ON SEQUENTIAL CONVERGENCE

BY

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1. **Introduction.** In first-countable topological spaces, i.e., those with a countable neighborhood-base at each point, one can restrict oneself to sequences in studying convergence and continuity. In practice, e.g., for groups, such spaces are all metrizable. However, for more general spaces it seems to be assumed that sequences are not enough and that more general nets or filters must be used. Many of the linear spaces considered in Schwartz's theory of distributions [12;13] are not first-countable, so that such spaces are of real importance in analysis at present. Thus the abstract theory of sequential convergence begun by Fréchet [7] has been more or less neglected, perhaps since it seemed unnecessary in metric spaces and insufficient elsewhere.

It is easy to prove, however, that for "bornological" locally convex linear spaces, a convex "sequentially open" set is open, so that a sequentially continuous linear mapping is continuous (Theorems 6.1 and 6.3 below). It is known that this is not true for *bi*linear mappings (see §9) and as Grothendieck has pointed out for the duals of certain *F*-spaces [8, p. 101], the closure of a set need not be obtained by taking all limits of convergent sequences of members of the set (once).

Still, it appears that in some senses sequences *are* adequate for all spaces considered up to now in analysis, including the theory of distributions. Also, the main theorems of integration theory (dominated convergence, monotone convergence, etc.) are true only for sequences. The sequential language is useful as an alternative in metric spaces, and finally there is a fact that the convergent sequence and its limit form a compact set, while this is not true for nets.

Thus there seems to be ample reason for direct study of sequential convergence, as in this paper.

We begin in §2 with general definitions and a discussion of correspondences between topologies and specifications of convergent sequences. In §3, analogous constructions for topologies and sequences, such as product spaces, are defined and compared. §4 discusses sequential convergence of abstract sets; its results are used in §6 on sequential convergence in linear spaces. §5 deals with a generalization of metric convergence called "quasimetric" convergence which is

considered on linear spaces in §§7-8; §9 applies the results to the theory of distributions.

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For the set-theoretic results in §4, some conversations with Dana Scott were also very helpful.

Notation. Some special notations used in §§7-9 are explained at the beginning of §7. Otherwise, I believe all notations are well known, except perhaps for $\sim$: $A \sim B$ is the set of elements of A not in B, for any two sets A and B, and $\{x: \cdots\}$: the set of all x such that $\cdots$.

2. **Basic definitions.**   There is a classical axiomatization of convergence for sequences (Fréchet's L-spaces and L^*-spaces; see [7; 15]) which has recently received new evidences of its success (see [11] and Theorem 2.2 below). It will be adopted here without changes.

DEFINITION.   If S is any set, a *sequential convergence C* on S is a relation between sequences $\{s_n\}_{n=1}^{\infty}$ of members of S and members s of S, denoted $s_n \to_C s$, such that

(1)   if $s_n = s$ for all n, $s_n \to_C s$ and

(2)   if $s_n \to_C s$ and $\{r_m\}$ is any subsequence $\{s_{n_m}\}$ of $\{s_n\}$, then $r_m \to_C s$.

DEFINITION.   If C is a sequential convergence such that

(3)   if $s_n \to_C s$ and $s_n \to_C t$, then $s = t$,

$$C \text{ is an } L\text{-convergence.}$$

DEFINITION.   If C is an L-convergence such that

(4)   if $s_n \nrightarrow_C s$ (i.e., it is false that $s_n \to_C s$) then there is a subsequence $\{r_m\}$ of $\{s_n\}$ such that for any subsequence $\{t_q\}$ of $\{r_m\}$, $t_q \nrightarrow_C s$,

$$C \text{ is an } L^*\text{-convergence.}$$

If C is an $L(*)$-convergence on a set S, the pair (S, C) will be called an $L(*)$-*space*. A statement $s_n \to_C s$ may be read "s_n converges to s (for C)."

