form $e_i R e_j$, when $1 = e_1 + \cdots + e_n$ (direct sum), each $e_i R e_j$ is a full ring of linear transformations, $e_i R e_j = 0$ if $i < j$, and $\sum_{i \neq j} e_i R e_j$ is the radical of R . Two distinguished rings are shown to be isomorphic iff their vector spaces are related in an obvious way.

1. Introduction. A module ${}_D M$ over a ring D has associated with it a lattice $L({}_D M)$ of all submodules and a ring $E({}_D M)$ of all endomorphisms. For each subring R of $E({}_D M)$, we shall always consider M to be a bimodule ${}_D M_R$. Associated with each sublattice J of $L({}_D M)$ is a subring $R = R(J) = \{r \in E({}_D M) \mid Nr \subset N \text{ for every } N \in J\}$ of $E({}_D M)$. Clearly $J \subset L(M_R)$. Let us call a sublattice J of ${}_D M$

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a *distinguished lattice* (*d-lattice*) of ${}_D M$ iff $J = L(M_R)$. Also, let us call a subring R of $E({}_D M)$ a *distinguished ring* (*d-ring*) iff $R = R(J)$ where $J = L(M_R)$.

The ring D is restricted in this paper to be a division ring and the module ${}_D M$ to be a unital module over D (i.e., a vector space over D). Thus, every *d-ring* is a ring of linear transformations of some vector space over a division ring. If $C(D)$ denotes the center of D , then we may consider $C(D) \subset R(J)$ for every $J \subset L({}_D M)$ if we define $xa = ax$ for all $a \in C(D)$ and $x \in M$. It is well known that there is associated with each basis B of ${}_D M$ a division ring $D_B \subset E({}_D M)$ isomorphic to D . Thus, for each $d \in D$ we define $d' \in D_B$ by $xd' = dx$, $x \in B$.

It is easily shown that $R(L({}_D M)) = C(D)$. Hence, $L({}_D M)$ is a *d-lattice* of ${}_D M$ iff D is a field. If B is a basis of ${}_D M$ and J is the sublattice of $L({}_D M)$ generated by all atoms of the form Dx and $D(x + y)$, $x, y \in B$, then it can be shown that $R(J) = D_B$. On the other hand, if J is generated by only the atoms Dx , $x \in B$, then $R(J) \cong \prod_{i \in \Delta} D_i$ where $\Delta = \text{card } B$ and $D_i = D_B$ for each $i \in \Delta$. At the other extreme, $\{0, M\}$ is a lattice of ${}_D M$ having $E({}_D M)$ as its *d-ring*.

Our primary interest in this paper is with finite *d-lattices* of ${}_D M$ and their associated *d-rings*. Clearly each *d-lattice* of ${}_D M$ is a complete, modular lattice containing 0 and M , and $R(J)$ is a ring with unity. For each $N \in J$, we denote the lattice dimension of N by $\dim N$ ($\dim N$ is the length of the longest chain in the interval $[0, N]$ of J) and let $\dim J = \dim M$. If $J = L({}_D M)$ then $\dim N$ is the usual vector space dimension of a subspace N of M .

If J is any finite-dimensional modular lattice, then $N \in J$ is called *irreducible* iff $N \neq 0$ and N is not a union of lower-dimensional elements of J . Each irreducible $P \in J$ covers a unique element of J which we will always designate by P^0 . Clearly $P^0 = 0$ iff P is an atom of J . The set of all irreducible elements of J is designated by $I(J)$. Given a lattice J and $N \in J$, we shall also let $I(N) = \{A \in I(J) \mid A \leq N\}$.

Given a finite-dimensional distributive lattice J , each nonzero $N \in J$ may be uniquely represented as an irreducible union of irreducible elements of J , namely as the union of the maximal elements of $I(N)$ [1, p. 142]. It is easily verified in this case that $I(A \cup B) = I(A) \cup I(B)$ and $I(A \cap B) = I(A) \cap I(B)$ for all $A, B \in J$. Incidentally, we shall use the notation $\dot{\cup}$ for *direct union* in a lattice.

The following theorem is of basic importance to the rest of this paper.

1.1 THEOREM. *Let L be a complete, complemented, modular lattice with identities 0 and 1 and J be a finite-dimensional sublattice of L containing 0 and 1. For each irreducible $P \in J$, let us select $\bar{P} \in L$ so that $P = \bar{P} \dot{\cup} P^0$. Then each nonzero $K \in J$ may be represented in the form $K = \bigcup_{i=1}^m \bar{P}_i$ for some subset $\{P_1, \dots, P_m\}$ of $I(K)$. If J is distributive, then $K = \bigcup_{i=1}^n \bar{P}_i$ where $I(K) = \{P_1, \dots, P_n\}$.*

Proof. The theorem is trivially true if $\dim K = 1$. Let us assume that the conclusion holds for every $K \in J$ of dimension $\leq n$ and let $K \in J$, $\dim K = n + 1$. If K is irreducible, then $K = K^0 \dot{\cup} \bar{K}$ and K is a direct union of elements of the

form $\bar{P}$, P irreducible, since K^0 is. If J is distributive, then $I(K) = \{K\} \cup I(K^0)$ so that $K = \bigcup_{P \in I(K)} \bar{P}$ by the inductive assumption.

If K is reducible, then $K = A \cup B$ where $A, B \in J$, A irreducible, and $\dim B \leq n$. We know that $A = A^0 \dot{\cup} \bar{A}$ and $A \cap B \leq A^0$. Since $A \cap (A^0 \cup B) = A^0 \cup (A \cap B) = A^0$, clearly $\bar{A} \cap (A^0 \cup B) \leq \bar{A} \cap A^0 = 0$. Hence, $K = \bar{A} \dot{\cup} (A^0 \cup B)$ and the first conclusion follows from the inductive assumption once we observe that $\dim(A^0 \cup B) \leq n$. If J is distributive, the desired conclusion follows from the observation that $I(K) = \{A\} \cup I(A^0 \cup B)$. This proves 1.1.

2. Distributive lattices of ${}_dM$. Every finite-dimensional distributive lattice is actually finite [1, p. 139]. A finite, distributive sublattice of $L({}_dM)$ containing 0 and M will be called a *FD-lattice of ${}_dM$* henceforth. If J is a FD-lattice of ${}_dM$ and $I(J) = \{P_1, \dots, P_n\}$, then by 1.1, $M = \bar{P}_1 \dot{\cup} \dots \dot{\cup} \bar{P}_n$ where $\bar{P}_i$ is any relative complement (in $L({}_dM)$) of P_i^0 in P_i , $i = 1, \dots, n$.

