CHARACTERIZATIONS OF THE CONTINUOUS IMAGES OF THE PSEUDO-ARC

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1. **Introduction.** At the Summer Institute on Set Theoretic Topology, Madison, Wisconsin, 1955, the question "What characterizes all continuous images of the pseudo-arc [1; 7; 10]?" was raised by R. H. Bing [3]. In particular it was asked whether or not there exists a characterization of the continuous images of the pseudo-arc analogous to the well-known Hahn-Mazurkiewicz characterization of the continuous images of the arc [5; 8]. The purpose of this paper is to investigate this question and to initiate an investigation of the properties of the class of continuous images of all chainable continua in general. A related question considered is whether or not every chainable continuum is a continuous image of the pseudo-arc.

In §3 of this paper two global characterizations of the continuous images of the pseudo-arc are established(1), and it is shown that if the definition of local connectedness is suitably reformulated, then these characterizations are analogous to that of Hahn and Mazurkiewicz for the arc. Furthermore, in §4, these characterizations prove to be readily applicable to showing that the pseudo-arc can be mapped continuously onto any chainable continuum. A consequence of this latter result is that the class of continuous images of the pseudo-arc and the class of continuous images of all chainable continua are identical.

Next we present one of the most important results of this paper with respect to answering the question of whether or not there exists a characterization of the continuous images of the pseudo-arc analogous to that of Hahn and Mazurkiewicz for the arc. Specifically, in §5, we show that there does not exist any local topological property which characterizes all continuous images of the pseudo-arc. This simultaneously indicates that the original question of R. H. Bing has an essentially negative answer and that, in this respect, the characterizations of §3 cannot be improved. Finally, in §6, a series of examples is given to show that the

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(1) The referee has pointed out that during the period of his examination of this paper an alternative development of one of these characterizations was given by A. Lelek, *On weakly chainable continua*, Fund. Math. **51** (1963), 271–283. However, since the preliminary theorems of the present development as well as this characterization are necessary to the completion, in §5, of the answer to the question of R. H. Bing, this paper is presented essentially in its original form.

major generalizations of the class of locally connected compact metric continua and the class of chainable continua fail to be contained in the class of continuous images of the pseudo-arc. In this regard, therefore, the results of §4 are also the best possible.

In a later paper to be presented by the author the characterizations of the continuous images of the pseudo-arc established in this paper will be used to develop properties of the class of continuous images of all chainable continua, including the results that this class is closed under the operations of finite connected union and countable topological product.

Continua with similar geometric characteristics to those described in §6 have been discussed by M. K. Fort, Jr. [4] and S. Waraszkiewicz [11]. While these papers and this present study have different primary objectives it is of interest to note, in their investigations of continua which fail to be continuous images of any members of rather broad classes of continua, that similar types of examples were developed. One of the results of §4 "that chainable continua are continuous images of the pseudo-arc," was also recently developed by J. Mioduszewski [9]. However, since the methods of this paper are very different from those of [9] and the result in question is a direct result and natural complement of the more general results of this paper, it is included here for completeness.

All continua considered in this paper will be separable metric continua.

2. **Preliminaries.** The more standard terms used in this paper are defined in [12] or in the other appropriately indicated references. In addition we shall define a number of special terms to be used throughout this paper. In general these latter terms and notations were suggested by those used by Bing in [1] and [2].

Definition 1. A *p-chain* will be defined to be a finite sequence of sets each of which, except the last, intersects its successor in the sequence. The members of the sequence are called *links* and the notation $P=(p_1,p_2,\cdots,p_n)$ will be used to denote the p-chain P whose links are $p_1,p_2,\cdots,p_n$.

Definition 2. If P is a p-chain, then the p-chain consisting of the links of P in reverse order will be said to be the *conjugate* of P and will be denoted by $\bar{P}$. A p-chain P such that P and $\bar{P}$ consist of the same sets in the same order will be said to be a *self-conjugate p-chain*. However, if $P=(p_1,p_2,\cdots,p_n)$ is a self-conjugate p-chain, the conjugate $\bar{P}=(p_n,p_{n-1},\cdots,p_1)$ of P is formally distinct from P and will be considered as a different p-chain.

Definition 3. The *p-chain sum* $P+Q$ of p-chains $P=(p_1,\ p_2,\cdots,p_n)$ and $Q=(q_1,q_2,\cdots,q_m)$ will be defined if and only if p_n and q_1 intersect. We will then define $P+Q$ to be the sequence of sets $p_1,p_2,\cdots,p_n,q_1,\ q_2,\cdots,q_m$.

Definition 4. We shall define the *sub-p-chain* $P(h,k)$, $1\leqq h\leqq k\leqq n$, of the p-chain $P=(p_1,p_2,\cdots,p_n)$ to be the sequence of links $p_h,\ p_{h+1},\cdots,p_k$. The notation $P(k,h)$, $1\leqq h\leqq k\leqq n$, will be used to denote the sub-p-chain of $\bar{P}$ consisting of the sequence of links $p_k,p_{k-1},\cdots,p_h$.

DEFINITION 5. If $P=(p_1,p_2,\cdots,p_n)$ and $Q=(q_1,q_2,\cdots,q_m)$ are p-chains and each link p_i of P is a subset of a link q_{x_i} of Q, then the sequence of ordered pairs of integers $(1,x_1),(2,x_2),\cdots,(n,x_n)$ will be said to be a *pattern* of P in Q. If, in this pattern, $|x_i-x_j|\leqq 1$ whenever $|i-j|\leqq 1$, $1\leqq i, j\leqq n$, then the pattern will be said to be an *r-pattern* of P in Q.

DEFINITION 6. A p-chain P will be said to be a *refinement* of a p-chain Q if there is an r-pattern of P in Q.

A number of types of refinements will be distinguished:

DEFINITION 7. If a p-chain $P=(p_1,p_2,\cdots,p_n)$ has an r-pattern of the form $(1,x_1=1)$, $(2,x_2),\cdots,(n,x_n=m)$ in a p-chain $Q=(q_1,q_2,\cdots,q_m)$ then P will be said to be a *normal refinement* of Q.

DEFINITION 8. If a p-chain $P=(p_1,p_2,\cdots,p_n)$ has an r-pattern $(1,x_1),(2,x_2),\cdots,$ (n,x_n) in a p-chain $Q=(q_1,q_2,\cdots,q_m)$ and for each i, $i=1,2,\cdots,n$, p_i is the same set as q_{x_i}, and if each link of Q corresponds to at least one link of P, then P will be said to be a *principal refinement* of Q.

If there is an r-pattern of P in Q satisfying the requirements of both 7 and 8 then P will be said to be a *principal normal refinement* of Q.

DEFINITION 9. If a p-chain $P=(p_1,p_2,\cdots,p_n)$ has an r-pattern $(1,x_1),(2,x_2),\cdots,$ (n,x_n) in a p-chain $Q=(q_1,q_2,\cdots,q_m)$, and for each pair of links p_h and p_k of P for which $|x_h-x_k|>2$ the sub-p-chain $P(h,k)$ of P, or $\bar{P}$, has links p_r and p_s such that $r<s$ if $h<k$ and $r>s$ if $h>k$, and such that $|x_h-x_s|=1$ and $|x_k-x_r|=1$, then P will be said to be *crooked in Q*.

In any type of refinement of a p-chain P in a p-chain Q, there may be several r-patterns of P in Q having the appropriate properties. However, in referring to a refinement whose existence has been hypothesized or otherwise established, we will assume that a particular r-pattern has been chosen and that this r-pattern will remain fixed throughout the given argument. Thus, in these circumstances, we will speak of "the" r-pattern of a p-chain P in a p-chain Q.

The concept of p-chainability of a continuum is now introduced.