If T is a topology on a set S, then a sequence $\{s_n\}$ is said to converge to an element s, or $s_n \to_{C(T)} s$, if whenever $s \in U \in T$, $s_n \in U$ for n sufficiently large. It is clear that $\to_{C(T)}$ satisfies (1), (2) and (4), so that $C(T)$ is a sequential convergence. If (S, T) is a Hausdorff space (distinct points have disjoint neighborhoods), then (3) is satisfied and $C(T)$ is an L^*-convergence.

Conversely, given *any* relation $B: x_n \to_B x$ between sequences and points of a set S, we can call a set U open for B if whenever $x \in U$ and $x_n \to_B x$, $x_n \in U$ for n sufficiently large. It is clear that these open sets form a topology $T(B)$, as was remarked by Garrett Birkhoff [2], with an acknowledgment to R. Baer.

If (S, B) is an L-space, $T(B)$ will be T_1 (will contain the complement of each one-point set), but even if (S, B) is an L^*-space $T(B)$ need not be Hausdorff, as is shown by the following example. Let S be the set of pairs (m, n) of nonnegative integers together with two distinct points a and b. Let $(m_k, n_k) \rightarrow_B a$ if $n_k \rightarrow \infty$ and $m_k \neq 0$ for k large enough, $(0, n_k) \rightarrow_B b$ if $n_k \rightarrow \infty$, and $(m_k, n_k) \rightarrow_B (0, n)$ if $m_k \rightarrow \infty$ and $n_k = n$ for k large enough. If $p \in S$, let $p_k \rightarrow_B p$ if $p_k = p$ for k large enough. Finally if $p_k \rightarrow_B p$ and $q_k \rightarrow_B p$ as already defined and $r_k = p_k$ or $r_k = q_k$ for k large, let $r_k \rightarrow_B p$. It is then easy to verify that $\rightarrow_B$ is an L^*-convergence and that a and b do not have disjoint $T(B)$ neighborhoods.

The above example seems to be available since the definition of L^*-space contains no condition on "iterated limits" such as, for example, the condition

(5)   if $p_n \rightarrow p$ and for each $n, p_{nm} \rightarrow p_n$, then for some function $m(\ )$, $p_{nm(n)} \rightarrow p$.

Such a condition is assumed in proving that each topology is uniquely determined by its "convergence class" of nets and conversely (Kelley [10, Chapter II, Theorem 9]).

However, (5) is not satisfied in certain very interesting L^*-spaces, and fortunately $C(T(C)) = C$ is true without it (Theorem 2.1 below). Hence we shall not assume it.

The well-known example in which S is the set of all Borel functions on an interval and C is pointwise convergence shows that if the pseudo-closure $pc(A)$ of a set $A \subset S$ is defined as the set of $x \in S$ such that for some $x_n \in A$ for all n, $x_n \rightarrow_C x$, we may have $pc(pc(A)) \neq pc(A)$ so that $pc(A)$ need not be the $T(C)$ closure of A even if C is an L^*-convergence. To obtain this closure $\bar{A}$, it is possible, and in this case necessary, to iterate the operation pc out to the first uncountable ordinal. If C satisfies (5), then $pc(A) = \bar{A}$, so that the example just given would be excluded. Of course, if C is any sequential convergence, a set is $T(C)$ closed if and only if it contains all limits of C-convergent sequences of its members.

The following basic theorem relating sequential convergence and topology was proved by J. Kisyński [11]:

THEOREM 2.1.   *If (S, C) is any L^*-space, then $C(T(C)) = C$.*

It follows from this theorem that if C is an L^*-convergence, so is $C(T(C))$, even though $T(C)$ may not be Hausdorff. It was also proved in [11] that if (S, C) is an L-space, then $C(T(C))$ is the smallest L^*-convergence containing C. (An interesting example of this is that if C is convergence almost everywhere of equivalence-classes of measurable functions on a nonatomic measure space, then C is an L-convergence but not an L^*-convergence, and $C(T(C))$ is convergence in measure.) More specifically, if C is an L-convergence, then $s_n \rightarrow_{C(T(C))} s$ if and only if every subsequence $\{r_m\}$ of $\{s_n\}$ has a subsequence $\{t_q\}$ such that $t_q \rightarrow_C s$. Hence $T(C(T(C))) = T(C)$.

