SEQUENTIALLY 1-ULC TORI

BY
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1. **Introduction.** A closed set X in Euclidean 3-space E^3 is called *tame* if there exists a homeomorphism h of E^3 onto itself such that $h(X)$ is a polyhedron. A set which is not tame is called *wild*. In this paper, we investigate conditions which determine tameness of an arc in E^3. Examples of wild arcs in E^3 are abundant; see, for example, [3; 8]. Also abundant are conditions implying tameness of an arc; see [7; 10].

Consider the following conditions placed on an arc $\mathscr{A}$ in E^3:

(1) $\mathscr{A}$ lies on a 2-sphere S in E^3.

(2) $\mathscr{A}$ lies on a simple closed curve J in E^3 which is the intersection of a nested sequence of (two-dimensional) tori plus their interiors.

This paper was motivated by a belief that (1) and (2) implied that $\mathscr{A}$ is tame. This turns out not to be the case; the wild arc constructed in [1] is a counterexample. With this in mind, we make the following definition. A sequence $\{M_1, M_2, \cdots\}$ of 2-manifolds in E^3 is *sequentially* 1-ULC if, given $\varepsilon > 0$, there exists a $\delta > 0$ and integer N such that: Whenever $n > N$, and α is a simple closed curve on M_n of diameter less than δ which bounds a disk on M_n, then α bounds a disk of diameter less than ε on M_n.

We now add another condition.

(3) The sequence of tori of condition 2 is sequentially 1-ULC.

Our primary result is that these three conditions imply tameness of the arc $\mathscr{A}$. This theorem yields as a corollary an answer to a question raised by Bing in [3]: No subarc of the "Bing sling" [3] lies on a disk.

A simple closed curve J is said to pierce a disk D if J links Bd D (boundary of D) and $J \cap D$ is a single point. As the "Bing sling" is the only example in the literature of a simple closed curve that pierces no disk, one is now led to a natural question. Can a different simple closed curve $\mathscr{K}$ be constructed where $\mathscr{K}$ pierces no disk, yet lies on a disk? In §3, we show the existence of such a simple closed curve $\mathscr{K}$. That $\mathscr{K}$ lies on a disk will be immediate from its construction. To show that $\mathscr{K}$ pierces no disk, we will use the following. Define $P_{\mathscr{A}}$ to be the set of points of an arc $\mathscr{A}$ at which $\mathscr{A}$ pierces a disk. We set up an alternate condition to (3) given above.

(3′) $P_{\mathscr{A}}$ is dense in $\mathscr{A}$.

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Conditions (1), (2), and (3′) are also shown to imply tameness. This result is then used to establish that $\mathscr{K}$ pierces no disk.

2. No subarc of the "Bing sling" lies on a disk.

THEOREM 1. *If $\mathscr{A}$ is an arc in E^3 such that*

(1) *$\mathscr{A}$ lies on a 2-sphere S in E^3;*

(2) *$\mathscr{A}$ lies on a simple closed curve J in E^3 which is the intersection of a decreasing sequence of tori plus their interiors;*

(3) *The sequence of tori of* (2) *is sequentially* 1-ULC;
then $\mathscr{A}$ is tame.

Proof. We assume without loss of generality that the 2-sphere S is locally polyhedral mod $\mathscr{A}$ [4]. We will use Theorem 6 of [5] to establish that S is locally tame at all non-endpoints of the arc $\mathscr{A}$. Toward this goal, we prove the following.

ASSERTION. Given a non-endpoint p of $\mathscr{A}$, and $\varepsilon > 0$, there exists a $\delta > 0$ such that: if β is a simple closed curve lying in a δ-neighborhood of p, $\beta \cap S = \varnothing$, and β bounds a disk B in E^3, then β bounds a disk B'' in $E^3 - S$ such that B'' lies in an ε-neighborhood of p. This Assertion is a bit weaker than the statement that $E^3 - S$ is locally simply connected at p, which is the hypothesis of Theorem 6 of [5]. However, the Assertion is sufficiently strong so that the proof of Theorem 6 of [5] still remains valid, showing that S is locally tame at p. We now prove the Assertion in six steps, numbered for convenience.