2.1 THEOREM. *If J is a FD-lattice of ${}_dM$ and, for each $P \in I(J)$, $\bar{P} \in L({}_dM)$ is chosen so that $P = P^0 \dot{\cup} \bar{P}$, then $R(J) = \{a \in E({}_dM) \mid \bar{P}a \subset P \text{ for every } P \in I(J)\}$.*

Proof. If $a \in R(J)$ then clearly $\bar{P}a \subset Pa \subset P$ for every $P \in I(J)$. Conversely, if $a \in E({}_dM)$ and $\bar{P}a \subset P$ for every $P \in I(J)$, then $K = \bigcup_{P \in I(K)} \bar{P}$ for every $K \in J$ and $Ka \subset \bigcup_{P \in I(K)} P = K$. Hence, $a \in R(J)$. This proves 2.1.

It is clear from 2.1 that if J is a FD-lattice of ${}_dM$, $I(J) = \{P_1, \dots, P_n\}$, and $P_i = \bar{P}_i \dot{\cup} P_i^0$ for some $\bar{P}_i \in L({}_dM)$, $i = 1, \dots, n$, then for any $a_i \in \text{Hom}({}_d\bar{P}_i, {}_dP_i)$, $i = 1, \dots, n$, there exists a unique $a \in R(J)$ such that $a \mid \bar{P}_i = a_i$, $i = 1, \dots, n$. In particular, we have the following result (if we select $\bar{P}_i$ to contain each x_i).

2.2 COROLLARY. *If J is a FD-lattice of ${}_dM$, $I(J) = \{P_1, \dots, P_n\}$, and $x_i \in P_i$, $x_i \notin P_i^0$, $i = 1, \dots, n$, then for any $y_i \in P_i$, $i = 1, \dots, n$, there exists some $a \in R(J)$ such that $x_i a = y_i$, $i = 1, \dots, n$.*

2.3 COROLLARY. *If J is a FD-lattice of ${}_dM$, then each $K \in J$ is a cyclic $R(J)$ module.*

Proof. If $K = P_1 \cup \dots \cup P_m$, each P_i irreducible, then select $x_i \in P_i$, $x_i \notin P_i^0$, $i = 1, \dots, m$, and let $x = x_1 + \dots + x_m$. If $y \in K$, say $y = y_1 + \dots + y_m$, $y_i \in P_i$, then by 2.2 $x_i a = y_i$, $i = 1, \dots, m$, for some $a \in R(J)$ and $xa = y$. Hence, $xR(J) = K$ and 2.3 is proved.

2.4 THEOREM. *Every FD-lattice of ${}_dM$ is a d -lattice of ${}_dM$.*

Proof. Let J be a FD-lattice of ${}_dM$ and $R = R(J)$. To prove that J is a d -lattice of ${}_dM$, we need only show that $L(M_R) \subset J$. To this end, let $x \in M$, $x \neq 0$, $K = xR$, and N be the least element of J containing x . Clearly $K \subset N$. Let $\{P_1, \dots, P_m\}$ be the set of maximal elements of $I(N)$, so that $N = P_1 \cup \dots \cup P_m$, and let $x = x_1 + \dots + x_m$, $x_i \in P_i$. If any $x_i \in P_i^0$, say $x_1 \in P_1^0$, then

$$x \in P_1^0 \cup P_2 \cup \cdots \cup P_m < N$$

since $P_1 \not\subset P_1^0 \cup P_2 \cup \cdots \cup P_m$. This contradicts the choice of N . Hence $x_i \notin P_i^0$, $i = 1, \dots, m$. It follows as in the proof of 2.3 that $K = xR = N$. Since each cyclic submodule of $L(M_R)$ is contained in J and J is complete, $L(M_R) \subset J$. This proves 2.4.

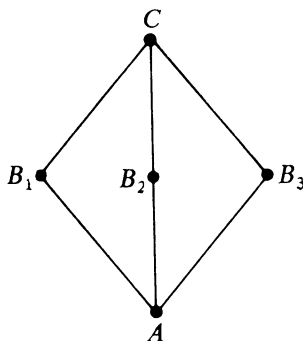
If J is a distributive lattice of finite dimension $n > 1$, then it is well known that $I(J)$ has n elements and that it may be imbedded in a Boolean algebra B of dimension n [1, pp. 139, 140]. Given an n -dimensional vector space ${}_D M$ and an atomic basis $\{A_1, \dots, A_n\}$ of $L({}_D M)$, the sublattice B' of $L({}_D M)$ generated by this basis is an n -dimensional Boolean algebra. Since $B \cong B'$, J is isomorphic to a sublattice J' of $L({}_D M)$. Clearly $0, M \in J'$ and J' is a FD -lattice of ${}_D M$. This proves the following result.

2.5 THEOREM. *Every distributive lattice J of finite dimension $n > 1$ is a FD -lattice of each n -dimensional vector space ${}_D M$.*

Not every finite d -lattice of a vector space ${}_D M$ need be distributive. For example, if D is a finite field and ${}_D M$ is finite-dimensional, then $L({}_D M)$ is a finite lattice that is not distributive. However, many finite d -lattices of a vector space are distributive according to the next result.

2.6 THEOREM. *If the division ring D has characteristic zero, then every finite d -lattice of ${}_D M$ is a FD -lattice of ${}_D M$.*

Proof. Let us assume that J is a finite d -lattice of ${}_D M$ that is not distributive. Then J must contain a sublattice of the type shown in the figure [1, p. 134]. Let



us select $B'_3 \in L({}_D M)$ such that $B_3 = A \dot{\cup} B'_3$ and let $\{z_i \mid i \in \Delta\}$ be a basis of B'_3 in ${}_D M$. Since $B_3 \subset B_1 + B_2$, $z_i = x_i + y_i$ for some $x_i \in B_1$ and $y_i \in B_2$, $i \in \Delta$. If B'_1 and B'_2 are the subspaces of B_1 and B_2 , respectively, generated by $\{x_i \mid i \in \Delta\}$ and $\{y_i \mid i \in \Delta\}$, then $B_1 = A \dot{\cup} B'_1$, $B_2 = A \dot{\cup} B'_2$, $\{x_i \mid i \in \Delta\}$ is a basis of B'_1 , and $\{y_i \mid i \in \Delta\}$ is a basis of B'_2 . For if $\sum d_i x_i \in A$ for some $d_i \in D$, then $\sum d_i z_i \in B_2 \cap B_3 = A$, $\sum d_i z_i = 0$, and each $d_i = 0$. Hence, $A \cap B'_1 = 0$ and $\{x_i \mid i \in \Delta\}$ is a basis of B'_1 . Similarly, $A \cap B'_2 = 0$ and $\{y_i \mid i \in \Delta\}$ is a basis of B'_2 .