Thus a topology T is of the form $T(C)$, where C is an L^*-convergence, if and only if $T = T(C')$ where C' is an L-convergence. We shall call such topologies "sequential." Clearly if T is sequential then $T(C(T)) = T$, and conversely if T is Hausdorff and $T(C(T)) = T$ then T is sequential.

In any case, it is clear that $T(C(T))$ is a finer topology than $T(T(C(T)) \supset T)$. If T is Hausdorff, $T(C(T))$ is the weakest sequential topology finer than T. For example, if T is the order topology on an uncountable well-ordered set with a supremum, then $T(C(T))$ is strictly finer than T. However, as will be seen later, there exist sequential topologies which do not have a countable neighborhood-base at any point.

The class of convergences $C(T)$, where T is T_1, includes the class of L^*-convergences, which in turn includes all convergences $C(T)$, T Hausdorff. It is an interesting problem to characterize in sequential terms the classes of convergences $C(T)$ where T is Hausdorff, T_1, or an arbitrary topology. Perhaps suitably weakened forms of axiom (5) would be useful.

If (S, ρ) is a metric space and $C(\rho)$ the usual convergence defined by ρ, then $T(C(\rho))$ is the usual topology defined by ρ and is sequential, $C(\rho)$ being an L^*-convergence. We shall see later that if S is a nonmetrizable topological linear space, with topology T, then $T(C(T))$ may be pathological, while its "locally convex part" is often equal to T.

Sequential continuity is naturally defined as follows:

DEFINITION. If C and C' are sequential convergences on sets S and S' respectively, a function f from S to S' is *continuous* for C and C' if and only if $f(x_n) \to_{C'} f(x)$ whenever $x_n \to_C x$.

It is well known that sequential continuity is equivalent to topological continuity in metric spaces. The following generalization is interesting and useful in proving its locally convex version:

THEOREM 2.2. *If C and C' are sequential convergences on sets S and S' respectively and f is continuous for C and C', then it is continuous for $T(C)$ and $T(C')$. If f is continuous for topologies T and T', then it is continuous for $C(T)$ and $C(T')$. If C' is an L^*-convergence, then continuity for C and C' is equivalent to continuity for $T(C)$ and $T(C')$.*

Proof. If f is continuous for C and C', $U \in T(C')$, $x \in f^{-1}(U)$, and $x_n \to_C x$, then $f(x_n) \to_{C'} f(x)$ so that $f(x_n) \in U$ and $x_n \in f^{-1}(U)$ for n large enough. Hence $f^{-1}(U) \in T(C)$ and f is topologically continuous.

If f is continuous for T and T', suppose $x_n \to_{C(T)} x$. Then if $f(x) \in U \in T'$, $x_n \in f^{-1}(U)$ for n large enough, so $f(x_n) \in U$, and $f(x_n) \to_{C(T')} f(x)$. Thus f is continuous for $C(T)$ and $C(T')$.

If $T = T(C)$, $T' = T(C')$, and C' is an L^*-convergence, then $C(T(C')) = C'$ by Theorem 2.1, so that if f is continuous for T and T' it is continuous for $C(T)$ and C'. Since $x_n \to_C x$ implies $x_n \to_{C(T)} x$, f is continuous for C and C', q.e.d.

Theorem 2.2 implies the already rather obvious fact that a theorem on sequential continuity of integration proved for convergence almost everywhere will extend to convergence in measure.

3. **Compactness, products, quotients and relativization.** In this section we explore the sequential analogues of various topological constructions, and observe that, even if the given topologies are sequential, corresponding sequential and topological constructions may yield different results.

DEFINITION. An L^*-space (S, C) is L^*-compact if every sequence in S has a C-convergent subsequence.

A topological space is called "sequentially compact" if every sequence has a convergent subsequence; thus (S, C) is L^*-compact if and only if $(S, T(C))$ is sequentially compact, and if (S, T) is a Hausdorff space, it is sequentially compact if and only if $(S, C(T))$ is L^*-compact. Of course, not every sequentially compact topology is sequential nor compact; examples are $\Omega + 1$ and Ω (minimal uncountable well-ordered sets with and without a supremum respectively, with the order topology; each has one property and not the other).

