ON DOMINANT DIMENSIONS OF QF-3 ALGEBRAS

BY

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Introduction. The complete cohomology theory of groups was extended by Nakayama [8] to the case of Frobenius algebras. He [10] also proposed to classify algebras in accordance with the length of a right augumented, two-sided, projective, injective resolution. One of the purposes of this paper is to give a partial extension of the above complete cohomology theory to the case of non-quasi-Frobenius algebras and another is to give an estimation of the length.

Let B be an associative algebra over a commutative field with unit element 1 and M a unital right B-module. If

$$0 \to M \to X_1 \to X_2 \to \cdots \to X_n \tag{1}$$

is a minimal injective resolution([1]) of M, then we shall say that M has dominant dimension $\geqq n$, provided that X_k is projective for all $k \leqq n$, where it should be noted even the zero module is considered to be projective. The largest such integer([2]) n is called in this paper the dominant dimension of M and denoted by domi.dim ${}_B M$. When no such integer exists the dimension is defined to be ∞. The modules we are chiefly concerned with are B_B which is B considered as a right module over itself and B_{B^e} which is B considered as a right module over $B^e = B \otimes B^0$. In fact domi.dim ${}_{B^e}B$ is the length considered by Nakayama. In both cases B is a QF-3 algebra in the sense of Thrall [12] if and only if domi.dim $B \geqq 1$ [11]. So our interest is confined to the class of QF-3 algebras.

On the other hand, for QF-3 algebras there exists a result established by Morita [6]. The result is stated as follows: For a given QF-3 algebra B let X be the unique minimal faithful right B-module. Denote by A the B-endomorphism algebra of X as a left operator domain. Then X is fully faithful as a left A-module and the A-endomorphism algebra B' of X is QF-3. Here a left A-module L is said to be fully faithful if every indecomposable projective left A-module as well as every indecomposable injective left A-module is A-isomorphic to a direct summand of L. The above settings about QF-3 algebras are fundamental in this paper and the meanings of notations A, B and B' will be retained throughout.

In §1 the correlation between domi.dim ${}_B M$ and domi.dim ${}_{B'} M \otimes_B B'$ is

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([1]) The injective resolution (1) is said to be minimal if Im $\delta_k \supseteq$ the socle of X_{k+1} (cf. Eilenberg [2]).

([2]) If X_1 is not projective, we shall say the dominant dimension of M is zero.

discussed and as an application we obtain that if domi. $\dim_{B^e} B > 1$, then $B = B'$. The upper bound of dominant dimension of modules is investigated in §2. Especially domi. $\dim_{B^e} B \leqq \dim A + 1$ holds, provided $0 < \dim A < \infty$.

In [10] Nakayama has conjectured that B is quasi-Frobenius if domi. $\dim_{B^e} B = \infty$ and proved it for the case of B being a generalized uni-serial algebra. The author is, however, likely inclined to conjecture that B is quasi-Frobenius if domi. $\dim_B B = \infty$. In §3 we shall prove that our conjecture holds for the case of A being generalized uni-serial. Since domi. $\dim_{B^e} B \leqq$ domi. $\dim_B B$, this assures Nakayama's conjecture is true for this case.

In the beginning of §4 we shall introduce as a generalization of Nakayama's automorphism of Frobenius algebra a ring-isomorphism between two subrings of B. Then following Kasch's argument in [4] we have the isomorphisms between (negative dimensional) cohomology groups over B^e-modules and (positive dimensional) homology groups over A^e-modules under a restriction of dimensions. Here it is noted that the restriction is determined by dominant dimension of a B^e-module.

1. **Reduction theorems.** Let B be a QF-3 algebra and eB the faithful, projective, injective right ideal of B, i.e., e is a suitable idempotent of B and eB is a dominant ideal in the sense of Thrall [12]. Let A be the B-endomorphism algebra of eB considered as a left operator domain of eB. Then $A = eBe$ and from Lemma 17.4 of [6] we know eB is a fully faithful left A-module (for definition cf. Introduction). Denote by B' the A-endomorphism algebra of eB as a right operator domain. Then by Theorem 17.2 of [6], B' is a QF-3 algebra and there exists an idempotent f of B' such that $eB = fB'$, and fB' is a faithful, projective, injective right B'-module. Let Be' be the dual representation module (i.e., $= \mathrm{Hom}_K(eB, K)$). Then by duality it is clear that there exists such an idempotent f' of B' that $B'f' = Be'$. Thus B is obtained as a subalgebra of B' which contains 1 and $B'f'$ and fB'.

We may assume throughout that B is self-basic, that is to say, the basic algebra of B is isomorphic to B itself, for dominant dimensions and homological dimensions are invariant under any category-isomorphism. Now we shall prove

PROPOSITION 1.1. *Let B and B' be* QF-3 *algebras just introduced above and M a right B-module. Let*

$$0 \longrightarrow M \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} X_2 \xrightarrow{\delta_2} \cdots \longrightarrow X_n \tag{2}$$

be an exact sequence of right B-homomorphisms δ_i, $0 \leqq i \leqq n$, where X_i is a direct sum of n_i-copies of eB. Then we have an exact sequence

$$M \otimes_B B' \xrightarrow{\Delta_0} Y_1 \xrightarrow{\Delta_1} Y_2 \xrightarrow{\Delta_2} \cdots \longrightarrow Y_n \tag{3}$$

of right B'-homomorphisms Δ_i, where Y_i is a direct sum of n_i-copies of fB'.

Proof. We shall prove that the induced sequence of right B'-homomorphisms

$$(3')\qquad M \underset{B}{\otimes} B' \xrightarrow{\Delta_0} X_1 \underset{B}{\otimes} B' \xrightarrow{\Delta_1} X_2 \underset{B}{\otimes} B' \xrightarrow{\Delta_2} \cdots \longrightarrow X_n \underset{B}{\otimes} B'$$

is exact. Put $\delta_i \otimes B' = \Delta_i$ and denote by 1 the unit element of B.

Consider right B-homomorphisms θ_i, $i \geqq 0: X_i \to X_i \otimes_B B'$ defined by $\theta_i(x_i) = x_i \otimes 1 \in X \otimes_B B'$ for all $x_i \in X$, $i \geqq 1$ and $\theta_0(x_0) = x_0 \otimes 1 \in M \otimes_B B'$ for all $x_0 \in M$. Then clearly every θ_i is a monomorphism. We shall express X_i, $i \geqq 1$, as a direct sum of n_i-copies of eB: $X_i = \bigoplus_{j=1}^{n_i} u_j^i eB$, $u_j^i eB \approx eB$. Then we have $x_i \otimes b' = \sum_{j=1}^{n_i} u_j^i eb_j \otimes b' = \sum_{j=1}^{n_i} u_j e \otimes eb_j b'$, where $b' \in B'$, $b \in B$, $j = 1,2,\cdots,n$ and $x_i \otimes b' \in X_i \otimes_B B'$. Since $eB = eB'$, $x_i \otimes b' = \sum_{j=1}^{n_i} u_j^i eb_j b' \otimes 1$ and it follows that θ_i, $i \geqq 1$, is an epimorphism and consequently an isomorphism. Thus we obtain the following commutative diagram:

$$(4)\qquad \begin{array}{ccccccccc} 0 \longrightarrow M & \xrightarrow{\delta_0} & X_1 & \xrightarrow{\delta_1} & X_2 & \xrightarrow{\delta_2} & \cdots & \longrightarrow & X_n \\ \downarrow{\scriptstyle\theta_0} & & \downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_2} & & & & \downarrow{\scriptstyle\theta_n} \\ M \underset{B}{\otimes} B' & \xrightarrow{\Delta_0} & X \underset{B}{\otimes} B' & \xrightarrow{\Delta_1} & X_2 \underset{B}{\otimes} B' & \xrightarrow{\Delta_2} & \cdots & \longrightarrow & X_n \underset{B}{\otimes} B' \end{array}$$

where the upper row is exact. Since θ_i, $i > 0$, is an isomorphism, the exactness of (3′) follows from (4).