DEFINITION 10. A continuum H will be said to be *p-chainable* if there is a sequence of p-chains $P_1,P_2,P_3,\cdots$ such that for each positive integer i:

(a) The union of the elements of P_i is H.

(b) P_{i+1} is a normal refinement of P_i.

(c) The diameter of each link of P_i is less than $1/i$.

(d) The closure of each link of P_{i+1} is a subset of the link of P_i to which it corresponds under the r-pattern of P_{i+1} in P_i.

A sequence of p-chains having these properties with respect to a continuum H will be said to be *associated* with H.

It will develop that the concepts of p-chain and p-chainability are of fundamental importance in this investigation of the continuous images of the pseudo-arc and in the related subsequent study mentioned in the Introduction.

3. **Characterizations of the continuous images of the pseudo-arc.** We shall show that the property of p-chainability and a modified somewhat stronger form of this property are each characterizations of the continuous images of the pseudo-arc. Furthermore it will be shown that if the property of local connectedness is suitably reformulated as a global property then these characterizations are analogous to that of Hahn and Mazurkiewicz for the arc. It will be established in §5 that no local characterization of the continuous images of the pseudo-arc exists so that in this respect these analogies cannot be improved.

A number of preliminary results will be needed, some of which will be of interest in themselves.

THEOREM 3.1. *Each of the relations, "refinement," "normal refinement" and "principal refinement," between ordered pairs of p-chains, is transitive.*

Proof. Let $P=(p_1,p_2,\cdots,p_n)$, $Q=(q_1,q_2,\cdots,q_m)$ and $T=(t_1,t_2,\cdots,t_k)$. Suppose first that P is a refinement of Q and Q is a refinement of T. Then there is an r-pattern $(1,x_1),(2,x_2),\cdots,(n,x_n)$ of P in Q and an r-pattern $(1,y_1)$, $(2,y_2),\cdots,(m,y_m)$ of Q in T. It follows from the definition of pattern that for each integer i, $1 \leqq i \leqq n$, the link p_i of P is a subset of the link of T with subscript y_{x_i}. Hence, $(1,y_{x_1}),(2,y_{x_2}),\cdots,(n,y_{x_n})$ is a pattern of P in T. Now, since $(1,y_1)$, $(2,y_2),\cdots,(m,y_m)$ is an r-pattern of Q in T, it follows that if i and j are integers, $1 \leqq i,\ j \leqq m$, such that $|\,i-j\,| \leqq 1$, then $|\,y_i - y_j\,| \leqq 1$. In particular, $|\,y_{x_a} - y_{x_b}| \leqq 1$ whenever $|\,x_a - x_b\,| \leqq 1$. Furthermore, since $(1,x_1)$, $(2,x_2),\cdots,(n,x_n)$ is an r-pattern of P in Q, $|\,x_a - x_b\,| \leqq 1$ whenever $|\,a-b\,| \leqq 1$, $1 \leqq a,\ b \leqq n$. It follows that $|\,y_{x_a} - y_{x_b}\,| \leqq 1$ whenever $|\,a-b\,| \leqq 1$. This is the condition that the pattern $(1,y_{x_1}),(2,y_{x_2}),\cdots,(n,y_{x_n})$ of P in T be an r-pattern. We conclude that P is a refinement of T and that the relation "refinement" is transitive.

If P is a normal refinement of Q and Q is a normal refinement of T, then there is an r-pattern of the form $(1,x_1=1)$, $(2,x_2),\cdots,(n,x_n=m)$ of P in Q, and there is an r-pattern of the form $(1,y_1=1)$, $(2,y_2),\cdots,(m,y_m=k)$ of Q in T. Now, as in the previous case, the pattern $(1,y_{x_1})$, $(2,y_{x_2}),\cdots,(n,y_{x_n})$ is an r-pattern of P in T. Furthermore $y_{x_1}=y_1=1$ and $y_{x_n}=y_m=k$. But this is the condition that the p-chain P be a normal refinement of the p-chain T. Hence the relation "normal refinement" is also transitive.

Finally, if P is a principal refinement of Q and Q is a principal refinement of T, then there is an r-pattern $(1,x_1)$, $(2,x_2),\cdots,(n,x_n)$ of P in Q, an r-pattern $(1,y_1),(2,y_2),\cdots,(m,y_m)$ of Q in T, and furthermore, $p_i=q_{x_i}$ for $i=1,3,\cdots,n$ and $q_i=t_{y_i}$ for $i=1,2,\cdots,m$. Again, $(1,y_{x_1})$, $(2,y_{x_2}),\cdots,(n,y_{x_n})$ is an r-pattern of P in T, and since also p_i is equal to the link of T with subscript y_{x_i}, $i=1,2,\cdots,n$, and each link of T corresponds to at least one link of P, it follows that P is a principal refinement of T. The relation "principal refinement," then, is also transitive.

The next theorem has a prominent function in the combinatorial and refinemental relationships among p-chains. In particular it is an important preliminary theorem to the principal results of both §§3 and 4.

THEOREM 3.2. *If a p-chain P is a normal refinement of a p-chain Q and T is a p-chain which is a principal normal refinement of Q, then there is a p-chain S such that S is a principal normal refinement of P and a normal refinement of T.*

Proof. Let $P=(p_1,p_2,\cdots,p_n)$, $Q=(q_1,q_2,\cdots,q_m)$ and $T=(t_1,t_2,\cdots,t_k)$, and let $(1,x_1=1),(2,x_2),\cdots,(n,x_n=m)$ be an r-pattern of P in Q and $(1,y_1=1)$, $(2,y_2),\cdots,(k,y_k=m),t_i=q_{y_i}$, $i=1,2,\cdots,k$, be an r-pattern of T in Q. Since P and T are each normal refinements of Q, we may choose two increasing sequences of integers, $a_1,a_2,\cdots,a_r$ and $b_1,b_2,\cdots,b_s$ as follows. For the sequence $a_1,a_2,\cdots,a_r$ we first define $a_1=1$. Then, if there exist integers j_2 and h_2 such that $j_2<h_2<n$, $x_{j_2}=m$ and $x_{h_2}<m$, we define a_2 to be the first integer such that $x_{a_2}=m$. Otherwise we set $a_2=a_r=n$, and in this case the definition of the sequence is complete. In the former case let j_3 be the first integer greater than a_2 such that $x_{j_3}=m$ and for which there is an integer h_3 between a_2 and j_3 such that $x_{h_3}<m$. We define a_3 to be the first integer greater than a_2 such that $x_{a_3}\leqq x_w$ for $a_2\leqq w\leqq j_3$. If there are integers j_4,h_4 greater than a_3 such that $j_4<h_4,x_{j_4}=m$, and $x_{h_4}<m$, then we define a_4 to be the first integer greater than a_3 such that $x_{a_4}=m$. Otherwise we set $a_4=a_r=n$. It may be seen that this process can be continued and that the sequence so defined will be an increasing sequence of integers whose last member a_r has the value n. We define the sequence $b_1,b_2,\cdots,b_s$ by replacing the letters x,a,r and n, in the definition of the sequence $a_1,a_2,\cdots,a_r$, by the letters y, b, s and k, respectively. Two other sequences of integers $c_1,c_2,\cdots,c_s$ and $d_1,d_2,\cdots,d_r$ are also chosen. The integer c_i is defined to be the greatest integer such that $x_{c_i}=y_{b_i}$, $i=1,2,\cdots,s$, and d_i is defined to be the greatest integer such that $y_{d_i}=x_{a_i}$, $i=1,2,\cdots,r$.

It is noted that the p-chain sums $P_1=P(a_1,a_2)+P(a_2,a_3)+\cdots+P(a_{r-1},a_r)$ and $T_1=T(b_1,b_2)+T(b_2,b_3)+\cdots+T(b_{s-1},b_s)$ are principal normal refinements of P and T, respectively.