If (S, C) is L^*-compact, then since $(S, T(C))$ is a T_1-space it is also "countably compact," i.e., every countable open cover of S has a finite subcover (see [10, Chapter V, problem E, p. 162]).

Although $(S, T(C))$ need not be compact, many of the standard properties of compact spaces hold, with suitable modifications, for L^*-compact spaces. For example, a $T(C)$-closed subspace of an L^*-compact space is L^*-compact, a sequentially continuous image of an L^*-compact space is L^*-compact (e.g., a continuous real-valued function is bounded), etc.

If $(S_\alpha, C_\alpha)_{\alpha \in I}$ is any family of L^*-spaces, it is natural to define a convergence C on the Cartesian product $S = \prod_{\alpha \in I} S_\alpha$ by letting $\{x_\alpha^{(n)}\} \to_C \{x_\alpha\}$ if and only if $x_\alpha^{(n)} \to_C x_\alpha$ for each α. If T is the product topology on S defined by the $T(C_\alpha)$, then $C = C(T)$ by Theorem 2.1 since each C_α is an L^*-convergence. Thus $T(C) = T(C(T)) \supset T$. The inclusion is strict if there are uncountably many nontrivial spaces S_α, and, as will be shown in §9, even for the product of the *two* spaces $\mathscr{D}$ and $\mathscr{D}'$ of Schwartz. For a countable product of metric spaces, $T(C(T)) = T$.

In the cases where $T(C(T)) \neq T$, we obtain a new topology on the product space, and there may be important functions on the product which are continuous for this topology but not for T. However, $T(C(T))$ may have the disadvantage of not being compatible with a product algebraic structure: see §6 and §9.

If (S, T) is a topological space and f is a function on S, a topology called the quotient topology is defined on the range of S, namely the strongest topology for which f is continuous (see [10, Chapter III, pp. 94–100]). Similarly, if C is a sequential convergence on S, one can define a "quotient convergence"

C_f by letting $f(x_n) \to_{C_f} f(x)$ whenever $x_n \to_C x$. Although C_f is a sequential convergence, neither of axioms (3) and (4) need be satisfied by C_f. Clearly C_f is an L-convergence if and only if $x_n \to_C x$ and $y_n \to_C y$ and $f(x_n) = f(y_n)$ for all n imply $f(x) = f(y)$. There are L-spaces (S, C) which are not L^*-spaces, e.g., $S = $ equivalence-classes of measurable functions on a nonatomic measure space, $C = $ convergence almost everywhere, and this and many other such spaces are quotients of L^*-spaces:

DEFINITION. An L-space (S_1, C_1) is an $L+$-space if whenever $s_n \to_{C_1} s$, and for sufficiently large m either $r_m = s$ or $r_m = s_{n(m)}$, where $\lim_{m \to \infty} n(m) = \infty$, then $r_m \to_{C_1} s$.

THEOREM 3.1. An L-convergence C_1 is of the form C_f where C is an L^*-convergence if and only if C_1 is an $L+$-convergence.

Proof. If there is a function f from S to S_1 such that $C_1 = C_f$, where (S, C) is an L^*-space and (S_1, C_1) is an L-space, suppose $s_n \to_{C_1} s$ and $r_m = s$ or $r_m = s_{n(m)}$. We have $x_n \to_C x$ in S with $f(x_n) = s_n$, $f(x) = s$. If $y_m = x$ or $y_m = x_{n(m)}$ according as $r_m = s$ or $r_m = s_{n(m)}$, then by definition of L^*-space $y_m \to_C x$, so $r_m \to_{C_1} s$ and C_1 is an $L+$-convergence.

Conversely, suppose (S_1, C_1) is an $L+$-space. Let S be the product space $S_1 \times A$ where A is the set of all sequences convergent for C_1 (regarded as functions from the positive integers to S_1). Let $(s_n, q_n) \to_C (s, q)$ if $q_n = q$ for n large enough and either

(a) $s_n = s$ for n large enough, or

(b) $q(m) \to_{C_1} s$ and there is a function $m( )$ with $\lim_{n \to \infty} m(n) = +\infty$ such such that $s_n = q(m(n))$ or $s_n = s$ for n large enough. (Then by definition of $L+$-space, $s_n \to_{C_1} s$.)