Finally, each $x \in B_1$ has the form $x = y + z$ for some $y \in B_2$ and $z \in B_3$. In turn, $z = z' + \sum d_i z_i$ for some $z' \in A$ and $d_i \in D$, so that $x = y + z' + \sum d_i x_i + \sum d_i y_i$. Hence, $x - \sum d_i x_i \in B_1 \cap B_2 = A$. Consequently, $B_1 = A \dot{\cup} B'_1$ and, similarly, $B_2 = A \dot{\cup} B'_2$.

For each $i \in \Delta$, let us select a nonzero $\gamma_i = (b_i, c_i) \in C(D) \times C(D)$, and let $B'(\gamma)$ be the subspace of ${}_D M$ generated by $\{b_i x_i + c_i y_i \mid i \in \Delta\}$. If $\sum d_i (b_i x_i + c_i y_i) \in A$ for some $d_i \in D$, then $\sum d_i b_i x_i \in B_2$ and $\sum d_i c_i y_i \in B_1$. It follows that $\sum d_i b_i x_i = \sum d_i c_i y_i = 0$ and $d_i b_i = d_i c_i = 0$ for each i . Since $\gamma_i \neq 0$, $d_i = 0$ for each i . We conclude that $B'(\gamma) \cap A = 0$.

We shall next prove that for each $\gamma = \{\gamma_i \mid i \in \Delta\}$, the subspace $B(\gamma) = A \dot{\cup} B'(\gamma)$ of ${}_D M$ is in J . First, we observe that $B_3 a \subset B_3$ for each $a \in R = R(J)$, so that $(x_i + y_i)a = u_i + d_i(x_i + y_i)$ for some $u_i \in A$ and $d_i \in D$, $i \in \Delta$. Since $x_i a \in B_1$ and $y_i a \in B_2$ for each $a \in R$, we must have $x_i a = v_i + d_i x_i$ and $y_i a = w_i + d_i y_i$ for some $v_i, w_i \in A$, $i \in \Delta$. Hence, $(b_i x_i + c_i y_i)a = b_i v_i + c_i w_i + d_i(b_i x_i + c_i y_i)$, $i \in \Delta$, and $B(\gamma)a \subset B(\gamma)$ for each $a \in R$. Therefore, $B(\gamma) \in J$.

We easily see that there are an infinite number of distinct subspaces of ${}_D M$ of the form $B(\gamma)$. For example, let B'_n be the subspace generated by $\{(n-2)x_i + y_i \mid i \in \Delta\}$ and let $B_n = A \dot{\cup} B'_n$, $n = 2, 3, \dots$. If $n \neq m$ and $u \in B_n \cap B_m$, then $u = v + \sum d_i [(n-2)x_i + y_i] = w + \sum d'_i [(m-2)x_i + y_i]$ for some $v, w \in A$ and $d_i, d'_i \in D$, $i \in \Delta$. Hence, $\sum [d_i(n-2) - d'_i(m-2)]x_i \in B_1 \cap B_2 = A$ and, similarly, $\sum (d_i - d'_i)y_i \in A$. Consequently,

$$\sum [d_i(n-2) - d'_i(m-2)]x_i = 0$$

and $\sum (d_i - d'_i)y_i = 0$. Therefore, $d_i(n-2) - d'_i(m-2) = 0$ and $d_i - d'_i = 0$ for each i . From these equations, we easily see that $d_i = d'_i = 0$ for each i . Hence, $B_n \cap B_m = A$ for all m and n , $m \neq n$, and J contains the infinite set $\{B_2, B_3, \dots\}$ of subspaces. This is contrary to assumption, and proves 2.6.

If J is a FD -lattice of ${}_D M$, then J has a center $C(J)$ consisting of all complemented elements of J . If $\{M_1, \dots, M_k\}$ is the set of atoms of the Boolean algebra $C(J)$ and J_i denotes the interval $[0, M_i]$ of J , $i = 1, \dots, k$, then $M = M_1 \dot{\cup} M_2 \dot{\cup} \dots \dot{\cup} M_k$ and $J \cong J_1 \times J_2 \times \dots \times J_k$. The d -rings $R = R(J)$ and $R_i = R(J_i)$, $i = 1, \dots, k$, are easily seen to be related by an isomorphism, $R \cong R_1 \times R_2 \times \dots \times R_k$, under the correspondence $a \leftrightarrow (a_1, a_2, \dots, a_k)$, $a \in R$, where $a_i = a \mid M_i$, $i = 1, \dots, k$.

Let us call the d -ring $R = R(J)$ *indecomposable* iff $C(J) = \{0, M\}$. It is evident that R is indecomposable iff R is not a direct union of two nonzero ideals of R . For if $R = A \dot{\cup} B$, A and B nonzero ideals, then $M = MA \dot{\cup} MB$ and $MA, MB \in C(J)$. Conversely, if $M = M' \dot{\cup} M''$ for some nonzero $M', M'' \in J$ then $R = A \dot{\cup} B$ where $A = \{r \in R \mid Mr \subset M'\}$ and $B = \{r \in R \mid Mr \subset M''\}$ are nonzero ideals of R . Clearly each ring R_i above is indecomposable. We state our remarks above in the following form.

2.7 THEOREM. Every d -ring is isomorphic to a finite direct product of indecomposable d -rings.

3. J -distributivity. If J is a FD -lattice of ${}_dM$ and $N \in L({}_dM)$, then N is called J -distributive iff $N \cap (A \cup B) = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$. It is easily checked that if N is J -distributive and $J_N = \{N \cap K \mid K \in J\}$, then J_N is a FD -lattice of ${}_dN$. Each element of J is J -distributive since J is a distributive lattice. It is equally clear that every subspace N of ${}_dM$ is J -distributive if J is a chain. A useful characterization of the J -distributive elements of $L({}_dM)$ is given below.