The theorem is now proved by induction on the number k of links of T. If $k=1$, then T and Q are the same p-chain and we need only set $S=P$. The p-chain S then has the required properties.

Suppose that the theorem has been established for all values of k less than some integer $f,f>1$, and consider the case that $k=f$. It will be assumed that $y_{k-1}=m-1$, since otherwise the induction step is trivial. Two cases will be considered.

Case I. $s=2$. In this case the p-chain $T(1,k-1)$ is a principal normal refinement of $Q(1,m-1)$. Let u be an integer such that $P(1,u)$ is maximal with respect to being a sub-p-chain of $P(a_1,a_2)$ and with respect to being a normal refinement of $Q(1,m-1)$. It is observed that the p-chain $T(k-1,k)$ and the

p-chain $Q(m-1, m)$ are identical and that $P(u, a_2)$ is a normal refinement of this p-chain. Since $P(1,u)$ is a normal refinement of $Q(1,m-1)$, and since $T(1,k-1)$ is a principal normal refinement of $Q(1,m-1)$ having less than k links, it follows by the induction hypothesis that there exists a p-chain S_{11} such that S_{11} is a principal normal refinement of $P(1,u)$ and a normal refinement of $T(1,k-1)$. It is easily seen that the p-chain sum $S_1 = S_{11} + S_{12}$, where $S_{12} = P(u,a_2)$, is a defined p-chain sum, that S_1 is a principal normal refinement of $P(a_1,a_2)$, and S_1 is a normal refinement of T.

Now, the p-chain $P(a_2,a_3)$ is a normal refinement of $Q(m,x_{a_3})$, and $T(k,d_3)$ is a principal normal refinement of $Q(m,x_{a_3})$. Furthermore, t_{d_3} is the first link of the p-chain $T(k,d_3)$ that corresponds to the last link of $Q(m,x_{a_3})$ under the pattern of T in Q, and p_{a_3} is the first link of $P(a_2,a_3)$ that corresponds to the last link of $Q(m,x_{a_3})$ under the pattern of P in Q. This is similar to the situation just considered. It follows that there is a p-chain S_2 such that S_2 is a principal normal refinement of $P(a_2,a_3)$ and a normal refinement of $T(k,d_3)$. Similarly, there is a p-chain S_3, such that S_3 is a principal normal refinement of $P(a_3,a_4)$, and such that S_3 is a normal refinement of $T(d_3,k)$. Proceeding in this way we obtain p-chains $S_1, S_2,\cdots,S_{r-1}$, which are principal normal refinements of $P(a_1,a_2)$, $P(a_2,a_3),\cdots,P(a_{r-1},a_r)$, respectively, and which are normal refinements of $T, T(k,d_3), T(d_3,k),\cdots, T(d_{r-1},k)$. Then the p-chain sum $S = S_1 + S_2 + \cdots + S_{r-1}$ is defined, and S is a principal normal refinement of P_1 and a normal refinement of T_1. It follows by Theorem 3.1 that S is a p-chain of the desired type.

Case II. $s > 2$. In this case, each of the p-chains $T(b_1,b_2)$, $T(b_2,b_3),\cdots$, $T(b_{s-1},b_s)$ has less than k links. Now these p-chains are respectively principal normal refinements of $Q(1,m)$, $Q(m,y_{b_3})$, $\cdots,Q(y_{b_{s-1}},y_{b_s}=m)$. In addition, the p-chains P, $P(n,c_3)$, $P(c_3,n),\cdots,P(c_{s-1},n)$ are normal refinements of $Q(1,m)$, $Q(m,y_{b_3})$, $Q(y_{b_3},m),\cdots,Q(y_{b_{s-1}},m)$, respectively. Then, by the induction hypothesis, there exist p-chains $S_1,S_2,S_3,\cdots,S_{s-1}$ which are principal normal refinements of $P, P(n,c_3)$, $P(c_3,n),\cdots,P(c_{s-1},n)$, respectively, and which are normal refinements of $T(b_1,b_2)$, $T(b_2,b_3)$, $\cdots,T(b_{s-1},b_s)$, respectively. Clearly, $S = S_1 + S_2 + \cdots + S_{s-1}$ is a defined p-chain, and S is a principal normal refinement of P and a normal refinement of T. This completes the proof.

THEOREM 3.3. *If a p-chain P is a normal refinement of a p-chain Q, then there is a principal normal refinement T of P such that T is crooked in Q and is a normal refinement of Q.*

Proof. Let $P=(p_1,p_2,\cdots,p_n)$, $Q=(q_1,q_2,\cdots,q_m)$ and let $(1,x_1=1)$, $(2,x_2),\cdots,(n,x_n=m)$ be an r-pattern of P in Q. We now define the p-chain $T=(t_1,t_2,\cdots,t_k)$ inductively with respect to the number, n, of links of P.

First, if $n=1$, then we set $T=P$ and observe that requirements of Definition 9 for T in P are vacuously satisfied. Next assume that for all values of n less than some integer s, $s>1$, T has been chosen to be a principal normal refinement of P, crooked in P, and consider the case that $n = s$. In this case, each of the p-chains $P(1,n-1)$, $P(n-1,2)$ and $P(2,n)$ has less than s links and we may choose p-chains T_1, T_2 and T_3 which are principal normal refinements crooked in $P(1,n-1)$, $P(n-1,2)$ and $P(2,n)$, respectively. We then define T to be the p-chain sum $T_1 + T_2 + T_3$. This completes the definition of the p-chain T such that T has an r-pattern in P satisfying the requirements of Definitions 7, 8 and 9.

It remains to show that T is crooked in Q. To do this, suppose that t_a and t_b are links of the p-chain T and let these links be chosen so that $a < b$ and $|x_{y_a} - x_{y_b}| > 2$. Then $|y_a - y_b| > 2$ and, since T is crooked in P, the sub-p-chain $T(a,b)$ of T contains links t_r and t_s such that $r < s$ and we have $|y_r - y_b| = 1$ and $|y_s - y_a| = 1$. It follows that $|x_{y_r} - x_{y_b}| \leqq 1$ and $|x_{y_s} - x_{y_a}| \leqq 1$. Hence, there are links t_u and t_v of $T(a,b)$ such that $u < v$, $|x_{y_u} - x_{y_b}| = 1$, and $|x_{y_v} - x_{y_a}| = 1$. We conclude that T is a principal normal refinement of P, T is a normal refinement of Q, and T is crooked in Q.

The following theorem, in addition to being a preliminary theorem to the characterizations of the continuous images of the pseudo-arc, is of some interest in itself. It is of direct use in investigating which classes of compact metric continua fail to be contained in the class of continuous images of the pseudo-arc.

THEOREM 3.4. *If a continuum H is a continuous image of the pseudo-arc and x is a point of H, then there is a sequence of p-chains associated with H such that the first link of each p-chain of the sequence contains x.*

Proof. Let M denote the pseudo-arc, let f be a continuous mapping of M onto H, and let p be a point of M such that $f(p) = x$. Now, by Theorem 10 [1], the continuum M is indecomposable and we may choose a point q from a composant of M which does not contain p. It then follows from the proof of Theorem 1 of [2] that M is chainable between p and q. Hence we may choose a sequence of chains $D_1, D_2, D_3, \cdots$ in M, considered as space, such that for each positive integer i, (1) D_{i+1} is a refinement of D_i, (2) the diameter of each link of D_i is less than $1/i$, (3) the closure of each link of D_{i+1} is a subset of some link of D_i, and (4) the union of the elements of D_i is M.