COROLLARY 1.2. *In Proposition* 1.1 *if it is assumed that* $n > 1$, *then* M *is a* B*-module which is obtained from a right* B'*-module by restricting its operator domain* B' *to* B.

Proof. From (4) we have $M \approx \operatorname{Ker}\delta_1 \approx \operatorname{Ker}\Delta_1$. But since Δ_1 is considered to be a B'-homomorphism, the conclusion follows.

LEMMA 1.3. *In Proposition* 1.1 *assume*

$$0 \longrightarrow M \underset{B}{\otimes} B' \xrightarrow{\Delta_0} X_1 \underset{B}{\otimes} B'$$

is exact; then M *is* B*-isomorphic to* $M \otimes_B B'$.

Proof. By Proposition 1.1 we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \xrightarrow{\delta_0} & X_1 & \xrightarrow{\delta_1} & X_2 & \longrightarrow \cdots \\ & \downarrow{\scriptstyle\theta_0} & & \downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_2} & \\ 0 \longrightarrow & M \underset{B}{\otimes} B' & \xrightarrow{\Delta_0} & X_1 \underset{B}{\otimes} B' & \xrightarrow{\Delta_1} & X_2 \underset{B}{\otimes} B' & \longrightarrow \cdots \end{array}$$

As θ_0 is a monomorphism, $\Delta_0\theta_0$ is a monomorphism. However, $\operatorname{Im}\Delta_0\theta_0 = \operatorname{Im}\theta_1\delta_0 = \operatorname{Ker}\Delta_1 = \operatorname{Im}\Delta_0$; hence θ_0 is an isomorphism.

THEOREM 1.4. *Let B and B' be* QF-3 *algebras as in Proposition* 1.1. *Assume* $\text{domi.dim}_B B > 1$; *then* $B = B'$.

Proof. From assumption the following exact sequence is given:

$$0 \to B \to X_1 \to X_2 \to \cdots \to X_n, \qquad n \geqq 2,$$

where X_i is a direct sum of n_i-copies of eB. Put $\Delta_i = \delta_i \otimes B'$, $i \geqq 0$. Then since $B \neq B'$ implies $B \approx B \otimes B'$, by Lemma 1.3 we have only to prove

$$(5) \qquad 0 \longrightarrow B \underset{B}{\otimes} B' \xrightarrow{\Delta_0} X \underset{B}{\otimes} B'$$

is exact. Let $\Delta_0(1 \otimes 1) = \sum_i u_i$, where u_i is an element of an indecomposable direct summand of $X_1 \otimes_B B'$. Then $\Delta_0(1 \otimes b') = \sum_i u_i b'$. Suppose $\Delta_0(1 \otimes b') = 0$ for some $b' \neq 0$, $b' \in B'$; then $u_i b' = 0$, hence $u_i b' B' = 0$ for all i. There exists clearly a minimal subideal r of $b'B'$ such that

$$(6) \qquad u_i r = 0 \qquad \text{for all } i.$$

Since B is self-basic, B' is also self-basic and QF-3. Hence $r \subseteq B'f'$. It follows that $r \subseteq B$, for $B'f' = Be' \subseteq B$. Then (6) contradicts to that δ_0 is a monomorphism. Thus (5) must be exact.

COROLLARY 1.5. *Let B and B' be* QF-3 *algebras as in Proposition* 1.1. *If B is properly contained in B', then*

$$\text{domi. dim}_{B'} B' \geqq \text{domi. dim}_B B = 1.$$

REMARK. The inequality of the above relation really holds. Let B, B' and B'' be subalgebras of full matrix ring K_{14} over a commutative field K such that the elements in the following table form K-basis respectively:

$B: e_1 = c_{11} + c_{22} + c_{33}$, $e_2 = c_{44} + c_{55}$, $e_3 = c_{66} + c_{77}$,
$e_4 = c_{88} + c_{99} + c_{10,10}$, $e_5 = c_{11,11} + c_{12,12}$,
$e_6 = c_{13,13} + c_{14,14}$, $c_{11,1} + c_{12,3}$, $c_{13,1} + c_{14,2}$, c_{51}, c_{71}, $c_{10,1}$, c_{94}
$+ c_{10,5}$, $c_{3,4}$, $c_{12,4}$, $c_{8\,6} + c_{10,7}$, c_{26}, $c_{14,6}$, $c_{12,9}$, $c_{14,8}$, $c_{10,11}$, $c_{10,13}$.

B'': annexing elements $c_{7,11}$, $c_{5,11}$, $c_{7,13}$, $c_{5,13}$ to the table of B,

B': annexing elements c_{39}, c_{28} to the table of B''. Here $c_{i,j}$ denotes the matrix with 1 in the (i,j)-position and zero elsewhere. Then $\text{domi. dim}_B B = \text{domi.dim}_{B''} B'' = 1$ and $\text{domi. dim}_{B'} B' = 3$.

REMARK. The dominant dimension of a QF-3 algebra which is isomorphic to an endomorphism algebra of a fully faithful module is not necessarily larger than 1 (cf. the example of Remark of Theorem 1.9).

Consider the enveloping algebra $B^e = B \otimes_K B^0$ of B, where B^0 is the opposite algebra of B. Then B may be regarded to be a right B^e-module by the following definition: $a \cdot b \cdot c = b(c \otimes_K a^0)$, for $a, b, c \in B$ and $a^0 \in B^0$. It was shown in [11] that domi. $\dim_{B^e} B \geqq 1$ if and only if B is QF-3. In this case B^e is also QF-3 and its unique minimal faithful right B^e-ideal is isomorphic to $eB \otimes_K e'^0 B^0$.

Similarly as Theorem 1.4 we have

THEOREM 1.6. *If* domi. $\dim_{B^e} B > 1$, *then* $B = B'$.