(1) There exists an integer N_1 and a positive number γ such that if α is any simple closed curve on $T_n, n > N_1$, and if α lies in a neighborhood of p of radius γ (which we abbreviate $\mathcal{O}_\gamma(p)$), then either $\alpha \cap S \neq \varnothing$, or α bounds a disk on T_n. Step 1 is devoted to a justification of this statement.

The arc $\mathscr{A}$ is now extended to form a simple closed curve K, $\mathscr{A} \subset K \subset S$. We assume $K \cap J = A$, i.e., $K \cup J$ is a θ-curve. Call the end points of $\mathscr{A}$ a and b, and call the sequence of tori given by our hypothesis $\{T_1, T_2, T_3, \cdots\}$; these may be taken to be polyhedral [4] and in general position. Let P be a plane missing p and separating a from b in E^3, with P in general position with respect to $\{T_1, T_2, \cdots\}$. For fixed n, $T_n \cap P$ is a collection of simple closed curves $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_k$. Let L_i be the disk on P bounded by λ_i. There exists an integer N_1 such that if n were chosen above to be larger than N_1, then L_i would not intersect both $\mathscr{A}$ and $J - \mathscr{A}$, for $i = 1, 2, \cdots, k$. This fact can be used to show that at least one of the λ_i's links J (in the sense of §9 of [4]). Thus, if $n > N_1$, at least one λ_I will link J. Note that such a λ_I could not bound a disk on T_n, but must lie on T_n in a nontrvial way.

There exists a positive number γ such that if a simple closed curve α lies in $\mathcal{O}_\gamma(p)$, then α does not intersect the plane P, and α does not link the simple closed curve $(J \cup K) - \text{Int } \mathscr{A}$. Now let us suppose that α lies on T_n, for $n > N_1$, and α does not bound a disk on T_n. Then α and the λ_I of the preceding paragraph can be joined by an annulus on T_n; since λ_I links J, α must also link J. Since α does not link $(J \cup K) - \text{Int}\,\mathscr{A}$, it follows from Theorem 9 of [4] that α links K. Since K lies on the 2-sphere S, it follows that $\alpha \cap S \neq \emptyset$, which completes step 1.

(2) We assume that diameter $\mathscr{A} > \varepsilon/3$. There exists a $\delta_1 > 0$ such that any δ_1-simple closed curve (a δ_1-set is a set of diameter less than δ_1) on S bounds an $\varepsilon/3$-disk on S. In particular $\delta_1 < \varepsilon/3$, of course. We use the sequential 1-ULC hypothesis to select a $\delta_2 > 0$ and integer N_2 such that any δ_2-simple closed curve on T_n, $n > N_2$, which bounds a disk on T_n bounds a $\delta_1/3$-disk on T_n; in particular $\delta_2 < \delta_1/3$.

(3) We now select a disk U on S containing $\mathscr{A}$ on its interior, "thin" enough so U has the following property: If W is any open set containing $\mathscr{A}$, and X is an open set containing S, then there exists a homeomorphism H of E^3 onto itself such that $H(S) = S$, $H =$ identity on $E^3 - X$, $H(U) \subset W$, and H moves no point of E^3 more than the minimum of the two numbers $\delta_2/3$ and $\gamma/2$.

The existence of such a disk U follows from the fact that S is locally tame, mod $\mathscr{A}$. To see this, note that if we had asked in the preceding paragraph that H be defined only on the 2-sphere S, then it is clear how to select U. In fact, in this case, H could be defined to be the identity on a small disk D_w, with $\mathscr{A} \subset \text{Int} D_w \subset D_w \subset \text{Int}\ U$. On the set $S - D_w$, where H is not the identity, H is isotopic to the identity. Furthermore, this set is tame, since it misses $\mathscr{A}$; hence it is bicollared in E^3. We now extend H to the bicollar in the obvious way, so that $H =$ identity, except on this bicollar. By choosing the bicollar to lie in X, we find H satisfies all required properties.