For each positive integer i, let D_i be the chain $(d_{i1}, d_{i2}, \cdots, d_{in_i})$ and let $f(D_i)$ denote the p-chain $(f(d_{i1}), f(d_{i2}), \cdots, f(d_{in_i}))$. We will now show that there is a sub-sequence of the sequence of p-chains $f(D_1), f(D_2), f(D_3), \cdots$ which is associated with H. To do this, we first note, since M is compact, that the continuous mapping f is uniformly continuous and hence, by property (2) above, we may choose a subsequence $f(D_{k_1}), f(D_{k_2}), f(D_{k_3}), \cdots$ of the sequence $f(D_1)$, $f(D_2)$, $f(D_3), \cdots$ such that, for each positive integer i, the diameter of each link of $f(D_{k_i})$ is less than $1/i$. Now, for $i = 1,2,3,\cdots, D_{k_{i+1}}$ is a refinement of D_{k_i} and, in addition, D_{k_i} and $D_{k_{i+1}}$ are chains from p to q such that, by property (3), the closure of each link of $D_{k_{i+1}}$ is a subset of some link of D_{k_i}. Hence we may choose a pattern $\pi(i, i+1)$ of $D_{k_{i+1}}$ in D_{k_i} which associates the first and last links of $D_{k_{i+1}}$ with the first and last links of D_{k_i}, respectively, and which associates each link of $D_{k_{i+1}}$ with a link of D_{k_i} which contains its closure. This pattern $\pi(i,i+1)$ is clearly an r-pattern of $f(D_{k_{i+1}})$ in $f(D_{k_i})$ and from the nature of $\pi(i,i+1)$ it follows that $f(D_{k_{i+1}})$ is a normal refinement of $f(D_{k_i})$ and that, furthermore, if the link $f(d_{k_is})$ of $f(D_{k_i})$ corresponds to the link $f(d_{k_{i+1}r})$ of $f(D_{k_{i+1}})$, then $\overline{f(d_{k_{i+1}r})} = f(\overline{d_{k_{i+1}r}}) \subset f(d_{k_is})$. Finally, from property (4) above, it is clear that the union of the elements of $f(D_{k_i})$ is H, for each positive integer i. Thus, the sequence of p-chains $f(D_{k_1})$, $f(D_{k_2}), f(D_{k_3}), \cdots$ satisfies each of the requirements of Definition 10 with respect to H and each p-chain of this sequence has x in its first element.

The theorem which follows constitutes the first characterization of the continuous images of the pseudo-arc and is one of the principal results of this section. In view of the factors mentioned in footnote 1, we shall give a condensed form of the proof of this theorem. In this proof, as well as in the development of the later sections of this paper, we shall make strong use of the combinatorial and refinemental properties of p-chains rather than special properties of chain coverings of the pseudo-arc.

THEOREM 3.5. *In order that a continuum H be a continuous image of the pseudo-arc it is necessary and sufficient that H be p-chainable.*

Proof of necessity. This follows from Theorem 3.4.

Proof of sufficiency. Let H be a p-chainable continuum and let $P_1, P_2, P_3, \cdots$ be a sequence of p-chains associated with H. We shall define two related sequences of p-chains $T_1, T_2, T_3, \cdots$ and $D_1, D_2, D_3, \cdots$ having the following properties: for each positive integer n, (1) T_n is a principal normal refinement of P_n, (2) D_n is a corresponding chain of open discs in the plane having the same number of links as T_n, and, if n is greater than 1, then (3) T_n is crooked in T_{n-1}, D_n has the same pattern in D_{n-1} as the r-pattern of T_n in T_{n-1}, each link of D_n has diameter less than $1/n$ and the closure of each link of D_n is contained in the corresponding link of D_{n-1}. In the case that $n = 1$, it is observed that these requirements may readily be satisfied. Next assume for each positive integer n less than some integer k, $k > 1$, that the requirements of this induction statement have been satisfied and consider the case that $n = k$.

First, by Theorem 3.2, we note that there is a p-chain S_k which is a principal normal refinement of P_k and a normal refinement of T_{k-1}. Furthermore, by Theorem 3.3, there is a principal normal refinement T_k of S_k such that T_k is crooked in T_{k-1}. Hence, noting Theorem 3.1, T_k is a principal normal refinement of P_k which is crooked in T_{k-1}. In addition, we may assume without loss in generality that T_k has a sufficient number of links so that we can choose a chain D_k of open discs in the plane which fulfills the remaining conditions of property (3) for $n = k$. It follows that T_k and D_k satisfy the requirements of the induction hypothesis and the definition of the sequences of p-chains $T_1, T_2, T_3, \cdots$ and $D_1, D_2, D_3, \cdots$ is complete. It is noted, from [2, §2], that the intersection of the sets of points of the chains $D_1, D_2, D_3, \cdots$ is a pseudo-arc M.

We now define a continuous mapping f of M onto H. To do this, let x be a point of M and let the links d_{1r_1} of D_1, d_{2r_2} of D_2, d_{3r_3} of $D_3, \cdots$ be a sequence of open sets closing down on x such that, for each positive i, $d_{i+1r_{i+1}}$ corresponds to d_{ir_i} under the pattern of D_{i+1} in D_i. Then, by conditions (c) and (d) of Definition 10, the intersection $\bigcap_{i=1}^{\infty} t_{ir_i}$ exists and is a single point. We define the mapping f by setting $f(x) = \bigcap_{i=1}^{\infty} t_{ir_i}$. To see that f is uniquely defined, suppose that d_{1r_1}, $d_{2r_2}, d_{3r_3}, \cdots$ and $d_{1s_1}, d_{2s_2}, d_{3s_3}, \cdots$ are two sequences of links closing down on x in the above described manner. Then, $| r_i - s_i | \leqq 1, i = 1,2,3,\cdots$, so that $t_{ir_i} \cup t_{is_i}$ has diameter less than $2/i$ and hence $\bigcap_{i=1}^{\infty}(t_{ir_i} \cup t_{is_i}) = \bigcap_{i=1}^{\infty} t_{ir_i} = \bigcap_{i=1}^{\infty} t_{is_i}$. To prove that f is a continuous mapping of M into H, let g be any open set in H and let k be an integer such that g contains three consecutive links t_{kc-1}, t_{kc} and t_{kc+1} of T_k. Now D_k is a chain and if $d_{1r_1}, d_{2r_2}, d_{3r_3}, \cdots$ is a sequence which closes down on a point of d_{kc}, then one of the links d_{kc-1}, d_{kc} and d_{kc+1} is a member of this sequence. Thus $d_{kc} \cap M$ which is open in M is mapped into $t_{kc-1} \cup t_{kc} \cup t_{kc+1}$ and hence into g under the mapping f. It follows that f is continuous and that $f(M)$ is everywhere dense in M. But M is compact so that the continuous transformation f is a mapping of M onto H, and the proof is complete.

The following theorem indicates an analogy between Theorem 3.5 and the Hahn-Mazurkiewicz characterization of the continuous images of the arc. It may be seen that it follows easily from the proof of the Hahn-Mazurkiewicz theorem [5; 8].

THEOREM 3.6. *In order that a compact continuum H be locally connected it is necessary and sufficient that H be p-chainable with p-chains whose links are connected open sets.*

With this formulation of the definition of local connectedness, the characterization of the continuous images of the pseudo-arc of Theorem 3.5 and the Hahn-Mazurkiewicz characterization of the continuous images of the arc differ only in that the links of the p-chains involved in the latter characterization are required to be connected open sets.

In the next and final theorem of this section we establish an alternative characterization of the continuous images of the pseudo-arc which, while it has certain disadvantages in investigating the combinatorial properties among these continua, involves a somewhat closer analogy to the Hahn-Mazurkiewicz theorem than does Theorem 3.5.