Proof. Let $0 \to B \to^{\gamma_0} Y_1$ be an exact sequence, where Y_1 is isomorphic to a direct sum of n-copies of $eB \otimes e'^0 B^0$. Then by Lemma 1.3 we have only to prove that $0 \to B \otimes_{B^e} (B')^e \to^{\Delta_0} Y \otimes_{B^e} (B')^e$ is also exact, because if it is proved, $B \approx B \otimes_{B^e} (B')^e$ and consequently $B = B'$. Since any element of $B \otimes_{B^e} (B')^e$ is expressed by $1 \otimes \beta$ for a suitable element β of $(B')^e$, we may take $1 \otimes \beta$ as an element of Ker Δ_0. Let D be a simple subideal of $\beta \cdot (B')^e$. Then $1 \otimes D \subseteq \operatorname{Ker} \Delta_0$. We shall show $D(e' \otimes e^0) = D$, where $e' \otimes e^0 \in (B')^e = B' \otimes_K (B')^0$. Any simple ideal of B' is monomorphic to $eB' \otimes_K (B'e')^0 = eB \otimes_K (B_{e'})^0$ and hence monomorphic to $e \cdot 1(N') \otimes_K e'^0 \cdot l(N'^0)$, where N' and $(N')^0$ are the radicals B' and $(B')^0$, and $1(N')$ and $1((N')^0)$ are the left annihilators of N' and $(N')^0$ in B' and $(B')^0$ respectively. Denoting $\overline{B'} = B'/N', \overline{(B^{0'})} = (B')^0/(N')^0$, we have $e \cdot 1(N') \otimes e'^0 \cdot 1((N')^0) \approx \bar{e}' \overline{B'} \otimes_K \overline{e^0 (B')^0}$. On the other hand the bottom Loewy constituent of $\bar{e}' \overline{B'} \otimes_K \overline{e^0 (B')^0}$ is isomorphic to the top Loewy constituent of $\bar{e}' \overline{B'} \otimes_K \overline{e^0 (B')^0}$, since $\overline{B'} \otimes_K \overline{(B')^0}$ is almost symmetric. Thus we obtain $D(e' \otimes e^0) = D$. Now let γ be an element of D and express $\gamma = \sum_\nu b_\nu \otimes c_\nu^0 \in B' \otimes_K (B')^0$, $b_\nu \in B'$ and $c_\nu^0 \in (B')^0$. Then $\gamma = \gamma(e' \otimes e^0) = \sum_\nu b_\nu e' \otimes c_\nu^0 e^0$. Here we notice $b_\nu e' \in Be'$ and $c_\nu^0 e^0 \in B^0 e^0$ because $B'e' = Be'$ and $(B')^0 e^0 = B^0 e^0$. Then it follows that

$$\Delta_0(1 \otimes \gamma) = \Delta_0 \left(\left(\sum_\nu b_\nu e' \otimes c_\nu^0 e^0 \right) \otimes (1 \otimes 1^0) \right)$$

$$= \delta_0 \left(\sum_\nu b_\nu e' \otimes c_\nu^0 e^0 \right) \otimes (1 \otimes 1^0) \neq 0,$$

where $1 \otimes 1^0$ is the unit of $(B')^e$. But this contradicts $1 \otimes \gamma \in \operatorname{Ker} \Delta_0$. This completes the proof.

COROLLARY 1.7. *Let B and B' be* QF-3 *algebras as in Proposition* 1.1. *If B is a proper subalgebra of B', then* domi. $\dim_{B^e} B = 1$.

REMARK. Consider the algebra A being fixed. Then so far as we estimate the upper bound of dominant dimensions of B's over QF-algebras B^e connected with A, we have only to consider such QF-3 algebras B' that B' consists of all A-endomorphisms of fully faithful A-modules.

Let A be an algebra, M_0 a fully faithful left A-module and M a left A-module. Denote by B the left A-endomorphism algebra of $M_0 \oplus M$ considered as a right operator domain and similarly denote by B_0 the left A-endomorphism algebra of M_0 as a right operator domain. Let N be a right B-module and

$$(7) \qquad 0 \longrightarrow N \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n$$

an exact sequence with X_k, $1 \leqq k \leqq n$, which is a projective, injective right B-module. Without loss of generality we may assume that X_k is isomorphic to a direct sum of n_k-copies of eB, where $M_0 \oplus M \approx eB$ and e is an idempotent of B. The projection $f: M_0 \oplus M \to M_0$ may be considered as an idempotent of B and it holds $B_0 = fBf$ and $M_0 = eBf$. Then multiplying (7) with f on the right hand we obtain

$$(8) \qquad 0 \longrightarrow Nf \longrightarrow X_1 f \longrightarrow X_2 f \longrightarrow \cdots \longrightarrow X_n f,$$

where $X_k f$ is a direct sum of n_k-copies of eBf.

Now we shall divide two cases.

Case 1. We shall take a right B-module B as N; then in place of (7) and (8) we obtain

$$(7') \qquad 0 \longrightarrow B \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} X_2 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} X_n,$$

and

$$(8') \qquad 0 \longrightarrow Bf \longrightarrow X_1 f \longrightarrow X_2 f \longrightarrow \cdots \longrightarrow X_n f.$$

eBf is right fBf-projective, injective and $Bf \approx fBf \oplus (1-f)Bf$. Hence we have

$$0 \longrightarrow B_0 \longrightarrow X_1' \longrightarrow X_2' \longrightarrow \cdots \longrightarrow X_n',$$

where X_1 is the injective hull of $\delta_0(B_0)$ and X_k' is the injective hull of $\delta_{k-1}(X_{k-1}')$. Here we notice that if $X_j' = 0$ for some j, then B_0 is projective, injective B_0-module for X_j is B_0-projective, and this implies B_0 is quasi-Frobenius.

Case 2. In this case we shall take B^e-module B as N; then we have to take in place of the ring B, B_0, projective, injective right B-modules X_i and the idempotent f, $B^e = B \otimes_K B^0$, $B_0^e = fBf \otimes_K (fBf)^0$, projective, injective right B^e-modules Y_i and an idempotent $(f \otimes_K f^0)$ of B^e. Then in place of (8) we obtain an exact sequence

$$(8'') \qquad 0 \longrightarrow B(f \otimes f^0) \longrightarrow Y_1(f \otimes f^0) \longrightarrow \cdots \longrightarrow Y_n(f \otimes f^0)$$

of B^e-homomorphisms, where $B(f \otimes f^0) = fBf = B_0$. Here we notice that if (8″) is split, B_0 is B_0^e-projective (injective) and B_0 is separable. Thus we have proved

THEOREM 1.8. *Let A be an algebra, M_0 a fully faithful left A-module and M a left A-module. Denote by B the left A-endomorphism algebra of $M_0 \oplus M$ considered as a right operator domain and similarly denote by B_0 the left A-endomorphism algebra of M_0 as a right operator domain. If B_0 is not quasi-Frobenius, then*

$$\text{domi. dim}_B B \leqq \text{domi. dim}_{B_0} B_0.$$

THEOREM 1.9. *Let B and B_0 be* QF-3 *algebras as in Theorem* 1.8. *If B_0 is not separable, then* domi. $\dim_{B^e} B \leqq$ domi. $\dim_{B_0^e} B_0$.

REMARK. The inequality of Theorem 1.8 really holds for the following example. Let A be a subalgebra of the full matrix K_4 over a commutative field K such that the elements

$$e_1 = c_{11}, \qquad e_2 = c_{22} + c_{33} + c_{44}, c_{21}, c_{31}$$

form a K-basis. Take $M_0 \approx Ae_1 \oplus Ae_2 \oplus (e_2A)^* \oplus (e_1A)^*$ and $M \approx Ae_1/Ac_{21}$ in Theorem 1.8. Then we have domi. $\dim_{B_0} B_0 = 2$ and domi. $\dim_B B = 1$.