(4) We now select the $\delta > 0$ required in the Assertion, by requiring that $\delta < \delta_2/6$, $\delta < \gamma/2$ and $\mathscr{O}_\delta(p) \cap [S - U] = \emptyset$. We now prove the Assertion. Let β be a simple closed curve in $\mathscr{O}_\delta(p)$ such that β bounds a disk B, and $\beta \cap S = \emptyset$. We may assume that B lies in $\mathscr{O}_\delta(p)$ simply by pushing it there without moving β.

(5) In this step, we show that β bounds a disk B' of diameter less than δ_1, and such that $B' \cap \mathscr{A} = \emptyset$. Let m be an integer, $m > N_1$, $m > N_2$, so that $[T_m \cup \text{Int}\, T_m] \cap \beta = \emptyset$. Let Int T_m be the open set W of step 3, and let X of step 3 be sufficiently small so that $X \cap \beta = \emptyset$. Step 3 guarantees the existence of a homeomorphism H, and the disk $H(B)$ has certain nice properties: Firstly, its diameter is less than diameter $B + \delta_2/3 + \delta_2/3 < \delta_2$. Secondly, β bounds $H(B)$, by choice of X. Most important, $S \cap T_m \cap H(B) = \emptyset$. This follows since $B \cap S \subset U$ so $H(B) \cap H(S) \subset H(U)$, but since $H(S) = S$, we have $H(B) \cap S \subset H(U)$, and since $H(U) \subset \text{Int}\, T_m$, we have $H(B) \cap S \cap T_m = \emptyset$.

We assume without loss of generality that $H(B)$ is polyhedral on its interior and in general position with respect to T_m. Thus, $H(B) \cap T_m$ is a collection of simple closed curves $\alpha_1, \alpha_2, \cdots, \alpha_r$. Since $H(B) \cap T_m \cap S = \emptyset$, each α_i does not intersect S. By step 1, each α_i bounds a disk on T_m. Since $H(B)$ has diameter less than δ_2, and $\alpha_i \subset H(B)$, it follows that each α_i bounds a $\delta_1/3$-disk on T_m, by step 2. The usual disk replacement process (see step 6 for details) is now performed on the disk $H(B)$, yielding a disk B', of diameter less than diameter $H(B) + \delta_1/3 + \delta_1/3 < \delta_1$. Furthermore $B' \cap \operatorname{Int} T_m = \emptyset$, so $B' \cap \mathscr{A} = \emptyset$.

(6) The disk B' is placed in general position with respect to S, and the usual disk replacement process used to modify B' into a new disk B''. That is, $B' \cap S$ is a collection of simple closed curves $l_1, l_2, \cdots, l_t$. Note that $l_i \cap \mathscr{A} = \emptyset$, $i = 1, 2, \cdots, t$. An "innermost" l_i on S is selected, and the disk it bounds on B' is replaced by the $\varepsilon/3$-disk it bounds on S. (See step 2 for why we have an $\varepsilon/3$-disk.) This new disk on B' is pushed slightly to one side of S; this can be done because the new disk cannot contain $\mathscr{A}$ on its interior, as diameter $\mathscr{A} > \varepsilon/3$. Thus, this new disk is polyhedral.

This process is continued with another l_i, until all intersection is eliminated, yielding B''. We have $B'' \cap S = \emptyset$, β is the boundary of B'', and diameter B'' $<$ diameter $B' + \varepsilon/3 + \varepsilon/3 < \varepsilon$. This establishes the Assertion.

It remains to show that $\mathscr{A}$ is tame at its end points a and b. Now that we know that $\mathscr{A}$ is locally tame mod $a \cup b$, it is easy to construct arbitrarily small 2-spheres around a (or b) out of the tori $\{T_i\}$, such that each 2-sphere intersects $\mathscr{A}$ in exactly one point. Thus $\mathscr{A}$ will be tame at its end points by satisfying Properties P and Q of [10]. We omit details of this construction as they are tedious, and similar to the proof of Theorem 2. Indeed, all that we really need to establish Corollary 1 is that $\mathscr{A}$ is locally tame on its interior.