THEOREM 3.7. *A necessary and sufficient condition that a compact continuum H be a continuous image of the pseudo-arc is that H be p-chainable with p-chains whose links are open sets.*

Proof of necessity. Let H be a continuous image of the pseudo-arc and, for convenience, let H be considered as space. Then, by Theorem 3.5, there is a sequence of p-chains $P_1, P_2, P_3, \cdots$ associated with H and we may without loss in generality assume that, for each positive integer n, each link of P_n has diameter less than $1/2n$. We now choose a sequence of p-chains $T_1, T_2, T_3, \cdots$ associated with H such that, for each positive integer n, T_n has the same number of links as P_n, T_{n+1} has a pattern in T_n which is the same as the r-pattern of P_{n+1} in P_n, and each link of T_n is an open set. To do this, let $P_n = (p_{n1}, p_{n2}, \cdots, p_{nr_n})$ and let $T_n = (t_{n1}, t_{n2}, \cdots, t_{nr_n})$ where t_{ni}, $i = 1, 2, \cdots, r_n$, is the set of all points which have distance less than $1/4n$ from the link p_{ni} of P. With this definition of T_n, it is clear that T_n is a p-chain whose links are open sets. Furthermore, for each positive integer n, the union of the elements of T_n is H, each link of T_n has diameter less than $1/n$, T_{n+1} has a pattern in T_n which is the same as the r-pattern of P_{n+1} in P_n, and, under this pattern, the closure of each link of T_{n+1} is a subset of the corresponding link of T_n. Thus $T_1, T_2, T_3, \cdots$ is a sequence of p-chains associated with H such that each link of each p-chain of this sequence is an open set and the proof is complete.

Proof of sufficiency. This follows from Theorem 3.5.

4. **Chainable continua.** In this section it will be shown that every chainable continuum is a continuous image of the pseudo-arc. The proof will primarily depend on Theorem 3.2 and the characterization of the continuous images of the pseudo-arc stated in Theorem 3.5.

THEOREM 4.1. *Every chainable continuum is a continuous image of the pseudo-arc.*

Before proceeding to give a proof of this theorem, it will be convenient to establish two lemmas.

LEMMA 4.1.1. *If P and Q are p-chains such that P is a refinement of Q and each link of Q corresponds to at least one link of P, then there is a p-chain T such that T is a principal refinement of P and a normal refinement of Q.*

Proof. Let the p-chain be $P(1,n)$ and let p_s and p_t be links of P which correspond to the first and last links of Q, respectively, under the r-pattern of P in Q. We will then define T to be the p-chain sum $P(s,1) + P(1,n) + P(n,t)$. It is clear that the p-chain T then has the required properties.

LEMMA 4.1.2. *If P, Q and T are p-chains such that T is a principal refinement of Q and P is a refinement of Q in which each link of Q corresponds to at least one link of P, then there is a p-chain S such that S is a principal refinement of P and S is a refinement of T in which each link of T corresponds to at least one link of S.*

Proof. First, from the preceding lemma we observe that there exist p-chains P_1 and T_1 which are principal refinements of P and T, respectively, and which are each normal refinements of the p-chain Q. In addition, from Theorem 3.1, T_1 is also a principal refinement of Q. It follows, by Theorem 3.2, that there is a p-chain S such that S is a principal normal refinement of P_1 and a normal refinement of T_1. Moreover, by Theorem 3.1, S is then a refinement of T and a principal refinement of P. Finally, since S is a normal refinement of T_1 and T_1 is a principal refinement of T, each link of T corresponds to at least one link of S. Thus, the p-chain S has the required properties and the proof is complete.

Proof of Theorem 4.1. Let H be a chainable continuum and, for convenience, let H be considered as space. Then there is a sequence of chains $D_1, D_2, D_3, \cdots$ such that for each positive integer i, (1) the set of points of D_i is H, (2) each link of D_i has diameter less than $1/i$, (3) no link of D_i is a subset of any other link of D_i, and (4) there is an r-pattern of D_{i+1} in D_i such that the closure of each link of D_{i+1} is a subset of the corresponding link of D_i. We now construct a corresponding sequence of p-chains $P_1, P_2, P_3, \cdots$ associated with H. First we define the p-chain P_1 to be the chain D_1. Then we observe, from property (3), above, and the connectedness of H, that each link of D_1 corresponds to at least one link of D_2, and therefore, by Lemma 4.1.1 we may choose P_2 to be a p-chain which is a principal refinement of D_2 and a normal refinement of P_1. Next, since each link of D_2 corresponds to at least one link of D_3, by Lemma 4.1.2 there is a p-chain S_3, such that S_3 is a principal refinement of D_3, and S_3 is a refinement of P_2 in which each link of P_2 corresponds to at least one link of S_3. It then follows by Lemma 4.1.1 that we may choose P_3 to be a principal refinement of S_3 and hence of D_3, and a normal refinement of P_2. Proceeding in this manner, we obtain a sequence of p-chains, $P_1, P_2, P_3, \cdots$ which are principal refinements of the chains $D_1, D_2, D_3, \cdots$, respectively, and which have the property that each is a normal refinement of the p-chain (if any) that precedes it in the sequence. This is clearly a sequence of p-chains associated with H and, by Theorem 3.5, we conclude that H is a continuous image of the pseudo-arc.

COROLLARY 4.1.1. *The class of continuous images of the pseudo-arc and the class of continuous images of all chainable continua are identical.*

5. **Nonexistence of a local characterization of the continuous images of the pseudo-arc.** In §3 we have established two characterizations of the continuous images of the pseudo-arc and have shown that, if the definition of local-connectedness is suitably reformulated as a global property, then these characterizations are analogous to the Hahn-Mazurkiewicz characterization of the continuous images of the arc [5; 8]. However, since an important characteristic of this Hahn-Mazurkiewicz theorem is the identification of the continuous images of the arc by a simple local property, the question arises, "Does there exist a local topological property which characterizes all continuous images of the pseudo-arc?" In this section we shall show that this question has a negative answer.

A number of additional special terms, some of which will also be of significance in §6, will be needed.

DEFINITION 11. If S is a metric space and x, y are points such that there is a shortest arc from x to y in S, then the *arc-distance* between x and y will be defined to be equal to the length of such a shortest arc. A unique arc from x to y in S will be denoted by $[x, y]$.

DEFINITION 12. If M is a set such that there is a defined arc distance between each two points in M, then we define the *arc-diameter* of a subset of M, and the *arc-distance between two sets* in M in the usual way in terms of arc-distance between points.

DEFINITION 13. An *ε-sequence* of points in an arc $[x, y]$ will be a finite sequence of points, lying in $[x, y]$, such that the arc-distance between adjacent members of the sequence is less than ε.

Next, we formalize the definition of "local topological property" which will be used in this section. This expression appears to be not yet completely standardized. However, the definition to be given will be one of the more natural forms in general use, and will include all of the properties normally referred to in this category.

DEFINITION 14. A space S will be said to have a *local topological property* π at a point x if for every open set U containing x there is an open set V such that V contains x and lies in U, and either V or the closure of V has the topological property π. The space S will be said to have the local topological property π if S has property π at each of its points.

Finally, we define a relationship between two topological spaces which essentially requires that any local topological property of one space necessarily must be a local topological property of the second space.

DEFINITION 15. Two topological spaces H and K will be said to be *locally homeomorphic* if (1) for every point x of H and every open set U containing x, there is an open set V such that V contains x and lies in U and V is homeomorphic to an open set in K, and (2) a similar statement can be made with the roles of H and K interchanged.

We now proceed to formulate the theorem which will be the principal result of this section.