A fully faithful left A-module M_0 is said to be minimal if each indecomposable summand of M_0 is either projective or injective and not isomorphic to others.

Let M_0 be a minimal fully faithful left A-module. Then the endomorphism algebra of M_0 is quasi-Frobenius if and only if A is a quasi-Frobenius algebra. Thus, to estimate the upper bound of dominant dimensions of modules B_B and B_{B^e} for a fixed algebra A we must divide the class of QF-3 algebras which are not quasi-Frobenius into two subclasses according as whether

(i) A is a quasi-Frobenius algebra, or

(ii) A is not a quasi-Frobenius algebra,

and by Theorems 1.8 and 1.9 we have only to consider QF-3 algebras B in each subclass such that

(i) B is the endomorphism algebra of $M_0 \oplus M$, where M_0 is a minimal faithful left A-module and M is an indecomposable, not projective, left A-module,

(ii) B is the endomorphism algebra of M_0, where M_0 is a minimal fully faithful left A-module.

2. **Dominant dimension and injective dimension.** Let M be a right B-module. Suppose the dominant dimension of M is infinite, that is to say, we have a minimal injective resolution of M: $0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$, where every X_k is a projective right B-module. If the injective dimension of M is finite, i.e., $X_{n-1} = 0$ for some n, then we obtain $X_{n-1} \approx \operatorname{Im} \delta_{n-1} \oplus \operatorname{Ker} \delta_{n-1}$ and obtain successively $X_{k-1} \approx \operatorname{Im} \delta_{k-1} \oplus \operatorname{Ker} \delta_{k-1}$ for X_k is projective, and consequently M is projective and injective. Thus we have

PROPOSITION 2.1. *Let M be a right B-module. Assume* domi. $\dim_B M = \infty$ *and* inj. $\dim_B M < \infty$. *Then M is projective and injective.*

COROLLARY 2.2. *If M is not a projective injective right B-module, then from condition* inj. $\dim_B M < \infty$ *it follows that* domi. $\dim_B M < \infty$.

Since self injective algebras are quasi-Frobenius, we obtain

COROLLARY 2.3. *Let B be a* QF-3 *algebra which is not quasi-Frobenius. If* inj. $\dim_B B < \infty$, *then* domi. $\dim_B B < \infty$.

REMARK. As an example of a QF-3 algebra which is not quasi-Frobenius and whose injective dimension $= \infty$ we can show a subalgebra B of the full matrix ring K_{10} over a commutative field K such that elements

$$e_1 = c_{11} + c_{22} + c_{33},\quad e_2 = c_{44} + c_{55} + c_{66},\quad e_3 = c_{77} + c_{88} + c_{99},$$
$$e_4 = c_{10,10}, c_{61}, c_{81} + c_{92}, c_{10,1}, c_{24} + c_{35}, c_{94}, c_{27}, c_{57} + c_{68}, c_{6,10}$$

form a K-basis.

COROLLARY 2.4. *If a* QF-3 *algebra B is not separable, then the condition* $0 < \text{inj. dim}_{B^e} B < \infty$ *implies* $\text{domi. dim}_{B^e} B < \infty$.

Let B be a QF-3 algebra and eB a minimal faithful right ideal, where e is an idempotent of B. There exists an idempotent e' of B such that $Be' \approx \text{Hom}_K(eB, K)$. Then $\text{Hom}_K(eBe, K)$ is left eBe-isomorphic to eBe'. Since eBe is a projective, faithful right eBe-module, eBe' is an injective, faithful left eBe-module and similarly eBe' is an injective, faithful right $e'Be'$-module. It is well known that by the contravariant functor $\text{Hom}(\quad, eBe')$ the duality holds between the category of finitely generated left eBe-modules and one of finitely generated right $e'Be'$-modules.

Now let β be a right B-homomorphism: $eB \to eB$. Then multiplying with the idempotent e' on the right-hand side we have right $e'Be'$-homomorphism $\alpha: eBe' \to eBe'$. Conversely for a right $e'Be'$-homomorphism $\gamma: eBe' \to eBe'$ we have a right B-homomorphism $\text{Hom}(Be', \gamma)$: $\text{Hom}_{e'Be'}(Be', eBe') \to \text{Hom}_{e'Be'}(Be', eBe')$ as follows: $(\text{Hom}(Be', \gamma)(\phi))(x) = \gamma(\phi(x))$ for $\phi \in \text{Hom}(Be', eBe')$, $x \in Be'$.

Further it holds $\text{Hom}_{e'Be'}(Be', eBe') \approx \text{Hom}_{e'Be'}(Be', \text{Hom}_K(e'Be', K)) \approx \text{Hom}_K(Be' \otimes_{e'Be'} e'Be', K) \approx \text{Hom}_K(Be', K) \approx eB$.

Now we shall prove

PROPOSITION 2.5. *Let α and β be just introduced homomorphisms. Then there exists an isomorphism σ: $\text{Hom}_{e'Be'}(Be', eBe') \to eB$ such that the following diagram is commutative*:

$$\begin{array}{ccc}
\text{Hom}_{e'Be'}(Be', eBe') & \xrightarrow{\text{Hom}(Be', \alpha)} & \text{Hom}_{e'Be'}(Be', eBe') \\
\sigma \downarrow & & \downarrow \sigma \\
eB & \xrightarrow{\quad\beta\quad} & eB.
\end{array}$$

Proof. To begin with we identify eB with $\text{Hom}_{e'Be'}(Be', eBe')$. Then by the assumption the following diagram is commutative:

$$\begin{array}{ccc}
\theta \in \text{Hom}_{e'Be'}(Be', eBe') & \longrightarrow & \beta(\theta) \in \text{Hom}_{e'Be'}(Be', eBe') \\
\rho \downarrow & & \downarrow \rho \\
\theta \circ e' \in \text{Hom}_{e'Be'}(e'Be', eBe') & \longrightarrow & \beta(\theta \circ e') \in \text{Hom}_{e'Be'}(e'Be', eBe'),
\end{array}$$

where

$$(\rho\theta)(e'xe') = (\theta \circ e')(xe')$$
$$= \theta(e'xe') \text{ for } xe' \in Be' \text{ and } \theta \in \mathrm{Hom}(Be', eBe').$$

In this case $\mathrm{Hom}(Be', eBe')$ is defined to be a left eBe-module by $(b * \theta)(xe') = b(\theta(xe'))$, for $b \in eBe$ and $xe' \in Be'$ and $\mathrm{Hom}(Be', eBe')$ is fully faithful as a left eBe-module. Hence the right B-endomorphism algebra of $\mathrm{Hom}(Be', eBe')$ is eBe and for a right B-homomorphism β there exists an element b_β of eBe such that $\beta(\theta) = b_\beta * \theta$. Thus we have

$$(\beta(\theta) \circ e')(xe') = \beta(\theta \circ e')(xe')$$
$$= \beta(\theta(e'xe'))$$
$$= (b_\beta * \theta)(e'xe')$$
$$= b_\beta(\theta(e'xe')).$$

On the other hand, by a correspondence $\tau : \theta \circ e' \leftrightarrow (\theta \circ e')(e')$, $\mathrm{Hom}_{e'Be'}(e'Be', eBe') \approx eBe'$. Hence β induces $e'Be'$-homomorphism $\alpha' : eBe' \to eBe'$ such that $\alpha'((\theta \circ e)(e')) = b_\beta(\theta \circ e')(e')$.