3.1 THEOREM. If J is a FD -lattice of ${}_dM$ and $N \in L({}_dM)$, then the following statements are equivalent:

- (a) N is J -distributive.
- (b) For each $P \in I(J)$ there exists $P' \in L({}_dM)$ such that $P' \subset P$, $P' \cap P^0 = 0$, and $N = \bigcup_{P \in I(J)} P'$.
- (c) $N = Me$ for some idempotent $e \in R(J)$.

Proof. Let N be J -distributive and for each $P \in I(J)$ let $P' \in L({}_dM)$ be selected so that $(N \cap P^0) \dot{\cup} P' = N \cap P$. Clearly $P' \cap P^0 = 0$, since

$$P' \cap P^0 \subset (N \cap P^0) \cap P' = 0.$$

We shall prove that

$$(1) \quad N \cap K = \bigcup_{P \in I(K)} P' \text{ for every nonzero } K \in J.$$

To prove (1), we note first that it holds if K is an atom of J , since then $K' = N \cap K$. Let us assume that (1) holds for every element of J of dimension n or smaller, and let $K \in J$ be of dimension $n + 1$. Since N is J -distributive, $N \cap K = \bigcup_{P \in I(K)} N \cap P = \bigcup_{P \in I(K)} [(N \cap P^0) \cup P']$. Now $\dim P^0 \leq n$ for every $P \in I(K)$, and therefore $N \cap P^0 = \bigcup_{Q \in I(P^0)} Q'$. Hence,

$$N \cap K = \bigcup_{P \in I(K)} \left[\bigcup_{Q \in I(P^0)} Q' \cup P' \right] = \bigcup_{P \in I(K)} P'$$

since $I(P^0) \subset I(K)$ for every $P \in I(K)$. This proves (1).

If we let $K = M$ in (1), we obtain (b).

If (b) holds, then for each $P \in I(J)$ we may choose $\bar{P}$, $P'' \in L({}_dM)$ such that $P = P^0 \dot{\cup} \bar{P}$ and $\bar{P} = P' \dot{\cup} P''$. Since $M = \bigcup_{P \in I(J)} \bar{P}$ by Theorem 1.1, there is a well-defined idempotent $e \in E({}_dM)$ such that $e|P' = \iota$ and $e|P'' = 0$ for each $P \in I(J)$, where ι designates the identity mapping. Clearly $e \in R(J)$ since $Pe \subset P$ for every $P \in I(J)$. Since $N = \bigcup_{P \in I(J)} P'$, $Me = N$ by the very definition of e . Thus, (b) implies (c).

Finally, let $e \in R(J)$ be an idempotent and $N = Me$. It is easily shown that $N \cap K = Ke$ for every $K \in J$. Hence, $N \cap (A \cup B) = (A + B)e = Ae + Be = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$, and N is J -distributive. Thus, (c) implies (a). This proves 3.1.

3.2 COROLLARY. *If J is a FD-lattice of ${}_D M$ and $x \in M$, $x \neq 0$, then Dx is J -distributive iff $x \in P$, $x \notin P^0$, for some $P \in I(J)$.*

This follows directly from 3.1, part(b).

For each idempotent $e \in R(J)$, $1 - e$ also is an idempotent in $R(J)$ and $M = Me \dot{\cup} M(1 - e)$. By Theorem 3.1, $Me = \bigcup_{P \in I(J)} P'$ and $M(1 - e) = \bigcup_{P \in I(J)} P''$ for some $P', P'' \in L({}_D M)$ such that $Me \cap P = (Me \cap P^0) \dot{\cup} P'$ and $M(1 - e) \cap P = [M(1 - e) \cap P^0] \dot{\cup} P''$ for every $P \in I(J)$. It is easy to show that $P = P^0 \dot{\cup} (P' \dot{\cup} P'')$ for each $P \in I(J)$ and hence that $M = \bigcup_{P \in I(J)} (P' + P'')$. Since $e|P' = \iota$ and $e|P'' = 0$ for every $P \in I(J)$, we have shown that every idempotent $e \in R(J)$ is arrived at in the way shown in the proof of 3.1.

3.3 THEOREM. *If J is a FD-lattice of ${}_D M$ and S is the set of all J -distributive elements of $L({}_D M)$, then $S = L({}_D M)$ iff J is a chain.*

Proof. It is evident that if J is a chain then $S = L({}_D M)$. If J is not a chain, then there exists a chain $0 = K_0 < K_1 < \dots < K_r$ contained in J such that (1) if $K \in J$ and $K \neq K_i$ for $i = 0, 1, \dots, r$, then $K > K_r$, and (2) K_r is covered by at least two distinct elements P and P' of J . Evidently $P, P' \in I(J)$ and $P^0 = P'^0 = P \cap P' = K_r$. Let us select nonzero $x \in P$ and $x' \in P'$ such that $x, x' \notin K_r$, and let $N = D(x + x')$. Clearly N is not J -distributive, since $N \cap (P \cup P') = N$ whereas $N \cap P = N \cap P' = 0$. This proves 3.3.

4. Projectors. Let J be a FD-lattice of ${}_D M$ and $R = R(J)$. For all $K, N \in J$, define the *projector of K into N* , $K^{-1}N$, to be the largest subset of R such that $K(K^{-1}N) \subset N$. Clearly $K^{-1}N$ is an ideal of R . Since $KR \subset K$, evidently $K^{-1}N = K^{-1}(K \cap N)$. Thus, we might as well assume that $K \supset N$ in discussing the projector of K into N . If A is an ideal of R and $N \in J$, then the *projector of A into N* , NA^{-1} , is the largest subset of M such that $(NA^{-1})A \subset N$. It is clear that $NA^{-1} \in J$ and $NA^{-1} \supset N$. Finally, if A and B are ideals of R , we can define the right and left *projectors of B into A* , $B^{-1}A$ and AB^{-1} , as the largest subsets of R such that $B(B^{-1}A) \subset A$ and $(AB^{-1})B \subset A$, respectively. Evidently $B^{-1}A$ and AB^{-1} are ideals of R containing A . Since $AB^{-1} = (A \cap B)B^{-1}$ and $B^{-1}A = B^{-1}(A \cap B)$, we might as well assume that $B \supset A$ in discussing the projectors $B^{-1}A$ and AB^{-1} . The properties of projectors we shall need are contained in the following theorem.