Corollary. 1. *No subarc of the "Bing sling" [3] lies on a disk.*

Proof. If some subarc does lie on a disk, then a smaller subarc lies on a 2-sphere S, by §5 of [5]. We observe that the "Bing sling" satisfies Properties 2 and 3 of Theorem 1, with the necessary tori being provided by its very construction. Thus, this small subarc is tame, by Theorem 1, which is a contradiction to the fact that it pierces no disk.

3. **The simple closed curve $\mathscr{K}$ which pierces no disk, yet lies on a disk.** Using a technique developed by Bing [2], one can construct a 2-sphere $\mathscr{S}$ in E^3 whose wild points from a wild, cellular arc in E^3, which we call ξ. For an exact description, see [1]. The arc ξ can be completed to a simple closed curve Z on $\mathscr{S}$, and the same argument which shows that ξ is cellular (see [9]) will establish that Z is the intersection of a decreasing sequence of tori plus their interiors (note: these tori cannot be sequentially 1-ULC, by Theorem 1).

If any non-endpoint x of ξ had the property that ξ pierced a disk at x, then one could use the symmetry given by the construction of ξ to show that ξ pierces a disk at a dense subset of itself, i.e., P_ξ is dense in ξ. This, however, gives us a contradiction on account of the following.

Theorem 2. *If $\mathscr{A}$ is an arc in E^3 such that*

(1) *$\mathscr{A}$ lies on a 2-sphere S in E^3;*

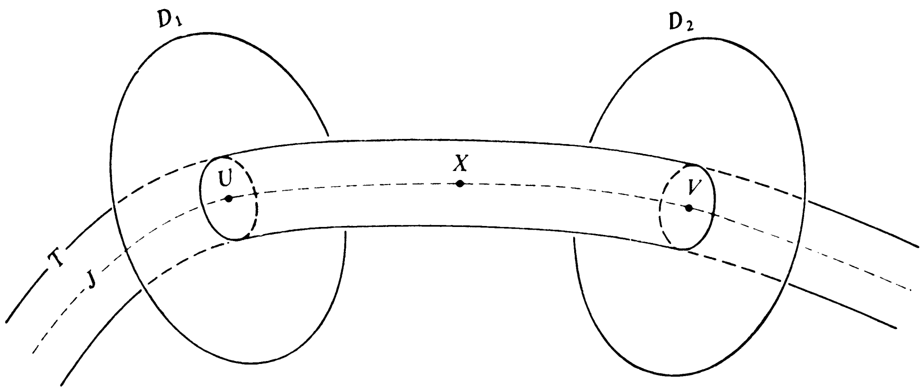
(2) *$\mathscr{A}$ lies on a simple closed curve J in E^3 which is the intersection of a nested sequence of tori plus their interiors*;

(3) *$P_{\mathscr{A}}$ is dense in $\mathscr{A}$*;

then $\mathscr{A}$ is tame.

Proof. Let x be any non-endpoint of $\mathscr{A}$. Given $\varepsilon > 0$, we will show that there exists a 2-sphere $\mathfrak{S}$ of diameter less than ε, such that $x \in \operatorname{Int} \mathfrak{S}$, and $\mathfrak{S} \cap J$ is a set of two points. This will establish that $\mathscr{A}$ is locally tame at all non-endpoints [10]. The endpoints of $\mathscr{A}$ can then be taken care of by the same method, with only slight changes necessary in the construction of $\mathfrak{S}$.

The 2-sphere $\mathfrak{S}$ will be constructed as shown in the figure. That is, an annulus of the torus T will connect two disks D_1 and D_2 which are pierced by J on opposite sides of x. The 2-sphere $\mathfrak{S}$ will consist of the annulus plus one subdisk of each D_i, $i = 1,2$. Of course, it must be justified that there is a torus and two disks which intersect as nicely as shown in the figure. This is done in eight steps.



(1) J cannot possibly be expressed as the intersection of a decreasing sequence of 3-cells in E^3. This follows easily from the fact that J is an absolute neighborhood retract. Thus, there exists a positive number ε_1 such that any 2-sphere lying completely in an ε_1-neighborhood of J cannot contain J in its interior. We assume that $\varepsilon < \varepsilon_1$, and that diameter $J > \varepsilon$.