Then it is sufficient to show that α' induces β.

Let $\eta \in \mathrm{Hom}_{e'Be'}(Be', eBe')$; then $\eta(xe') \in eBe'$. There exists $\theta \in \mathrm{Hom}_{e'Be'}(Be', eBe')$ such that $\eta(xe') = (\theta \circ e')(e')$, because we have only to put $\theta \circ e' = \tau^{-1}(\eta(xe'))$. Then we have

$$(\mathrm{Hom}(Be', \alpha')(\eta))(xe') = \alpha'\eta(xe')$$
$$= \alpha'((\theta \circ e')(e'))$$
$$= b_\beta((\theta \circ e')(e'))$$
$$= b_\beta(\eta(xe'))$$
$$= (b_\beta * \eta)(xe')$$
$$= \beta(\eta)(xe'), \text{ for all } xe' \in Be'.$$

This completes the proof.

COROLLARY 2.6. *Let B be a* QF-3 *algebra. Let M_i, $i = 1, 2$, be a submodule of X_i respectively and X_i free right B-modules. Let ϕ be a right B-homomorphism: $M_1 \to M_2$ and ψ a right $e'Be'$-homomorphism: $M_1e' \to M_2e'$ defined by $\psi(xe') = \phi(x) \cdot e'$ for $x \in M_1$. Then there exists a natural isomorphism σ such that the following diagram is commutative*:

$$\begin{array}{ccc} M_1 & \xrightarrow{\quad\phi\quad} & M_2 \\ \sigma\downarrow & & \sigma\downarrow \\ \mathrm{Hom}_{e'Be'}(e', M_1e') & \xrightarrow{\mathrm{Hom}(Be',\psi)} & \mathrm{Hom}_{e'Be'}(Be', M_2e'). \end{array}$$

Proof. Since B is a QF-3 algebra, for a suitable integer n, X_i can be embedded in a direct sum X of n-copies of eB and we have a right B-homomorphism $\Phi: X \to X$ such that the following diagram is commutative:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\phi} & M_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\Phi} & X_2 . \end{array}$$

Then from Proposition 2.5 the conclusion follows.

Let

$$(9) \qquad 0 \longrightarrow M \xrightarrow{\beta_0} X_1 \xrightarrow{\beta_1} X_2 \xrightarrow{\beta_2} \cdots \longrightarrow X_n$$

be an exact sequence of right B-homomorphisms β_p, where X_p, $1 \leqq p \leqq n$, are projective, injective right B-modules. By multiplying X_p with e' on the right-hand side we have an exact sequence of right $e'Be'$-homomorphisms:

$$(10) \qquad 0 \longrightarrow Me' \xrightarrow{\alpha_0} X_1e' \xrightarrow{\alpha_1} X_2e' \xrightarrow{\alpha_2} \cdots \longrightarrow X_ne',$$

with X_pe' being injective right $e'Be'$-modules. Then from Proposition 2.5 and Corollary 2.6 there exist natural isomorphisms σ_i, $i = 0, 1, 2, \cdots, n$:

$$(11) \qquad \begin{array}{ccccccc} 0 \longrightarrow M & \xrightarrow{\beta_0} & X_1 & \xrightarrow{\beta_1} & \cdots & \longrightarrow & X_n \\ \sigma_0 \downarrow & & \sigma_1 \downarrow & & & & \sigma_n \downarrow \\ 0 \to \mathrm{Hom}(Be', Me') & \xrightarrow{\mathrm{Hom}(Be', \alpha_0)} & \mathrm{Hom}(Be', X_1e') & \xrightarrow{\mathrm{Hom}(Be', \alpha_1)} & \cdots & \to & \mathrm{Hom}(Be', X_ne'). \end{array}$$

If (9) is minimal in the sense that $\mathrm{Im}\, \beta_i \supseteq$ the socle of X_{i+1}, then (10) is minimal in the sense of that $\mathrm{Im}\, \alpha_i \supseteq$ the socle of $X_{i+1}e'$. We have now proved

PROPOSITION 2.7. *Let B be a* QF-3 *algebra with Be' and eB as unique minimal faithful left and right B-modules respectively. Let M be a right B-module. Then from the assumption* $0 < \text{inj. r. dim}_{e'Be'} Me' < \infty$, *it follows that*

$$\text{domi. dim}_B M \leqq \text{inj. r. dim}_{e'Be'} Me' + 1.$$

THEOREM 2.8. *Let B be a* QF-3 *algebra as in Proposition* 2.7. *Then*

$$\text{domi. dim}_B B \leqq \text{inj. r. dim}_{e'Be'} Be' + 1 = \text{proj. l. dim}_{eBe} eB + 1,$$

provided $0 < \text{inj. r. dim}_{e'Be'} Be' < \infty$.

COROLLARY 2.9. *Let A be an algebra which is not quasi-Frobenius. If the projective dimension of a fully faithful injective left A-module is finite, then the dominant dimension of* QF-3 *algebra B which is connected with A is also finite.*

THEOREM 2.10. *Let B be a* QF-3 *algebra as in Proposition* 2.7. *From the assumption* $0 < \dim eBe$ $(= \text{proj. dim}_{eBe^e} eBe) < \infty$ *it follows that* domi. $\dim_{B^e} B \leqq \dim eBe + 1$.

Proof. Let us consider again B to be a right B^e-module. Then in place of e', $e'Be'$ we can take $e' \otimes_K e^0$ and $e'Be' \otimes_K e^0B^0e^0$. Hence from the assumption $0 <$ inj. r. $\dim_{e'Be' \otimes e^0B^0e^0} eBe' < \infty$ it follows that domi. $\dim_{B^e} B$ $<$ inj. r. $\dim_{e'Be' \otimes e^0B^0e^0} eBe' + 1$. On the other hand, eBe' is left eBe-isomorphic to $\text{Hom}_K(eBe, K)$ and right $e'Be'$-isomorphic to $\text{Hom}_K(e'Be', K)$. Hence $\text{Hom}_K(eBe', K)$ is right eBe-isomorphic to eBe and is left $e'Be'$-isomorphic to $e'Be'$. Thus inj. r. $\dim_{e'Be' \otimes e^0B^0e^0} eBe' =$ proj. l. $\dim_{eBe \otimes e^0B^0e^0} eBe = \dim eBe$. And we have proved domi. $\dim_{B^e} B \leqq \dim eBe + 1$.

3. **In case A is generalized uni-serial.** Throughout this section we assume A is a generalized uni-serial algebra and B is an endomorphism algebra of a fully faithful A-module. We shall prove that if B is not quasi-Frobenius, the dominant dimension of B, considered as a right B-module, is finite. Since domi. $\dim_{B^e} B \leqq$ domi. $\dim_B B$, this implies Nakayama's conjecture holds for the same situation.

Let B be a QF-3 algebra and eB and Be' unique minimal faithful right ideal and unique minimal faithful left ideal, respectively.