4.1 THEOREM. *If J is a FD-lattice of ${}_D M$ and $R = R(J)$, then:*

- (1) $K(K^{-1}N) = N$ and $N(K^{-1}N)^{-1} = K$ if $K \supset N$, $K, N \in J$.
- (2) $N_1^{-1}N_2 = (N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}$ and $N_2^{-1}N_3 = (N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)$ if $N_1 \supset N_2 \supset N_3$, $N_i \in J$.

Proof. (1) There exists an idempotent $e \in R$ such that $Me = N$. Hence, $K \cap Me = Ke = N$, $e \in K^{-1}N$, and $K(K^{-1}N) = N$. To prove the second part, we first observe that $N(K^{-1}N)^{-1} \supset K$. Let us assume that $N(K^{-1}N)^{-1} = K' \neq K$.

Then $K' \in J$ and there exists some $P \in I(K')$ such that $P \notin K$. Hence, we can select $P' \in L({}_dM)$ so that $P' \cup P^0 = P$. Therefore $P' \cap K = 0$ and we can select $e \in R$ so that $e|K = 0$ and $e|P' = \iota$. Clearly $e \in K^{-1}N$ and $K'(K^{-1}N) \not\subset N$ since $K'e \supset P'$. However, $[N(K^{-1}N)^{-1}](K^{-1}N) \subset N$ according to the definition of $N(K^{-1}N)^{-1}$. This contradiction proves that $N(K^{-1}N)^{-1} = K$.

(2) Clearly

$$(N_1^{-1}N_2)(N_1^{-1}N_3) \subset N_1^{-1}N_3,$$

so that $N_1^{-1}N_2 \subset (N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}$ and $N_2^{-1}N_3 \subset (N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)$. On the other hand, $[(N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}](N_2^{-1}N_3) \subset N_1^{-1}N_3$ so that $N_1[(N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}](N_2^{-1}N_3) \subset N_3$. Hence, $N_1[(N_1^{-1}N_3)(N_2^{-1}N_3)^{-1}] \subset N_3(N_2^{-1}N_3)^{-1} = N_2$ and $(N_1^{-1}N_3)(N_2^{-1}N_3)^{-1} \subset N_1^{-1}N_2$. Therefore, $(N_1^{-1}N_3)(N_2^{-1}N_3)^{-1} = N_1^{-1}N_2$. Similarly, $(N_1^{-1}N_2)[(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)] \subset N_1^{-1}N_3$, $N_1(N_1^{-1}N_2)[(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)] \subset N_3$, $N_2[(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3)] \subset N_3$, and $(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3) \subset N_2^{-1}N_3$. Therefore, $(N_1^{-1}N_2)^{-1}(N_1^{-1}N_3) = N_2^{-1}N_3$, and 4.1 is proved.

If J is a FD -lattice of ${}_dM$ and $R = R(J)$, then an ideal of R of the form $A^{-1}0$, resp. $0A^{-1}$, for some ideal A of R is called a *right*, resp. *left*, *annihilating ideal* of R . Let us designate by $L_r(R)$, resp. $L_l(R)$, the set of all right, resp. left, annihilating ideals of R .

4.2 THEOREM. *If J is a FD -lattice of ${}_dM$ and $R = R(J)$, then $L_r(R) = \{K^{-1}0 \mid K \in J\}$ and $L_l(R) = \{M^{-1}K \mid K \in J\}$. Thus, the mapping $J \xrightarrow{\alpha} L_r(R)$ defined by $\alpha K = K^{-1}0$, $K \in J$, is a dual isomorphism and the mapping $J \xrightarrow{\beta} L_l(R)$ defined by $\beta K = M^{-1}K$, $K \in J$ is an isomorphism.*

Proof. By 4.1, (2), $K^{-1}0 = (M^{-1}K)^{-1}(M^{-1}0) = (M^{-1}K)^{-1}0$ and therefore $K^{-1}0 \in L_r(R)$ for each $K \in J$. Similarly, $M^{-1}K = (M^{-1}0)(K^{-1}0)^{-1} = 0(K^{-1}0)^{-1}$ and $M^{-1}K \in L_l(R)$ for each $K \in J$. If $A \in L_r(R)$ and $B = 0A^{-1}$, then $B \in L_l(R)$ and $A = B^{-1}0$. If we let $K = MB$, then $K \in J$ and $K^{-1}0 = B^{-1}0 = A$. Thus, A has the desired form. Also, $M(M^{-1}K)A = 0$, $(M^{-1}K)A = 0$, and $B \subset M^{-1}K \subset 0A^{-1} = B$. Hence, $B = M^{-1}K$ and B has the desired form. If $K^{-1}0 = N^{-1}0$ for some $K, N \in J$, then $K = 0(K^{-1}0)^{-1} = 0(N^{-1}0)^{-1} = N$ by 4.1, (1). If $M^{-1}K = M^{-1}N$ for some $K, N \in J$, then $K = M(M^{-1}K) = M(M^{-1}N) = N$ by 4.1, (1). Thus, α and β are bijections. This proves 4.2.

If J is a FD -lattice of ${}_dM$ and $R = R(J)$, and if $K, N \in J$ with $K \supset N$, then $K^{-1}N = (K^{-1}0)(N^{-1}0)^{-1} = (M^{-1}K)^{-1}(M^{-1}N)$ by 4.1, (2). That is, $K^{-1}N = (\alpha K)(\alpha N)^{-1} = (\beta K)^{-1}(\beta N)$ if we use the mappings α and β of 4.2. This shows that the ideals of the form $K^{-1}N$, where $K, N \in J$, may be defined intrinsically in the ring R itself. We shall use this fact later on in the paper.

If J is a FD -lattice of ${}_dM$ and $K, N \in J$ with $K \supset N$, then evidently $J' = \{N' - N \mid N' \in [N, K]\}$ is a FD -lattice of ${}_d(K - N)$. There is a natural mapping $R \xrightarrow{\phi} R'$ of $R = R(J)$ onto $R' = R(J')$ defined by: $(x + N)(c\phi) = xc + N$

for all $c \in R$ and $x + N \in K - N$. It is easily seen that ϕ is an epimorphism and $\ker \phi = K^{-1}N$. We state this result as follows.