It is easy to verify that C is an L-convergence. If $(s_n, q_n) \not\to_C (s, q)$, then either $q_n \neq q$ for infinitely many n, or $q(m) \not\to_{C_1} s$, or there is an r such that for infinitely many n, s_n is neither equal to s nor of the form $q(m)$ for $m \geq r$. In any case there is a subsequence of which no further subsequence converges to (s, q), so C is an L^*-convergence.

Let f be the projection of S onto $S_1 : f((s, q)) = s$. If $s_n = q(n) \to_{C_1} s$, then $(s_n, q_n) \to_C (s, q)$ if $q_n = q$ for all n, so $s_n \to_{C_f} s$. Conversely, if $(s_n, q_n) \to_C (s, q)$ then $s_n \to_{C_1} s$. Thus $C_1 = C_f$, q.e.d.

If (S, T) is a topological space and $B \subset S$, one defines the relative topology T_B of T on B to be the class of all sets $U \cap B$, $U \in T$. Likewise if C is a sequential convergence on S, one can define a "relative convergence" C_B on B by restricting all ranges and limits of sequences to B. It is easy to check that C_B is an L^*-convergence if C is. Clearly $C(T(C)_B) = C_B$ so that $T(C)_B \subset T(C_B)$ and $T(C)_B$ is sequential if and only if it is equal to $T(C_B)$. If there is a subset A of S such that $pc(pc(A)) \neq pc(A)$, where pc is the pseudo- or sequential closure, let $p \in pc(pc(A))$

$\sim pc(A)$ and let $B = A \cup \{p\}$. Then $\{p\} \in T(C_B)$ but $\{p\} \notin T(C)_B$ so the inclusion $T(C)_B \subset T(C_B)$ is strict and $T(C)_B$ is not sequential.

Of course the difference between topological closure and sequential closure pc is another example of a disparity between corresponding topological and sequential constructions. Here at least the topological closure can be obtained by (possibly uncountable) iteration of pc, if the topology is sequential.

A recent paper of Hörmander [9] considers sequential continuity on subspaces of spaces of test functions, apparently for technical reasons. In the cases in question the subspaces turn out to be the full spaces, so that there is no actual relativization. Also, according to §§6 and 9 below, sequential continuity is equivalent to continuity.

If B is closed for $T(C)$, then if $V \in T(C_B)$ it is clear that $V \cup (S \sim B) \in T(C)$ so that $V \in T(C)_B$. Hence $T(C)_B = T(C_B)$, and this is also clearly true for B open.

4. Convergence of sets. If S is a set and S_n are subsets of S, $n = 1, 2, \cdots$, then one defines

$$\liminf_{n \to \infty} S_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} S_n,$$

and

$$\limsup_{n \to \infty} S_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} S_n.$$

If $\liminf S_n = A = \limsup S_n$, then we say $\lim_{n \to \infty} S_n = A$ or $S_n \to_C A$. This natural convergence C is easily verified to be an L^*-convergence on the class $P(S)$ of all subsets of S, or any subclass $\mathscr{A}$ of $P(S)$. It defines a topology $T(C)$ on $P(S)$ and a topology $T(C_\mathscr{A})$ on $\mathscr{A}$ (see §3).

$\mathscr{A}$ is a σ-algebra in S if it contains S and is closed under complementation and countable unions and intersections; equivalently, closed under finite Boolean operations and a $T(C)$-closed subset of $P(S)$. Thus in particular $T(C_\mathscr{A}) = T(C)_\mathscr{A}$. A finite countably additive measure is a finitely additive, C-continuous function on such a subclass.

A $(\sigma$-$)$ algebra $\mathscr{A}$ in S is a ring with the operations $A + B = (A \sim B) \cup (B \sim A)$, $AB = A \cap B$. An *ideal* in $\mathscr{A}$ is a nonempty subset I of $\mathscr{A}$ which is closed under finite unions and such that if $A \subset B \in I$, $A \in \mathscr{A}$, then $A \in I$. Our discussion of ideals below owes much to the paper [14] of Tarski.