Let

$$\cdots \longrightarrow X_{i-1} \xrightarrow{\delta_{i-1}} X_i \xrightarrow{\delta_i} X_{i+1} \xrightarrow{\delta_{i+1}} \cdots \tag{12}$$

be an exact sequence of right $e'Be'$-homomorphisms with X_i being isomorphic to a direct sum of n_i-copies of eBe'. As was shown in §2 we have a sequence of right B-homomorphisms

$$\cdots \to \text{Hom}(Be', X_{i-1}) \xrightarrow{\Delta_{i-1} = \text{Hom}(Be', \delta_{i-1})} \text{Hom}(Be', X_i) \xrightarrow{\Delta_i = \text{Hom}(Be', \delta_i)} \text{Hom}(Be', X_{i+1}) \to \cdots. \tag{13}$$

This sequence, however, is not always exact. For the sake of (13) being exact what condition does (12) satisfy?

Let ϕ be an element of $\text{Ker}\,\Delta_i$, i.e., $\delta_i\phi$ is the zero mapping. On the other hand, let ψ be any element of $\text{Im}\,\Delta_{i-1}$. Then there exists a homomorphism $\Psi: Be' \to X_{i-1}$ such that $\psi = \delta_{i-1}\Psi$. Since it holds that $\text{Im}\,\delta_{i-1} = \text{Ker}\,\delta_i$ and $\text{Im}\,\Delta_{i-1} \subseteqq \text{Ker}\,\Delta_i$, we have

PROPOSITION 3.1. *In order that* $\text{Ker}\,\Delta_i = \text{Im}\,\Delta_{i-1}$, *it is necessary that* δ_{i-1} *satisfies the following condition: For a homomorphism* $\phi: Be' \to \text{Im}\,\delta_{i-1}$, *there always exists a homomorphism* $\psi: Be' \to X_{i-1}$ *such that the following diagram is commutative*:

(14)
$$\begin{array}{ccc} & & X_{i-1} \\ & \overset{\psi}{\nearrow} & \downarrow \delta_{i-1} \\ Be' & \xrightarrow{\phi} & \operatorname{Im}\delta_{i-1}. \end{array}$$

Remark. Since $Be' \approx (1-e')Be' \oplus e'Be'$ and $e'Be'$ is $e'Be'$-projective, (14) can be replaced by the following diagram:

$$\begin{array}{ccc} & & X_{i-1} \\ & \overset{\psi}{\nearrow} & \Big| \delta_{i-1} \\ (1-e')Be' & \xrightarrow{\phi} & \operatorname{Im}\delta_{i-1}. \end{array}$$

Hereafter we assume eBe $(=A)$ is a generalized uni-serial algebra. And we assume the following sequence of right B-homomorphisms:

$$0 \longrightarrow B \xrightarrow{\Delta_0} Z_1 \xrightarrow{\Delta_1} \cdots \longrightarrow Z_n \xrightarrow{\Delta_n} \cdots$$

is infinite and exact with Z_i being isomorphic to n_i-copies of eB. As is shown in §2, we have an exact sequence of $e'Be'$-homomorphisms

$$0 \longrightarrow Be' \xrightarrow{\delta_0} Z_1e' \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_n} Z_ne' \longrightarrow \cdots.$$

Then from Proposition 2.5 there exists a natural equivalence σ:

(15)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\Delta_0} & Z_1 & \xrightarrow{\Delta_1} & \cdots \longrightarrow & Z_n & \xrightarrow{\Delta_n} \cdots \\ & & \sigma\Big\downarrow\ \operatorname{Hom}(Be',\delta_0) & & \sigma\Big\downarrow\ \operatorname{Hom}(Be',\delta_1) & & & \sigma\Big\downarrow\ \operatorname{Hom}(Be,\delta_n) & \\ 0 & \longrightarrow & \operatorname{Hom}(Be',Be') & \longrightarrow & \operatorname{Hom}(Be',Z_1e') & \longrightarrow & \cdots \longrightarrow & \operatorname{Hom}(Be',Z_ne') & \longrightarrow \cdots. \end{array}$$

Let us impose further an assumption that B is not quasi-Frobenius. Then there exists at least an indecomposable summand S of Be' which is not injective as a right $e'Be'$-module and for S we have an exact sequence of homomorphisms

(16) $$0 \longrightarrow S \xrightarrow{\tau_0} Y_1e' \xrightarrow{\tau_1} \cdots \longrightarrow Y_ne' \xrightarrow{\tau_n} \cdots,$$

where τ_i are minimal and Y_ie' is nonzero, indecomposable summand of Z_ie', for S is not injective and $e'Be'$ is generalized uni-serial. By (15),

(17) $$0 \longrightarrow \operatorname{Hom}(Be',S) \longrightarrow \operatorname{Hom}(Be',Y_1e') \longrightarrow \cdots$$

must be exact. But this is a contradiction.

To show the reason why this is a contradiction, we shall divide our discussion.

(i) $e'Be'$ is quasi-Frobenius.

In this case $e' = e$, $(1-e')Be' = (1-e)Be$ and from Theorem 1.8 we may assume that $(1-e)Be$ is indecomposable and $S = (1-e)Be$. Since the sequence (16) is infinite, there exist two integers i and $i+k$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_ie & \longrightarrow & Y_{+k}e & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Ker}\,\tau_i & \longrightarrow & \mathrm{Ker}\,\tau_{i+k} & \longrightarrow & 0 \\ & & \uparrow & & & & \\ & & 0 & & 0 & & \end{array}$$

is commutative, where all row and column sequences of homomorphisms are exact.

Since all $Y_1e, Y_2e, \cdots, Y_{i+k}e$ are projective, $S \approx \mathrm{Im}\,\tau_{i+p}$ for some p, $1 \leqq p \leqq k$, because (16) is minimal. Then for the following diagram

$$\begin{array}{ccccc} & & Y_{i+p}e & & \\ & & \big\downarrow \tau_{i+p} & & \\ 0 \longrightarrow (1-e)Be & \xrightarrow{\ \phi\ } & \mathrm{Im}\,\tau_{i+p} \approx S \approx (1-e)Be & \longrightarrow & 0, \end{array}$$

there is no homomorphism $\psi: (1-e)Be \to Y_{i+p}e$ such that $\tau_{i+p}\psi = \phi$, because otherwise $(\phi^{-1}\delta_{i+p})\psi$ is an isomorphism: $(1-e)Be \to (1-e)Be$ and $(1-e)Be$ is isomorphic to a direct summand of $Y_{i+p}e$. Thus S is injective. But this contradicts the selection of S. It follows from Proposition 3.1 that (17) is not exact. We arrive at the above stated contradiction.

(ii) $e'Be'$ is not quasi-Frobenius.

In this case we may assume by Theorem 1.8 that every indecomposable injective right $e'Be'$-module which is not projective is isomorphic to a direct summand of $(1-e')Be'$. We can select S among them.