(2) Select points u and v of $\mathscr{A}$ such that the subarc $\overline{uxv}$ of J with endpoints u and v and non-endpoint x has diameter less than $\varepsilon/2$. The other subarc of J with endpoints u and v will be denoted by $\overline{uyv}$. The points u and v are also selected so that J pierces a disk D_1 at u and a disk D_2 at v. We assume that D_1 and D_2 are sufficiently small so that diameter $(\overline{uxv} \cup D_1 \cup D_2) < \varepsilon/2$ and so that D_1 and D_2 are disjoint.

(3) In this step we select the torus T of the figure. Select a ray R starting at x, such that R and $[\overline{uxv} \cup D_1 \cup D_2]$ are disjoint. R may be taken to be locally polyhedral mod x. There is a positive number η, such that if A is an arc in E^3 of diameter less than η, and if A intersects both $\overline{uxv}$ and $\overline{uyv}$, then A intersects $D_1 \cup D_2$. This follows from the fact that J pierces the disks D_1 and D_2 at u and v, respectively. We also assume that $\eta <$ dist $(R, \overline{uyv})$ and $\eta < \varepsilon$.

Let D_1' be an $\eta/8$-disk with $u \in \text{Int } D_1' \subset D_1' \subset \text{Int } D_1$; let D_2' be similarly situated in D_2. We choose γ sufficiently small so that a γ-neighborhood of J intersects D_i only in a subset of D_i', $i = 1, 2$. We have $\gamma < \eta/8$, of course. The torus T is now selected from our sequence so T lies in this γ-neighborhood of J. By applying [4], we may assume that T is polyhedral, that D_1 is locally polyhedral mod u, that D_2 is locally polyhedral mod v; furthermore, we assume that T, D_1, D_2 and R are in general position.

(4) At the present time, T may intersect D_1 and D_2 very differently from the way indicated in the figure. We now simplify this intersection.

Let us examine a simple closed curve L of $T \cap [D_1 \cup D_2]$. L may be classified thus:

1. L bounds a disk on T;

2. L does not bound a disk on T.

L may also be classified in a different way. Assume for convenience that $L \subset D_1$.

1′. L bounds a subdisk E_1 of D_1 which does not contain u.

2′. The subdisk E_1 of D_1 bounded by L does contain u.

We show that L is of Type 1 if and only if L is of Type 1′, that is, these classifications are really the same.

If L is of Type 1, then L does not link J. Thus, L is also of Type 1′.

If L is of Type 1′, then using techniques of Theorem 1 of [6], one can show that L bounds a disk which does not intersect J, and whose interior does not intersect T. If L were of Type 2, then by cutting T along L and inserting two copies of this disk, one could construct a 2-sphere in contradiction to step 1. Thus, if L is of Type 1′, then L must also be of Type 1.

(5) All L of Type 1 are now removed. That is, we suppose that L is an "innermost" simple closed curve of Type 1 in D_1. The subdisk E_1 of D_1 bounded by L will not contain any simple closed curves of Type 2. This is obvious from the equivalence of Types 1 and 2 with Types 1′ and 2′. Thus, T may be altered by removing the disk bounded by L on T, replacing it by E_1, then pushing to one side slightly. This process is repeated until all Type 1 simple closed curves have been removed, forming a new torus T'.

We now show that $J \subset \text{Int } T'$. The first stage of the alteration of T consisted of interchanging two disks. This will change Int T only by adding to or subtracting from it the 3-cell bounded by these two disks. J cannot lie in this 3-cell, by step 1. The same line of reasoning is continued during each alteration in the construction of T', showing that $J \subset \text{Int } T'$.