4.3 THEOREM. *If J is a FD-lattice of ${}_D M$ and $K, N \in J$ with $K \supset N$, and if $J' = \{N' - N \mid N' \in [N, K]\}$, then J' is a FD-lattice of ${}_D(K - N)$ and $R(J') \cong R(J)/(K^{-1}N)$.*

5. Baer rings. While we defined the projector $K^{-1}N$ above only for $K, N \in J$, it is clear that $K^{-1}N$ may be defined in the same way for any subsets K and N of M or R . In particular, if J is a FD-lattice of ${}_D M$ and $R = R(J)$, then we can easily describe $K^{-1}N$ for any J -distributive subspaces K and N of M . Thus, $K = Me$ and $N = Mf$ for some idempotents $e, f \in R$, and $K^{-1}N \supset (1 - e)R + Rf$. On the other hand, if $a \in R$ then $a \in K^{-1}N$ iff $Mea \subset N$, or iff $ea = ea$. That is, $a \in K^{-1}N$ iff $a - af \in (1 - e)R$, or iff $a \in (1 - e)R + Rf$. Hence, $K^{-1}N = (1 - e)R + Rf$. It follows that $M^{-1}K = Re$ and $K^{-1}0 = (1 - e)R$ if $K = Me$. Evidently $M^{-1}K = 0(K^{-1}0)^{-1}$ and $K^{-1}0 = (M^{-1}K)^{-1}0$; that is, $M^{-1}K$ is an annihilating left ideal and $K^{-1}0$ is an annihilating right ideal of R . These are the only annihilating right or left ideals of R generated by idempotents according to the next result.

5.1 THEOREM. *Let J be a FD-lattice of ${}_D M$ and $R = R(J)$. If A is a right ideal and B is a left ideal of R such that $A = B^{-1}0$ and $B = 0A^{-1}$, then A and B are generated by idempotents iff either $0A^{-1}$ or MB is a J -distributive subspace of M .*

Proof. If $A = eR$ for some idempotent $e \in R$, then $B = R(1 - e)$ and $0A^{-1} = MB = M(1 - e)$. Hence, $0A^{-1} (= MB)$ is J -distributive by 3.1. Conversely, if either $0A^{-1}$ or MB is J -distributive, then either $0A^{-1} = Mf$ or $MB = Me$ for some idempotents $e, f \in R$. If $0A^{-1} = Mf$, then $A \subset (1 - f)R$ and $B \subset Rf$ since $MB \subset Mf$. Hence, $A = B^{-1}0 \supset (Rf)^{-1}0 = (1 - f)R \supset A$ and $A = (1 - f)R$, $B = Rf$. If $MB = Me$, then $(MB)^{-1}0 = B^{-1}0 = (1 - e)R = A$ and $B = 0A^{-1} = Re$. This proves 5.1.

A ring R is called a *Baer ring* iff every annihilating right or left ideal of R is generated by an idempotent. In a recent paper [2], Wolfson proved the following result.

5.2 THEOREM. *If J is a finite chain in $L({}_D M)$ containing 0 and M , then $R(J)$ is a Baer ring.*

We point out that this theorem follows directly from 3.3 and 5.1. Thus, for every right ideal A and left ideal B of $R(J)$ such that $A = B^{-1}0$ and $B = 0A^{-1}$, MB is J -distributive by 3.3 and therefore A and B are generated by idempotents according to 5.1.

It is evident that if the FD-lattice J of ${}_D M$ is isomorphic to a direct product of chains, then $R(J)$ is isomorphic to a direct product of Baer rings and hence is a

Baer ring. However, these are not the only distinguished rings that are Baer rings. We shall not give the details, but it may be shown that the ring R of all 3×3 matrices over a division ring D of the form $d_{11}e_{11} + d_{22}e_{22} + d_{31}e_{31} + d_{32}e_{32} + d_{33}e_{33}$, where e_{ij} are the usual matrix units, is a Baer ring. In this case, we may also represent R as $R(J)$, where $J = \{0, Dx_1, Dx_2, Dx_1 + Dx_2, M\}$ and $M = Dx_1 + Dx_2 + Dx_3$, a 3-dimensional vector space over D . If the lattice J is slightly more complicated, then $R(J)$ is not a Baer ring as the following theorem shows.

5.3 THEOREM. *If the FD-lattice J of ${}_D M$ contains three irreducible elements P_1, P_2, P_3 such that $P_1 \cup P_2 \subset P_3$ and $P_1 \cap P_2$ is different from 0, P_1 , and P_2 , then $R = R(J)$ is not a Baer ring.*

Proof. Let $\{P_1, \dots, P_n\} = I(J)$ and let $P'_i \in L({}_D M)$ be selected so that $P_i^0 \dot{\cup} P'_i = P_i$, $i = 1, \dots, n$. Select $x_i \in P'_i$, $x_i \notin P_i^0$, $i = 1, 2, 3$, and let $N = D(x_1 + x_2)$. Since $N \cap (P_1 \cup P_2) = N$ whereas $(N \cap P_1) \cup (N \cap P_2) = 0$, N is not J -distributive. If we imagine that $x_3 \in B_3$, a basis of P_3' , then there exists $b \in R$ such that $x_3 b = x_1 + x_2$, $x b = 0$ for $x \in B_3$, $x \neq x_3$, $P'_i b = 0$ if $i \neq 3$. Clearly $Mb = N$, and therefore $N^{-1}0 = b^{-1}0 = A$ is an annihilating right ideal of R . If A were generated by an idempotent, then $K = 0A^{-1}$ would be a J -distributive subspace of M , $K \supset N$. Now for any nonzero $y \in P_1 \cap P_2$, there exists some $a \in R$ such that $x_1 a = y$, $x_2 a = -y$, and $a|P'_i = \iota$ for $i = 3, \dots, n$. Since $(x_1 + x_2)a = 0$, evidently $a \in A$. However, $P_i a \neq 0$ for $i = 1, \dots, n$, and therefore $P_i \not\subset K$, $i = 1, \dots, n$. It follows that $K = 0$, contrary to the fact that $K \supset N$. Consequently, A is not generated by an idempotent. This proves 5.3.