THEOREM 5.1. *There does not exist a local topological property which characterizes all continuous images of the pseudo-arc.*

In order to establish this theorem we shall consider a second result, Theorem 5.2. This latter theorem will show that there does not exist a local topological property of a continuum at any one of its points or on any collection of its points, which characterizes it as a continuous image of the pseudo-arc. In particular Theorem 5.1 will be a direct consequence of Theorem 5.2.

THEOREM 5.2. *There exist locally homeomorphic compact metric continua H and K such that H is a continuous image of the pseudo-arc, but K is not a continuous image of the pseudo-arc.*

Proof. Since this proof will involve a rather complicated development it will be convenient to make a preliminary outline of the steps we shall follow.

First, the compact metric continua H and K will be defined. These continua will be considered as spaces and, for the reference needs of the later stages of the proof, will be described in precise detail. However, the essential features of these continua, which will be sufficient for the needs of the early stages of the proof, will be more readily seen from the accompanying figures. Then, we shall give arguments to show, in turn, that (a) H is a continuous image of the pseudo-arc, and that (b) H and K are locally homeomorphic continua. Finally, the most difficult part of the proof will be considered. Specifically, we shall use Theorem 3.4 and the concepts introduced in the preceding segments of this section to establish that (c) the continuum K is not a continuous image of the pseudo-arc.

Definition of H. In terms of plane cartesian co-ordinates the continuum H will be defined to be the closure of the ray

$$
\begin{aligned}
A = \bigcup_{n=4}^{\infty} [&\{(u,v): u = \pm 2^{-n},\ 2^{-n} \leqq v \leqq 2+2^{-n}\}\\
&\cup \{(u,v): -2^{-n} \leqq u \leqq 2^{-n},\ v = 2+2^{-n}\}\\
&\cup \{(u,v): -2-2^{-n} \leqq u \leqq -2^{-n}, 2^{-n} \leqq u \leqq 2+2^{-n}-2^{-(n+2)}|(\sin n\pi/2)|, v=2^{-n}\}\\
&\cup \{(u,v): -2-2^{-n} \leqq u \leqq 2+2^{-n}-2^{-(n+2)}|\cos(n\pi/2)|,\ v = -2^{-n}\}\\
&\cup \{(u,v): u = -2-2^{-n}, -2^{-n} \leqq v \leqq 2^{-n}\}\\
&\cup \{(u,v): 2+2^{-(n+1)}, \leqq u \leqq 2+3\cdot 2^{-(n+2)}, v=0\}\\
&\cup \{(u,v): u = 2+2^{-n}-2^{-(n+2)}|\sin(n\pi/2)|, 0 \leqq v \leqq 2^{-n}\}\\
&\cup \{(u,v): u = 2+2^{-n}-2^{-(n+2)}|\cos(n\pi/2)|, -2^{-n} \leqq v \leqq 0\}].
\end{aligned}
$$

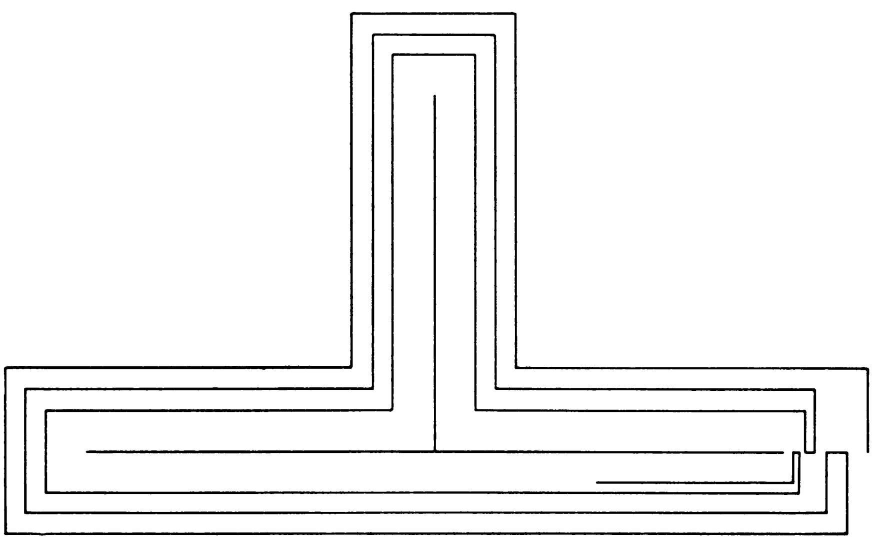

Figure 1

DEFINITION OF K. In terms of plane cartesian co-ordinates the continuum K will be defined to be the closure of the ray

$$\begin{aligned} B = &\{(w,z): w = 2 + 2^{-4}, 0 \leqq z \leqq 2^{-4}\} \\ &\cup \Big[\bigcup_{n=4}^{\infty} (\{(w,z): 2^{-n} \leqq w \leqq 2 + 2^{-n}, -2 - 2^{-n} \leqq w \leqq -2^{-n},\ z = 2^{-n}\} \\ &\cup \{(w,z): w = \pm 2^{-n}, 2^{-n} \leqq z \leqq 2 + 2^{-n}\} \\ &\cup \{(w,z): -2^{-n} \leqq w \leqq 2^{-n}, z = 2 + 2^{-n}\} \\ &\cup \{(w,z): w = -2 - 2^{-n}, -2^{-n} \leqq z \leqq 2^{-n}\} \\ &\cup \{(w,z): -2 - 2^{-n} \leqq w \leqq 2 + 2^{-(n+1)}, z = -2^{-n}\} \\ &\cup \{(w,z): w = 2 + 2^{-(n+1)}, -2^{-n} \leqq z \leqq 2^{-(n+1)}\}) \Big]. \end{aligned}$$

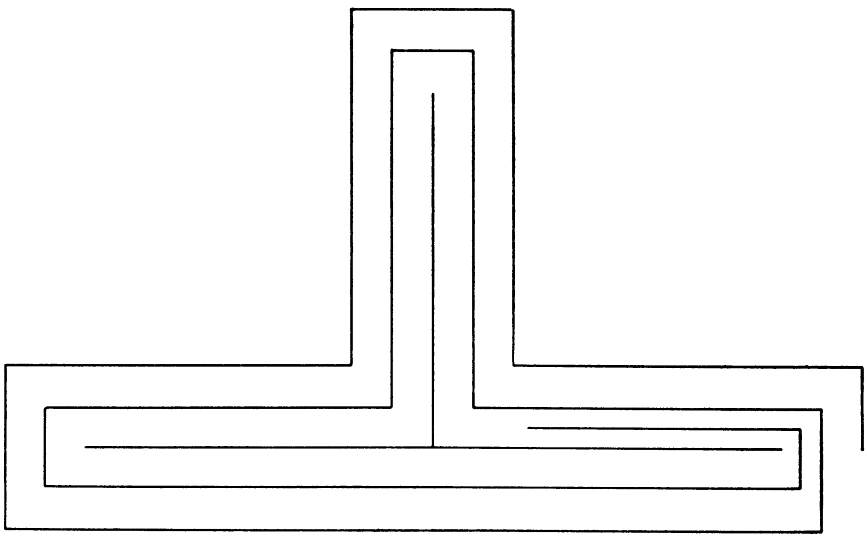

FIGURE 2

(a) Proof that H is a continuous image of the pseudo-arc. If C is the continuum which is the closure of the ray $R = \{(x,y): y = \sin 1/x,\ 0 < x \leqq 1\}$, there is a continuous mapping of the arc $C - R$ onto the simple triad $H - A$ which can be extended in a natural manner to a continuous mapping of C onto H. Furthermore, C is a chainable continuum. Hence, by Theorem 4.1, it follows that H is a continuous image of the pseudo-arc.