(a) If we have in (16) the following commutative diagrams for all j, $1 \leqq j \leqq i$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_je' & \longrightarrow & Y_{j+k}e' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Ker}\,\tau_j & \longrightarrow & \mathrm{Ker}\,\tau_{j+k} & \longrightarrow & 0 \end{array}$$

in which rows are exact and column mappings are injections, then as in (i) we have a diagram

$$\begin{array}{ccccc} & & Y_{i+p}e' & & \\ & & \big\downarrow \tau_{i+p} & & \\ 0 \longrightarrow S & \xrightarrow{\ \phi\ } & \mathrm{Im}\,\tau_{i+p} & \longrightarrow & 0, \end{array}$$

in which there exists no homomorphism $\psi : S \to Y_{i+p}e'$ such that $\tau_{i+p}\psi = \phi$, for otherwise $S \approx Y_{i+p}e'$ and $\operatorname{Im}\tau_{i+p-1} = 0$, which contradicts (11). So we arrive again at the above contradiction.

(b) Now, the case remains where we have the next diagram:

$$\begin{array}{ccc} & Y_{i+p}e' & \\ & \Big\downarrow \tau_{i+p} & \\ Y_je' \xrightarrow{\ \tau\ } & \operatorname{Im}\tau_{i+p} \approx \operatorname{Im}\tau_j & \longrightarrow 0 \\ & \Big\downarrow & \\ & 0 & \end{array}$$

where Y_je', $1 \leqq j < i$, is injective but not projective and not isomorphic to Y_{i+p}. Hence Y_je' is considered as a direct summand of $(1-e')Be'$. Since $e'Be'$ is generalized uni-serial, we may assume there exists an epimorphism $\theta: Y_{i+p}e' \to Y_je'$ such that $\tau_j\theta = \tau_{i+p}$. Then we know that there exists no homomorphism $\psi : Y_je' \to Y_{i+p}$ such that $\tau_{i+p}\psi = \tau_j$, for otherwise $\tau_j\theta\psi(x) = \tau_j(x)$ holds for all $x \in Y_je'$; then $Y_je' = \theta\psi(Y_je') + \operatorname{Ker}\tau_j$. But Y_je' is uni-serial, and hence $\theta\psi$ must be an isomorphism. It follows that Y_je' is isomorphic to a direct summand of $Y_{i+p}e'$, and consequently $Y_je' \approx Y_{i+p}e'$. But this is a contradiction. By Proposition 3.1, (17) is not exact. Thus in this case too we arrive at the first pointed contradiction.

THEOREM 3.2. *Let A be a generalized uni-serial algebra and B a* QF-3 *algebra connected with A. If B is not quasi-Frobenius, then the dominant dimension of B is finite.*

REMARK. The class of QF-3 algebras, each of them obtained as the endomorphism algebra of a fully faithful module over a generalized uni-serial algebra, contains properly the class of generalized uni-serial algebras. In fact, the example at the remark of Theorem 1.8. is an endomorphism algebra of a fully faithful module over a generalized uni-serial algebra, but not generalized uni-serial.

4. **Isomorphisms between cohomology groups and homology groups.** To begin with we shall intend to generalize the notion "Nakayama's automorphism of Frobenius algebra." Let B be a QF-3 algebra with eB with the unique minimal faithful right B-ideal. Then the unique minimal faithful left B-ideal Be' is B-isomorphic to $\operatorname{Hom}_K(eB,K)$ and by an identification of Be' with $\operatorname{Hom}_K(eB,K)$, eB and Be' form an orthogonal pair (inner product) with respect to K: for any element $x \in eB$ and any element $y \in Be'$ there corresponds an element (x,y) of K, and $(xb,y) = (x,by)$ holds for any element $b \in B$.

Then for an element ρ of eBe, $(\rho x, y) = (\rho, xy)$ holds, where $xy \in eBe'$. In this pairing it is immediate that eBe and eBe' also form an orthogonal pair, and since K is contained in the center of B this makes it possible for us to identify every element ρ of eBe with an element of $\mathrm{Hom}_K(eBe',K)$, that is to say, for any elements x, y there exists an element ρ' of $e'Be'$ such that

$$(\rho, xy) = (xy, \rho').$$

Since in the right pairing eBe' and $e'Be'$ form an orthogonal pair, the correspondence: $\rho \to \rho'$ is unique and gives ring-isomorphism σ: $eBe \to e'Be'$. When B is a Frobenius algebra, σ is Nakayama's automorphism. It is known that σ is determined uniquely up to inner automorphisms of $e'Be'$.

Let $l_1, l_2, \cdots, l_k$ be a left K-basis of eB and $r_1, r_2, \cdots, r_k$ its dual right K-basis of Be'; that is to say, in the above pairing $(l_i, r_k)_k = \delta_{ik}$ holds. Then, if $\rho l_i = \sum_{i=1}^k \lambda_{ij} 1_j$ for $\rho \in eBe$, we have $r_j \rho' = \sum_{i=1}^k r_i \lambda_{ij}$.

Next we shall define the notion "Spur" in the sense of Kasch. Let M be a left B-module and f an element of $\mathrm{Hom}_K(M,K)$. The Spur f is defined to be the following mapping: $M \to Be'$ such that

$$\mathrm{Spur}\, f : M \ni x \to \sum_{i=1}^{k} r_i f(l_i x) \in Be'.$$

It is easy to show that $\mathrm{Spur}\, f \in \mathrm{Hom}_B(M, Be')$, and

$$\mathrm{Spur}\,(f\rho)(x) = (\mathrm{Spur}\, f)(x)\rho' \text{ for } \rho \in eBe \text{ and } \rho' \in e'Be'.$$

Now, similarly as Kasch we can prove several propositions for preparation.

PROPOSITION 4.1. *Let $e\rho e$ be an element of* $\mathrm{Hom}_K(Be,K)$ *defined by* $e\rho e(x) = (e\rho e, x)$ *for* $x \in Be'$. *Then the correspondence*:

$$e\rho e \to \mathrm{Spur}\, e\rho e \in \mathrm{Hom}_B(Be', Be')$$

gives the ring-isomorphism σ.

We shall omit the proof (cf. Kasch [4, Hilfssatz 1]).

Let X be a direct sum of m-copies of Be' : $X = \bigoplus_{j=1}^m Be'x_j$ and $\sum_{j=1}^m \gamma_j x_j$, $\gamma_j \in Be'$ be an element of X. Then we have

PROPOSITION 4.2. *Let X^* be a direct sum of m-copies of eBe: $X^* = \bigoplus_{j=1}^m \varepsilon_j eBe$, $\varepsilon_j eBe \approx eBe$. And by $\sum_{j=1}^m \varepsilon_j \rho_j$, $\rho_j \in eBe$, we shall denote an element of* $\mathrm{Hom}_K(X,K)$ *defined by*

$$\left(\sum_k \varepsilon_k \rho_k\right)\left(\sum_j \gamma_j x_j\right) = \sum_k (\rho_k, \gamma_k).$$

Then the mapping $X^* \to \mathrm{Hom}_B(X, Be')$ *defined by*

$$X^* \ni f \to \mathrm{Spur}\, f \in \mathrm{Hom}_B(X, Be')$$

gives a semi-linear isomorphism connected with σ.

We omit the proof (cf. Kasch [4, Hilfssatz 2]).

From left B-module N a left $e'Be'$-module $e'N$ is obtained by restricting its operator domain B to $e'Be'$. Now by $e'N^{\theta}$ we shall denote an additive group having the following one-one correspondence with $e'N : e'N^{\theta} \ni y \to {}^{\theta}y \in e'N$. If we shall define the multiplication $\circ$ with elements of eBe as follows : $\rho \circ y^{\theta} = \rho^{\sigma} \cdot y$, where σ is Nakayama's ring-isomorphism, then N^{θ} is considered to be a left eBe-module.