An ideal I in a σ-algebra $\mathscr{A}$ is called a σ-*ideal* if it is closed under countable unions, and will be called *countably saturated* if whenever B_n, $n = 1, 2, \cdots$, are disjoint sets in $\mathscr{A}$, all but finitely many B_n belong to I.

LEMMA 4.1. *Let $\mathscr{A}$ be a σ-algebra in S and I a countably saturated ideal in $\mathscr{A}$. Suppose $C_n \in \mathscr{A}$, $n = 1, 2, \cdots$, and $C_n \cap C_m \in I$ whenever $n \neq m$, n, $m = 1, 2, \cdots$. Then $C_n \in I$ except for at most finitely many n.*

Proof. The sets $C_n \sim \bigcup_{j=1}^{n-1} C_j$ are disjoint for $n = 2, 3, \cdots$, so for some n_0

they belong to I for $n \geqq n_0$, and since $(C_n \cap C_j) \in I$ for $j < n$, $C_n \in I$ for $n \geqq n_0$, q.e.d.

For $\mathscr{A} = P(S)$, at least, the above result is included in Satz 4.7 of [14], the conclusion being that I is "$\aleph_0$-saturated" in the sense of that paper.

THEOREM 4.2. *If I is an ideal in a σ-algebra $\mathscr{A}$ in S, the following three assertions are equivalent:*

(i) $I \in T(C_{\mathscr{A}})$.

(ii) *If $A_n \in \mathscr{A}$ and $A_n \to_C \varnothing$, where $\varnothing$ is the empty set, then $A_n \in I$ for n large enough.*

(iii) *I is a countably saturated σ-ideal in $\mathscr{A}$.*

Proof. Clearly (i) implies (ii). Assuming (ii), suppose that B_n are disjoint sets. Then $B_n \to_C \varnothing$, so $B_n \in I$ for n large enough, and I is countably saturated. Also if $A_n \in I$ for all n, then $(\bigcup_{n=1}^{\infty} A_n) \sim (\bigcup_{n=1}^{m} A_n) \to_C \varnothing$ as $m \to \infty$. Thus these sets are in I for m large enough, so since $\bigcup_{n=1}^{m} A_n$ is in I, $\bigcup_{n=1}^{\infty} A_n \in I$ so I is a σ-ideal and (iii) holds.

Now suppose (iii) holds and let $A_n \to_C A$, $A \in I$. Then $\bigcap_{m=1}^{\infty} B_m = A$ where $B_m = \bigcup_{n=m}^{\infty} A_n$. $\{B_m\}$ is a decreasing sequence of sets, so the sets $B_m \sim B_{m+1}$ are disjoint for all m. Thus for some M, $B_m \sim B_{m+1} \in I$ for $m \geqq M$ since I is countably saturated, and $B_M = A \cup \bigcup_{m=M}^{\infty} (B_m \sim B_{m+1}) \in I$ since I is a σ-ideal. Thus $A_m \in I$ for $m \geqq M$ since $A_m \subset B_M$, and $I \in T(C)$, q.e.d.

We shall call a σ-algebra $\mathscr{A}$ in S σ-*atomic* if there exist sets A_k in $\mathscr{A}$, $k = 1, 2, \cdots$, such that for any set Y of positive integers,

$$\left(\bigcap_{k \in Y} A_k \right) \cap \bigcap_{k \notin Y} (S \sim A_k)$$

contains at most one point. Clearly if $\mathscr{A}$ is σ-atomic, S has at most the cardinal of the continuum, and the converse holds if $\mathscr{A} = P(S)$.

THEOREM 4.3. *If $\mathscr{A}$ is a σ-atomic σ-algebra in S, I is an ideal in $\mathscr{A}$, and $I \in T(C_{\mathscr{A}})$, then there is a finite subset F of S such that $S \sim F \in I$ and $B \in I$ if and only if $B \in \mathscr{A}$ and $B \subset S \sim F$.*

Proof. Let A_k be given according to the definition of σ-atomic σ-algebra. For any set Y of positive integers and integer n, let

$$S(Y, n) = \bigcap_{k=1}^{n} B_k$$

where $B_k = A_k$ for $k \in Y$, $B_k = S \sim A_k$ for $k \notin Y$.