(b) Proof that the compact continua H and K are locally homeomorphic. First, we observe that each point of H, other than the point $h = (u = 2, v = 0)$, is contained in a neighborhood which is homeomorphic to a neighborhood of K

under the identity mapping. Similarly, each point of K other than the point $k = (w = 2,\ z = 0)$ is contained in a neighborhood which is homeomorphic to a neighborhood of H under the identity mapping. Now consider the neighborhoods N_h in H with center h and radius r, $r < 2$, and N_k in K with center k and the same radius r. We shall complete the proof of this section of the theorem by describing a homeomorphism f of N_h onto N_k. First, f will be required to map the component of N_h which contains h onto the component of N_k which contains k, in an identical manner. Then, if C_n is the component of N_h which is nth in the decreasing order of distances of the components of N_h from the point h, and D_n is the similarly determined component of N_k, we shall define f over the part C_n of its domain to be a particular homeomorphism of C_n onto D_n. Specifically, let e_1 be the end-point of the arc $\bar{C}_n$ having smallest abscissa, let e_2, be the point of $\bar{C}_n$ which has abscissa equal to 2 and is closest to e_1, let e_3 be the point of $\bar{C}_n$ which has abscissa equal to 2 and is farthest from e_1, and finally let e_4 be the remaining end-point of $\bar{C}_n$. In addition, let g_1, g_2, g_3 and g_4 be the similarly chosen points of $\bar{D}_n$. Then we shall define a transformation of $\bar{C}_n$ onto $\bar{D}_n$ by mapping $[e_1, e_2]$ onto $[g_1, g_2]$ in an order preserving and arc-distance preserving manner, mapping $[e_2, e_3]$ onto $[g_2, g_3]$ in an order preserving manner and mapping $[e_3, e_4]$ onto $[g_3, g_4]$ in an order preserving and arc-distance preserving manner. The restriction of this transformation to the domain C_n is then a homeomorphism. The transformation f over the part C_n of its domain will be defined to be this homeomorphism. Thus, the definition of the mapping f of N_h onto N_k is complete. We observe that f is clearly continuous at all points of N_h at which this set is locally connected. Furthermore, if x is a point of N_h at which N_h fails to be locally connected and $x_1, x_2, x_3, \cdots$ is any sequence of points of N_h which converges to x then it is easily verified that $f(x_1), f(x_2), f(x_3), \cdots$ converges to the appropriate point $f(x)$ of N_k. Since, in addition, corresponding statements can be made for the inverse transformation f^{-1} with respect to its domain N_k, we conclude that f is a homeomorphism. This completes the proof that the compact continua A and K are locally homeomorphic.

(c) Proof that K is not a continuous image of the pseudo-arc. For the purpose of investigating the continuum K we choose a sequence of reference points $a_0, a_1, a_2, \cdots$ where $a_0 = (w = 2 + 2^{-4},\ z = 0)$ and is referred to as the first point of the ray B, the points $a_0, a_2, a_4, \cdots$ are the successive points of intersection of B with the union of the w-axis and the positive z-axis, and $a_1, a_3, a_5, \cdots$ are the mid-points of the arcs $[a_0, a_2]$, $[a_2, a_4]$, $[a_4, a_6], \cdots$ respectively. We now proceed to show that K is not a continuous image of the pseudo-arc.

Suppose, on the contrary, that the continuum K is a continuous image of the pseudo-arc. Then, by Theorem 3.5, K is p-chainable and, by Theorem 3.4, we may choose a sequence of p-chains associated with K such that the first link of each p-chain of the sequence contains the point a_0. Let $P = (p_1, p_2, \cdots, p_n)$ and $Q = (q_1, q_2, \cdots, q_m)$ be two p-chains of such a sequence and let these p-chains be chosen so that P is a refinement of Q with an r-pattern $(1, x_1 = 1)$, $(2, x_2), \cdots$, (n, x_n) in Q, each link of P has diameter less than $\varepsilon = 2^{-m}$, and each link of Q has diameter less than $1/4$. For the purposes of this argument we need to observe, since the elements of Q form a cover for K, that m is greater than 7.

From Definition 10 and the choice of restriction of the diameters of the links of P it follows that the first link of P has arc-diameter less than ε and, if p_r is the first link of P with arc-distance less than 2ε from a_m then each link of the sub-p-chain $P(1,r)$ is contained in the arc $[a_0, a_m]$ and has arc-diameter less than 2ε. Hence, we may select a 2ε-sequence of points $c_1, c_2, \cdots, c_r$ in $[a_0, a_m]$ such that for each integer i, $1 \leqq i \leqq r$, c_i is an element of the link p_i of P. With this sequence of points and a corresponding sequence of points to be chosen from the links of Q it will be shown that K is not a continuous image of the pseudo-arc. The sequence $d_1, d_2, \cdots, d_s$ of points of the links of Q is defined by first setting $d_1 = a_0$ and then, if $i \geqq 1$ and d_i has been chosen, and there is a link of $P(1,r)$ which fails to correspond to a link of $Q(1,i)$ under the r-pattern of P in Q, we define d_{i+1} to be a point of the first such link of $P(1,r)$. Then, $d_1, d_2, \cdots, d_s$ is a 1-sequence of points in $[a_0, a_m]$ and for each integer j, $1 \leqq j \leqq s$, the point d_j is contained in the link q_j of the p-chain Q.

Now, the r-pattern of P in Q defines a natural correspondence between the points $c_1, c_2, \cdots, c_r$ of the links of $P(1,r)$ and the points $d_1, d_2, \cdots, d_s$ of the links of $Q(1,s)$. Furthermore under this r-pattern each point c_i of $c_1, c_2, \cdots, c_r$ has distance less than $1/4$ from the corresponding point d_{x_i} of $d_1, d_2, \cdots, d_s$.

For the next step in our proof, we shall prove by induction that, for each integer i, $1 \leqq i \leqq r$, the point c_i has arc-distance less than 6 from d_{x_i}.

In the case that $i = 1$ we need only observe that c_1 and $d_{x_1} = d_1$ are each points of the link p_1 of P and p_1 has arc-diameter less than ε which is less than 6.

Next, assume that the induction statement has been established for all values of i less than or equal to some integer t, $1 \leqq t < r$, and consider the situation that $i = t + 1$. We let $[a_u, a_{u+1}]$ be an arc containing c_t and let z be the mid-point of this arc. Two cases will now be considered.

In the first case, we suppose that c_t is an element of whichever of the arcs $[a_u, z]$ and $[z, a_{u+1}]$ is nearest to the origin. Then, by the induction hypothesis, c_t and d_{x_t} have arc-distance apart less than 6 and, by the nature of the r-pattern, c_t and d_{x_t} have distance apart less than $1/4$. It is easily seen from the geometrical nature of K that these two conditions can only obtain if c_t and d_{x_t} have arc-distance apart of less than $4\frac{1}{2}$. But the arc-distance between c_t and c_{t+1} is less than 2ε and the arc-distance between d_{x_t} and $d_{x_{t+1}}$ is less than 1. Hence, the arc-distance between c_{t+1} and $d_{x_{t+1}}$ is less than 6.

In the second case, we suppose that c_t is an element of whichever of the two arcs $[a_u, z]$ and $[z, a_{u+1}]$ is farthest from the origin. Here, the restrictions on the distance and arc-distance between the points c_t and d_{x_t} and the geometrical nature of K imply that the arc-distance between c_t and d_{x_t} is less than 3. Then, as in the previous case the arc-distance between the points c_{t+1} and $d_{x_{t+1}}$ is less than 6. This completes the induction.