For given left B-modules M and N we shall define a homomorphism ϕ: $\mathrm{Hom}_K(eM, K) \otimes_{eBe} e'N^{\theta} \ni f \otimes y^{\theta} \to (M \ni x \to (\mathrm{Spur}\ f)(x) y) \in \mathrm{Hom}_B(M, N)$. Then it is proved similarly as in [4] that ϕ is functorial in M and N.

THEOREM 4.3. *Let X be a direct sum of m-copies of Be'. Then* $\mathrm{Hom}_K(eX, K) \approx X^*$ *and ϕ, defined by*

$$\phi : X^* \otimes_{eBe} e'N^{\theta} \ni f \otimes y^{\theta} \to (X \ni x \to (\mathrm{Spur}\ \ f)(x)\, y) \in \mathrm{Hom}_B(X, N),$$

is an isomorphism.

Proof. First we shall show ϕ is a monomorphism. Let $\phi(f \otimes y^{\theta}) = 0$; this means $(\mathrm{Spur}\ f)(x) y = 0$ for all x. Denote f by $\sum_{j=1}^{m} \varepsilon_j \rho_j$, $\rho_j \in eBe$ and x by $\sum_j b_j e' x_j$, $b_j e' \in Be'$. Then $0 = (\mathrm{Spur} f)\ (e'x_j) y = (\mathrm{Spur} \sum \varepsilon_j \rho_j)\ (e'x_j) \cdot y = (\mathrm{Spur}\ \varepsilon_j \rho_j)\ (e'x_j) y = (e' \rho_j^{\sigma}) y$. Hence $f \otimes y^{\theta} = \sum \varepsilon_j \rho_j \otimes y^{\theta} = \sum \varepsilon_j \otimes (\rho_j^{\sigma} y)^{\theta} = 0$. This proves that ϕ is a monomorphism.

Next we shall show that ϕ is an epimorphism. Let f be any element of $\mathrm{Hom}_B(X, N)$. Then it suffices to prove

$$\phi\left(\sum_j \varepsilon_j e \otimes f(e'x_j)\right) = f,$$

i.e.,

$$f(x) = \sum_j ((\mathrm{Spur}\ \varepsilon_j e)(x)) f(e'x_j) \text{ for all } x \in X.$$

On the other hand, for $x = \sum_j b_j e' x_j$ we have $\sum_j ((\mathrm{Spur}\ \varepsilon_j e)\ (x)) f(e'x_j) = \sum_j\ ((\mathrm{Spur}\ \ \varepsilon_j e)\ \ (\sum_i b_i e' x_i))\ \ f(e'x_j) = \sum_j (b_j e')\ \ f(e'x_j) = \sum_j f(b_j e' x_j) = f(\sum_j b_j e' x_j) = f(x)$. Thus the proof is completed.

Now we arrive at a place to prove

THEOREM 4.4. *Let M and N be left B-modules. If* $\mathrm{domi.\,dim}_B M = n$ *and* $\mathrm{domi.\,dim}_B \mathrm{Hom}_K(M, K) = m$, *then*

$$\mathrm{Tor}_k^{eBe}(\mathrm{Hom}_K(eM, K), e'N^{\theta}) \approx \mathrm{Ext}_B^{-(k+1)}\ (M, N).$$

Proof. Let

$$(18) \qquad \begin{array}{c} \to X_1 \to \cdots \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots \to X_{-(k+1)} \to \\ \searrow \ \nearrow \\ M \\ \nearrow \ \searrow \\ 0 \qquad 0 \end{array}$$

be a commutative diagram, where the row is exact with projective, injective left B-modules for $l \leqq m, k+1 \leqq n$, and σ and τ are epimorphism and monomorphism respectively. Without loss of generality X_i, $-n < i \leqq m$, may be assumed to be isomorphic to a direct sum of n_i-copies of Be'. Multiplying X_i with e on the left-hand side we obtain an exact sequence

$$\begin{array}{ccccc} \to eX_l \to \cdots \to eX_1 \to & eX_0 & \to & eX_{-1} & \to eX_{-2} \to \cdots \to eX_{-(k+1)} \to \\ & \searrow & & \nearrow & \\ & & eM & & \\ & \nearrow & & \searrow & \\ & 0 & & 0 & \end{array} \tag{19}$$

of eBe-homomorphisms with eX_i injective eBe-modules. Taking the dual of M and X_i, $i = -n, \cdots, m$, we obtain an exact sequence

$$\begin{array}{ccccc} \to (eX)^*_{-(k+1)} \to \cdots \to (eX_{-2})^* \to & (eX_{-1})^* & \to & (eX_0)^* & \to (eX_1)^* \to \cdots \to (eX_l)^* \to \\ & \searrow & & \nearrow & \\ & & (eM)^* & & \\ & \nearrow & & \searrow & \\ & 0 & & 0 & \end{array} \tag{20}$$

of right eBe-homomorphisms with $(eX_i)^*$ projective right eBe-modules. Then from Theorem 4.3 it follows that

$$X^*_{i+1} \underset{eBe}{\otimes} e'N^\theta \approx \mathrm{Hom}_B(X_{-(i+1)}, N).$$

However, since we can take (18) and (20) as parts of complete projective resolutions of ${}_BM$ and $(eM)^*_{eBe}$ respectively, we obtain

$$\mathrm{Tor}_i^{eBe}((eM)^*, e'N^\theta) \approx \mathrm{Ext}_B^{-(i+1)}(M, N).$$

This completes the proof.

Now, if we take ${}_{B^e}B$ as M, then for a left B^e-module N we obtain

$$H_k^{eBe \otimes_K (e'Be')^0}((eBe')^*, e'Ne^\theta) \approx H_{B^e}^{-(k+1)}(B, N).$$

Let η be a ring-isomorphism: $eBe \otimes_K (e'Be')^0 \to eBe \otimes_K (eBe)^0$ defined by $\eta(\rho \otimes \rho'^0) = \rho \otimes (\rho'^{\sigma^{-1}})^0$, where σ is Nakayama's ring-isomorphism. Then using η, $(eBe)^*$ and $e'N^\theta$ can be considered as left eBe^e-module and it is easy to show that

$$(eBe')^* \overset{eBe^e}{\approx} eBe$$

and $(\rho_1 \otimes \rho_2^0)x^\theta = \rho_1^\sigma x \rho_2^\sigma$ for $x \in N$, $x^\theta \in e'Ne^\theta$, $\rho_1, \rho_2 \in eBe$.

THEOREM 4.5. *Let B be a* QF-3 *algebra with eB and Be' as unique minimal faithful right and left ideals respectively and N a two sided B-module. If* domi. $\dim_{B^e} B = n$, *then*

$$H_k^{eBe}(eBe, e'Ne^{\theta}) \approx H_B^{-(k+1)}(B, N), \qquad 0 < k < n,$$

where the above groups denote the homology group of eBe with coefficients in N^{θ} and the cohomology group of B with coefficients in N respectively.

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