If I is not all of $\mathscr{A}$, there exists at least one set Y such that $S(Y, n) \notin I$ for all n; call such a set "outer." Suppose there are infinitely many distinct outer sets; then by König's lemma there is an outer set Y such that for all n there is an outer set $Y(n)$ with $m \in Y$ if and only if $m \in Y(n)$ for $m \leqq n$ but $i(n) \in (Y \sim Y(n)) \cup (Y(n) \sim Y)$ for some $i(n) > n$. Then the sets

$$S(Y(1), i(1)), \ S(Y(i(1)), \ i(i(1))), \cdots,$$

are disjoint and none belongs to I, a contradiction. Thus there are only finitely many outer sets Y_r, $r = 1, \cdots, R$.

Let $P_r = \bigcap_{n=1}^{\infty} S(Y_r, n)$, $r = 1, \cdots, R$; P_r contains exactly one point. Then for each r there is an $n = n(r)$ such that $S(Y_r, n) \sim P_r$ belongs to I, since these sets converge to $\varnothing$ as $n \to \infty$. Thus if F is the union of the P_r, $S \sim F$ belongs to I, since if $N = \max_r n(r)$, $S(Y, N) \sim F$ belongs to I for all Y and $S \sim F$ is a finite union of such sets. Since $P_r \notin I$ for all r, $A \in I$ if and only if $A \in \mathscr{A}$ and $A \subset S \sim F$, q.e.d.

We also have

THEOREM 4.4. *If* $\mathscr{A} = P(S)$, *the cardinal of S is weakly accessible* (*not strongly inaccessible*), *I is an ideal in* $\mathscr{A}$, *and* $I \in T(C_{\mathscr{A}})$, *then* $I = P(S \sim F)$ *for some finite subset F of S.*

This is a consequence of Satz 4.14 of [14], bearing in mind Lemma 4.1 and Theorem 4.2 above and Korollar 4.9 of [14]. A proof of Satz 4.14 uses transfinite induction, passing from a cardinal b to 2^b as in Theorem 4.3 (where the set of A_k has cardinal b), using the fact that a countably saturated σ-ideal is b-additive, with an easier proof for accessible limit cardinals.

It is clear without the continuum hypothesis that any strongly inaccessible cardinal is very large, e.g., much larger than $c + 2^c + 2^{2^c} + \cdots$ where c is the cardinal of the continuum.

5. **Quasi-metric spaces.** Many of the L^*-spaces arising in analysis have convergence of the type about to be defined (mentioned previously in [5]).

DEFINITION. A *quasi-metric space* is a triple (S, ρ, F) where S is a set, ρ is a metric on S, and F is a set of nonnegative real-valued functions on S. If (S, ρ, F) is a quasi-metric space, the *quasi-metric convergence* $C = C(\rho, F)$ on S is defined by $x_n \to_C x$ if and only if $\lim_{n \to \infty} \rho(x_n, x) = 0$ and $\{f(x_n)\}$ is a bounded sequence of real numbers for each $f \in F$. $T(C(\rho, F))$ will be called the *quasi-metric topology*.

Clearly any quasi-metric convergence is an L^*-convergence. The description "quasi-metric" has been applied by several Portuguese authors to structures defined by one function satisfying conditions less restrictive than those for a metric. There is no evident connection between such structures and those just defined.

If (S, ρ) is a metric space, then the convergence $C(\rho)$ is equal to $C(\rho, F)$ where F is the null set or contains only the function $x \to \rho(x_0, x)$ where x_0 is a fixed point of S.

If F is finite, then clearly $C(\rho, F) = C(\rho, G)$ where G has the one element $g = \Sigma_{f \in F} f$. Quasi-metric spaces (S, ρ, F) where F has only one element f will be called "simple." In this case we will write $(S, \rho, F) = (S, \rho, f)$, $C(\rho, F) = C(\rho, f)$, etc.