Now, from this preceding induction result, we are able to obtain a contradiction to the initial hypothesis that K is p-chainable by showing that Q must have more than m links. Specifically, we note that c_r has arc-distance less than 4 from a_m and d_{x_r} has arc-distance less than 6 from c_r. Thus a_m and d_{x_r} have arc-distance apart of less than 7. But d_{x_r} is a point of a 1-sequence of points $d_1, d_2, \cdots, d_s$, where $d_1 = a_0$, in $[a_0, a_m]$, and the length of the arc $[a_0, a_m]$ is greater than $2m$. It follows that the 1-sequence of points $d_1, d_2, \cdots, d_s$ in $[a_0, a_m]$ has more than $2m - 7$ members and, since m is greater than 7, we have the result that s is greater than m. This involves the desired contradiction and we conclude that K is not a continuous image of the pseudo-arc.

6. **Continua which are not continuous images of the pseudo-arc.** In §4, we have shown that the pseudo-arc can be mapped continuously onto all chainable continua and it is observed that, of course, all locally connected compact metric continua are continuous images of the pseudo-arc. Furthermore, the pseudo-arc is highly complex [2; 7; 10] and, thus, it might be conjectured that the principal generalizations of the class of chainable continua and the class of locally connected compact metric continua are also contained in the class of continuous images of the pseudo-arc. However, this is not the case and in this section we shall give examples to show that the classes of tree-like continua, arc-wise connected compact metric continua, semi-locally connected compact metric continua, and aposyndetic [6] compact metric continua, each fail to be contained in the class of continuous images of the pseudo-arc.

EXAMPLE 6.1. A tree-like plane continuum which is not a continuous image of the pseudo-arc.

The plane continuum K which was discussed in Theorem 5.2, and represented in Figure 2, is clearly tree-like and is not a continuous image of the pseudo-arc.

EXAMPLE 6.2. An arc-wise connected, compact metric continuum C which is not a continuous image of the pseudo-arc.

DEFINITION OF C. In terms of spherical co-ordinates, the continuum C is defined to be the closure of the ray

$$R = \{(r,\theta,\phi): r = 2 - 2^{-\theta/2\pi},\ \theta \geqq 4\pi,\ \phi = \pi/2\}$$

together with the arc

$$A = \{(r,\theta,\phi): r\sin\phi = 2, \theta = 0, \pi/4 \leqq \phi \leqq \pi/2\}$$
$$\cup \{(r,\theta,\phi): r\sin\phi = 2 - 2^{-4},\ \theta = 0,\ \pi/4 \leqq \phi \leqq \pi/2\}$$
$$\cup \{(r,\theta,\phi): 2\sqrt{2} \geqq r \geqq (2 - 2^{-4})\sqrt{2},\ \theta = 0,\ \phi = \pi/4\};$$

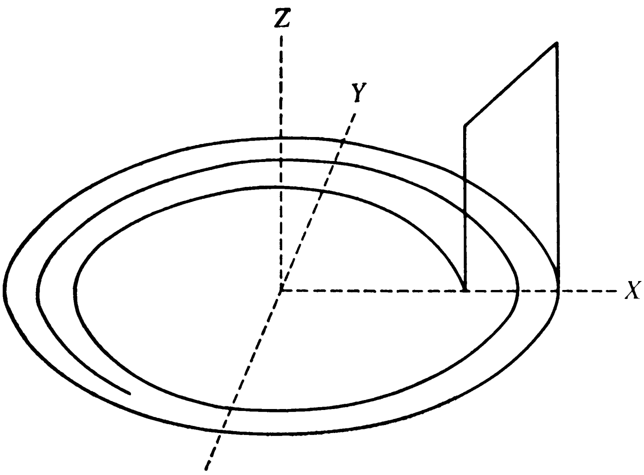

FIGURE 3

It is clear that the compact continuum C is arc-wise connected.

To show that C is not a continuous image of the pseudo-arc, we first choose a sequence of reference points $a_0, a_1, a_2, \cdots$ of the ray R, where a_0 is the point whose spherical co-ordinates are $(2 - 2^{-4}, 0, \pi/2)$ and is referred to as the first point of R, and $a_0, a_1, a_2, \cdots$ are the successive points of intersection of R with the set consisting of the union of the x and y axes.

Now, suppose that the continuum C is a continuous image of the pseudo-arc and, noting Theorem 3.4, let $P = (p_1, p_2, \cdots, p_n)$ and $Q = (q_1, q_2, \cdots, q_m)$ be two p-chains of a sequence associated with C such that the first link of each p-chain of the sequence contains the point a_0. In particular, we choose the p-chains P and Q such that each link of P has diameter less than $\varepsilon = 2^{-2m}$, each link of Q has diameter less than $1/4$ and P is a normal refinement of Q. Then, from this choice of restriction of the diameters of the links of P, there is either (1) a sub-p-chain $P(h,k)$ of P such that p_h has arc-distance less than 2ε from a_0, p_k has arc-distance less than 2ε from a_m and each link of $P(h,k)$ lies in $[a_0, a_m]$ and has arc-diameter less than 2ε, or (2) a similar statement can be made in which a_0 is replaced by a_{2m}. In either case we may choose a 2ε-sequence of points $c_h, c_{h+1}, \cdots, c_k$ such that each point c_i of this sequence is an element of the link p_i of P and the arc-diameter of the set of points of this sequence is greater than m. But this situation is similar to the one considered in the discussion of K in the previous section, so that we may choose a corresponding sequence of points from the links of Q and proceed using the same method of proof as that used in Theorem 5.2(c). Therefore, we conclude that C is not a continuous image of the pseudo-arc.

EXAMPLE 6.3. A compact metric continuum D which is semi-locally connected and aposyndetic, but which is not a continuous image of the pseudo-arc.

DEFINITION OF D. In terms of cylindrical co-ordinates the continuum D is defined to be the closure of the set $\{(r, \theta, z): r = 2 - 2^{-\theta/2\pi},\ \theta \geqq 4\pi,\ 0 \leqq z \leqq 1\}$.

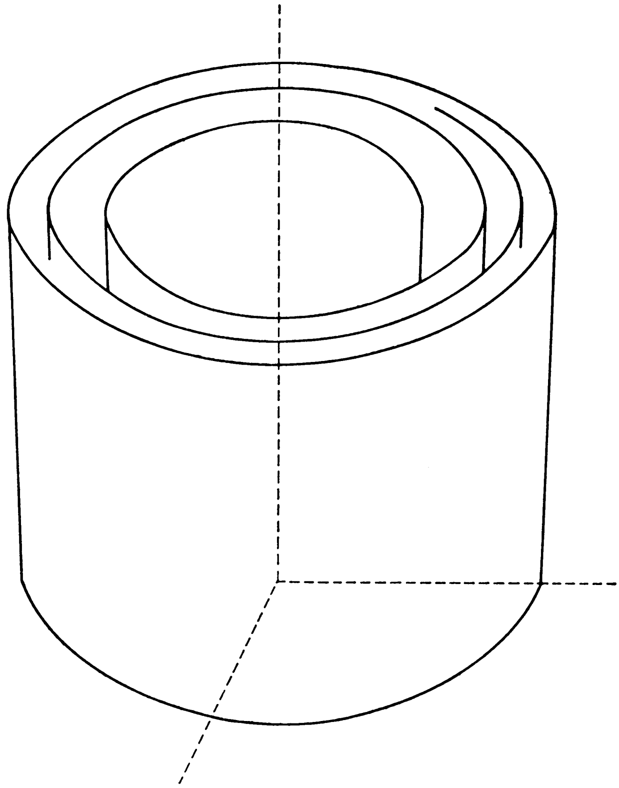

FIGURE 4

It is easily seen that the compact continuum D is semi-locally connected and can be shown that the projection of this continuum onto the $(z=0)$-plane fails to be a continuous image of the pseudo-arc. Therefore, since the projection mapping is continuous, D is not a continuous image of the pseudo-arc.

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