(6) If t is a point of $T' - D_1 - D_2$, then t can be joined to J by an arc $A(t)$ which is disjoint from $D_1 \cap D_2$, and which is of diameter less than $\eta/4$. To see this we examine two cases: either t lies on T, or t lies very close D_1' or D_2'. The latter case is clear from the choice of D_1' and D_2', in fact, the arc will have diameter less than $\eta/8$. In the former case, we begin by joining t to J with an $\eta/8$ arc, which may intersect D_1' (or D_2'). If it does, we modify it by bending it just before it hits D_1' so it instead runs down the side of D_1' to J. This bent arc will have diameter less than $\eta/8 + \eta/8 = \eta/4$, as desired.

(7) We now look at the components $C_1, C_2, \cdots, C_m$ of $T' - D_1 - D_2$. If t and t' are both points of the same component, say C_1, then $A(t)$ and $A(t')$ both intersect the same component of $J - u - v$. Otherwise, let $\overline{tt'}$ be an arc of C_1 joining t and t'. Let s and s' be two points of this arc such that $A(s)$ and $A(s')$ intersect different components of $J - u - v$, and such that the subarc $\overline{ss'}$ of $\overline{tt'}$ has diameter less than $\eta/2$. The path $[A(s) \cup \overline{ss'} \cup A(s')]$ has diameter less than η, contradicting the definition of η. Thus, each C_i lies in an $\eta/4$-neighborhood of either $\overline{uxv}$ or $\overline{uyv}$.

(8) The desired 2-sphere $\mathfrak{S}$ may now be selected. Since the ray R hits T' an odd number of times, and since $R \cap [D_1 \cup D_2] = \emptyset$, R hits some component C_N of $T' - D_1 - D_2$ an odd number of times. C_N cannot lie in an $\eta/4$-neighborhood of $\overline{uyv}$, since $C_N \cap R \neq \emptyset$, and $\eta < \operatorname{dist}(R, \overline{uyv})$. Thus, C_N lies in an $\eta/4$-neighborhood of $\overline{uxv}$.

C_N cannot be all of T', as

$$\operatorname{diam}(C_N) < \operatorname{diam}(\overline{uxv}) + \eta/4 + \eta/4 < \varepsilon,$$

whereas Int T' contains J, a set of diameter larger than ε, so T' has diameter larger than ε. Thus, C_N will be an annulus of T', with two boundary simple closed curves of Type 2', by steps 4 and 5. Furthermore, both of these simple closed curves do not lie on D_1. If they did, the annulus lying between them on D_1 could be added to C_N to produce a torus T'' disjoint from J. Since $R \cap T''$ would contain an odd number of points. J would lie in Int T''. Thus diameter $(T'') > \varepsilon$. But diameter $(T'') <$ diameter $(\overline{uxv} \cup D_1 \cup D_2) + \eta/4 + \eta/4 < \varepsilon$ which gives us a contradiction. By similar reasoning, both of these simple closed curves do not lie on D_2.

Let $\mathfrak{S}$ be the 2-sphere composed of C_N plus the subdisk of D_1 bounded by a boundary simple closed curve of C_N, plus the subdisk of D_2 bounded by the other boundary simple closed curve of C_N. Then,

$$\operatorname{diam}(\mathfrak{S}) < \operatorname{diam}(\overline{uxv} \cup D_1 \cup D_2) + \eta/4 + \eta/4 < \varepsilon, \text{ and}$$

$\mathfrak{S} \cap J$ will be just the two points u and v. That x lies in Int $\mathfrak{S}$ follows from the fact that $\mathfrak{S} \cap R$ consists of an odd number of points. This completes the proof of Theorem 2.

It is a simple matter to construct a simple closed curve $\mathscr{K}$ such that $\mathscr{K}$ looks locally just like the arc ξ, and with $\mathscr{K}$ lying on a 2-sphere in E^3. To do this the construction of [1] is simply performed with eyebolts hooking in a circular fashion at each stage. Thus, $\mathscr{K}$ lies on a 2-sphere in E^3, yet pierces no disk in E^3.

QUESTION. Is $\mathscr{K}$ homogeneously embedded in E^3? Precisely, given points p and q in $\mathscr{K}$, is there a homeomorphism h of E^3 onto itself such that $h(\mathscr{K}) = \mathscr{K}$ and $h(p) = q$?

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