For example, if $M = Dx_1 + Dx_2 + Dx_3 + Dx_4$ is a 4-dimensional vector space over D and $J = \{0, Dx_1, Dx_1 + Dx_2, Dx_1 + Dx_3, Dx_1 + Dx_2 + Dx_3, M\}$, then $R = R(J)$ is the ring of all 4×4 matrices over D of the form

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

In the notations of 5.3, $P_1 = Dx_1 + Dx_2$, $P_2 = Dx_1 + Dx_3$, and $P_3 = M$. Letting $N = D(x_2 + x_3)$ as we did in the proof of 5.3, $N^{-1}0 = A = e_{11}R + (e_{21} - e_{31})R + e_{44}R$ and $B = 0A^{-1} = R(e_{42} + e_{43})$. Since $B^2 = 0$, evidently B , and therefore also A , is not generated by an idempotent.

6. Subrings of distinguished rings. Let J be a FD-lattice of ${}_D M$ and $R = R(J)$. An idempotent $e \in R$ is called *simple* iff $e \neq 0$ and there exists an irreducible $P \in J$ such that $Me \subset P$ and $Me \cap P^0 = 0$. A sum $e = e_1 + \dots + e_m$ of idempotents

$e_i \in R$ is called *direct* iff $e_i e_j = \delta_{ij} e_j$, $i, j = 1, \dots, m$. Evidently a direct sum of nonzero idempotents of R is a nonzero idempotent of R .

We have seen that if $\{P_1, \dots, P_n\} = I(J)$, then associated with each nonzero idempotent $e \in R$ is a splitting $P_i = P_i^0 \dot{\cup} P_i' \dot{\cup} P_i''$, $i = 1, \dots, n$, such that $Me = \bigcup_{i=1}^n P_i'$ and $M(1-e) = \bigcup_{i=1}^n P_i''$. If we define $e_i \in R$ as follows: $e_i|P_i' = \iota$, $e_i|P_j' = e_i|P_k'' = 0$, $i \neq j$, $i, j, k = 1, \dots, n$, then clearly $e_i e_j = \delta_{ij} e_j$, $i, j = 1, \dots, n$, and $e = e_1 + \dots + e_n$. Since each nonzero e_i is simple, we have proved the following result.

6.1 THEOREM. *If J is a FD-lattice of ${}_dM$, then each nonzero idempotent in $R(J)$ is a direct sum of simple idempotents.*

If $e = e_1 + \dots + e_m$ is a direct sum of simple idempotents of $R = R(J)$, then the subring eRe of R is a direct sum of the subrings $e_i Re_j$. Let us now describe the structure of each subring $e_i Re_j$.

6.2 THEOREM. *If J is a FD-lattice of ${}_dM$ and e, f are nonzero idempotents of $R = R(J)$, then $eRf \cong S = \{a \in \text{Hom}({}_d(Me), {}_d(Mf)) \mid (Ke)a \subset Kf \text{ for every } K \in J\}$.*

Proof. We can consider $S \subset E({}_dM)$ by letting $a|M(1-e) = 0$ for each $a \in S$. Since $(Ke)eRf \subset Kf$ for each $K \in J$, evidently $eRf \subset S$. Now $KS = [Ke + K(1-e)]S = (Ke)S \subset Kf \subset K$ for each $K \in J$, and therefore $S \subset R$. Since $xa = x e a f$ for all $x \in M$ and $a \in S$, $S = eSf \subset eRf$. Thus, $S = eRf$ and 6.2 is proved.

6.3 COROLLARY. *If J is a FD-lattice of ${}_dM$, e is a nonzero idempotent of $R = R(J)$, and $N = Me$, then $eRl \cong R(J_N)$.*

We recall from §3 that $J_N = \{N \cap K \mid K \in J\}$. It follows from 6.3 that every subring of a d -ring R of the form eRe , e an idempotent, is also a d -ring.

6.4 THEOREM. *Let J be a FD-lattice of ${}_dM$ and e, f be simple idempotents of $R = R(J)$ such that $Me \subset P$, $Me \cap P^0 = 0$, and $Mf \subset Q$, $Mf \cap Q^0 = 0$, for given irreducible $P, Q \in J$. Then $eRf \neq 0$ iff $Q \subset P$. If $eRf \neq 0$, then $eRf \cong \text{Hom}({}_d(Me), {}_d(Mf))$.*

Proof. If $Q \subset P$ then $MeRf = Pf \neq 0$ and $eRf \neq 0$. If $Q \not\subset P$ then $Pf \subset Mf \cap Q \cap P \subset Mf \cap Q^0 = 0$, $MeRf = 0$, and $eRf = 0$. In case $Q \subset P$ and $K \in J$, we have $Ke \subset K \cap P \neq 0$ iff $K \supset P$. Hence, $(Ke)a \subset Kf$ for all $a \in \text{Hom}({}_d(Me), {}_d(Mf))$ and $K \in J$, and $eRf \cong \text{Hom}({}_d(Me), {}_d(Mf))$ in view of 6.2. This proves 6.4.

Let us call ring R a *full ring* iff $R = E({}_dN)$ for some vector space ${}_dN$. If J is a FD-lattice of ${}_dM$ and $R = R(J)$, then R is a full ring iff $J = \{0, M\}$. Hence, the result below follows directly from 6.3.

6.5 THEOREM. If J is a FD-lattice of ${}_D M$ and e is a nonzero idempotent of $R = R(J)$, then eRe is a full ring iff e is simple.

6.6 COROLLARY. The ring eRe of 6.5 is a division ring iff $\dim(Me) = 1$ in $L({}_D M)$. If eRe is a division ring, then $eRe \cong D$.

That $eRe \cong D$ is seen as follows. Since $\dim(Me) = 1$, $Me = Dx$ for some $x \in Me$. Now $x(eRe) \in L({}_D M)$ and therefore $x(eRe) = Dx$. Hence, the mapping $eRe \xrightarrow{\alpha} D$ defined by: $xere = (ere)^\alpha x$, $r \in R$, is an isomorphism.

If J is a FD-lattice of ${}_D M$ and $\{P_1, \dots, P_n\} = I(J)$, then we can order this set so that $P_j \not\subset P_i$ if $j > i$, $i, j = 1, \dots, n$ (for example, order $I(J)$ so that $\dim(P_i) \leq \dim(P_{i+1})$ in J). Let $1 = e_1 + \dots + e_n$ be a direct sum of simple idempotents of $R = R(J)$, with $Me_i \cup P_i^0 = P_i$, $i = 1, \dots, n$. Then $e_i Re_j = 0$ if $j > i$ and $R = \sum_{j \leq i=1}^n e_i Re_j$. Thinking in terms of a matrix representation of R , this says that the representation can be broken up into n^2 blocks with the blocks down the main diagonal full matrix rings and the blocks above the main diagonal all zero. The subring

$$A = \sum_{j < i=1}^n e_i Re_j$$

clearly is an ideal of R . Since $A^n = 0$ and $R/A \cong e_1 Re_1 + \dots + e_n Re_n$, a direct sum of primitive rings, evidently A is the Jacobson radical of R . We have proved part of the following theorem.

6.7 THEOREM. Let J be a FD-lattice of ${}_D M$ of dimension n and $R = R(J)$. Then 1 can be expressed as a direct sum of n simple idempotents of R , $1 = e_1 + \dots + e_n$, in such a way that R is a direct sum of subrings

$$R = \sum_{j \leq i=1}^n e_i Re_j.$$

The subring

$$A = \sum_{j < i=1}^n e_i Re_j$$

is the Jacobson radical of R , and

$$R/A \cong e_1 Re_1 + \dots + e_n Re_n,$$

a direct sum of full rings. The ideal A is nilpotent with index of nilpotency one less than the length of the longest chain in $I(J)$.

Proof. We need only find the index of nilpotency of A to complete the proof. Let $I(J) = \{P_1, \dots, P_n\}$, with $\dim P_i \leq \dim P_{i+1}$, $i = 1, \dots, n-1$. By 6.4, $e_i Re_j \neq 0$ iff $P_i \supset P_j$; and if $P_i \supset P_j$ then $e_i Re_j \cong \text{Hom}({}_D(Me_i), {}_D(Me_j))$. If $e_i Re_j \neq 0$ and $e_j Re_k \neq 0$, then for every nonzero $a \in e_i Re_j$ there exists some $b \in e_j Re_k$ such that $ab \neq 0$. It follows that $A^{m-1} \neq 0$ iff there exists a chain

$P_{i_1} \supset P_{i_2} \supset \cdots \supset P_{i_m}$ in $I(J)$. Hence, the index of nilpotency of A is the length of the longest chain in $I(J)$.

7. Isomorphic theorems. Distinguished rings are isomorphic iff the obvious isomorphisms hold between their underlying vector spaces, as we shall show below. Our results are extensions and generalizations of those of Wolfson [2, Theorem 7].

7.1 THEOREM. *Let J and J' be FD-lattices of ${}_D M$ and ${}_D M'$, respectively, and let $R = R(J)$ and $R' = R(J')$. Then $R \cong R'$ iff there exist isomorphisms $D \xrightarrow{\alpha} D'$ and $J \xrightarrow{\beta} J'$ such that $\dim_D(P - P^0) = \dim_{D'}(\alpha P - \alpha P^0)$ for every $P \in I(J)$.*

Proof. Let $R \xrightarrow{\phi} R'$ be an isomorphism. Then the mapping $J \xrightarrow{\alpha} J'$ defined by: $\alpha K = M'[(M^{-1}K)\phi]$, $K \in J$, is an isomorphism by 4.2. If $P \in I(J)$ and $P' = \alpha P$, then $E_D(P - P^0) \cong R/(P^{-1}P^0)$ and $E_{D'}(P' - P'^0) \cong R'/(P'^{-1}P'^0)$ by 4.3. Since $(P^{-1}P^0)\phi = [(M^{-1}P)^{-1}(M^{-1}P^0)]\phi = (M'^{-1}P')^{-1}(M'^{-1}P'^0) = (P'^{-1}P'^0)$, we conclude that $E_D(P - P^0) \cong E_{D'}(P' - P'^0)$ for every $P \in I(J)$. By a classical result of ring theory [3, p. 79], we have that $D \cong D'$ and $\dim_D(P - P^0) = \dim_{D'}(P' - P'^0)$ for every $P \in I(J)$.

Conversely, let us assume that $D \xrightarrow{\alpha} D'$ and $J \xrightarrow{\beta} J'$ are isomorphisms such that $\dim_D(P_i - P_i^0) = \dim_{D'}(\alpha P_i - \alpha P_i^0)$, $i = 1, \dots, n$, where $I(J) = \{P_1, \dots, P_n\}$. We may select $\bar{P}_i \in L_D(M)$ and $\bar{P}'_i \in L_{D'}(M')$ such that $P_i = \bar{P}_i \dot{\cup} P_i^0$ and $\alpha P_i = P'_i = \bar{P}'_i \dot{\cup} P_i'^0$, $i = 1, \dots, n$. By assumption, there exist bases $\{x_{ij} | j \in \Delta_i\}$ of $\bar{P}_i$ and $\{x'_{ij} | j \in \Delta_i\}$ of $\bar{P}'_i$, $i = 1, \dots, n$. Each element of M is uniquely represented as a finite sum of the form $\sum_{i,j} d_{ij}x_{ij}$, $d_{ij} \in D$, by 1.1. Thus, there is a unique mapping $M \xrightarrow{\theta} M'$ defined by: $(\sum d_{ij}x_{ij})\theta = \sum d_{ij}\alpha x'_{ij}$. It is clear that θ is a 1-1 semilinear mapping of M onto M' that maps J onto J' . Hence the mapping $E_D(M) \xrightarrow{\psi} E_{D'}(M')$ defined by: $a\psi = \theta^{-1}a\theta$, $a \in E_D(M)$, is an isomorphism [3, p. 45]. If $r \in R$, so that $P_i r \subset P_i$, $i = 1, \dots, n$, then $P'_i(r\psi) = P'_i(\theta^{-1}r\theta) = P_i(r\theta) \subset P_i \subset P'_i$, $i = 1, \dots, n$, and $r\psi \in R'$. Conversely, if $r\psi \in R'$ then $P'_i(r\psi) = P_i(r\theta) \subset P'_i$, $P_i r \subset P'_i \theta^{-1} = P_i$, $i = 1, \dots, n$, and $r \in R$. Thus, if we let $\phi = \psi | R$, the mapping $R \xrightarrow{\phi} R'$ is an isomorphism. This proves 7.1.

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