If F is countably infinite we shall call (S, ρ, F) and $C(\rho, F)$ "countably quasi-metric." Such spaces are considered in [5] and later in this paper.

If F is infinite, a sequence is convergent for $C(\rho, F)$ if and only if it is convergent for each $C(\rho, f), f \in F$. However, it need not be true that $T(C(\rho, F))$ is the weakest topology stronger than each $T(C(\rho, f))$. For example, let S be the set of all ordered pairs $p = (p_1, p_2)$ of integers, both positive or both 0, let $a(x) = 1/x$ for $x \neq 0$ and $a(0) = 0$, and $\rho(p, q) = a|(p_1 + p_2) - a(q_1 + q_2)| + |a(p_2) - a(q_2)|$. Let $f_1(p) = p_1$ and, for $j > 1$, $f_j(p) = p_2$ for $p_1 \leq j$, $f_j(p) = 1$ for $p_1 > j$. Let F be the set of all $f_j, j = 1, 2, \cdots$. Then since the topology of ρ is discrete except at $(0,0)$ and there are no sequences convergent to $(0,0)$ for $C(\rho, F)$ except the eventually constant ones, $T(C(\rho, F))$ is discrete. However, the set whose only member is $(0,0)$ is not in $T(C(\rho, G))$ for any finite subset G of F.

The space $\mathscr{D}(R^k)$ of C^∞ functions with compact support on k-space R^k has a simple quasi-metric structure $(\mathscr{D}, \rho, f)$ where convergence for ρ is equivalent to uniform convergence of all partial derivatives and, for $\phi \in \mathscr{D}$, $f(\phi)$ is the greatest distance from the origin to any point in the support of ϕ (or 0 if $\phi \equiv 0$). Then $C(\rho, f)$ is the usual sequential convergence or "pseudo-topology" of $\mathscr{D}$. It will be shown in §9 that $T(C(\rho, f))$ is *not* the usual topology of $\mathscr{D}$, but that the class of *convex* sets in this topology is a base for the usual topology.

PROPOSITION 5.1. *The product convergence C on a countable product $S = \prod_{n=1}^\infty S_n$ of quasi-metric spaces (S_n, ρ_n, F_n) is quasi-metric.*

Proof. For $x = \{x_n\}$, $y = \{y_n\} \in S$ let $\rho(x, y) = \sum_{n=1}^\infty [\text{arc tan } \rho_n(x_n, y_n)]/2^n$, and let F be the set of functions f of the form $f(\{x_m\}) = g(x_n)$, $g \in F_n$, $n = 1, 2, \cdots$. Then ρ is a metric and $C = C(\rho, F)$, q.e.d.

If each F_n is countable, then F is countable. Also,

PROPOSITION 5.2. *The product convergence on a finite product $S = \prod_{n=1}^N S_n$ of simple quasi-metric spaces (S_n, ρ_n, f_n) is simple quasi-metric.*

Proof. Let $f(\{x_n\}) = \sum_{n=1}^N f_n(x_n)$, and $\rho(\{x_n\}, \{y_n\}) = \sum_{n=1}^N \rho_n(x_n, y_n)$; then $C = C(\rho, f)$, q.e.d.

6. Linear spaces.

DEFINITION. If S is a real or complex linear space and (S, C) is an L^*-space, then (S, C) is an L^*-*linear space* if and only if $(a, b, f, g) \to af + bg$ is L^*-continuous from $A \times A \times S \times S$ into S, where A is the real or complex numbers with the usual convergence.

If (S, T) is a topological linear space in the usual sense, then $(S, C(T))$ is an L^*-linear space. However, if (S, C) is a nonmetric L^*-linear space, then $(S, T(C))$ need not be a topological linear space, as will be shown below. (Note that the product topology on $S \times S$ is not obviously sequential.) On the other hand, a locally convex uniform topology always yields a topological linear space structure, and many or most of the functions on S to be considered will be linear. Thus we make the following definitions:

DEFINITION. If (S, C) is an L^*-linear space, then $T_c(C)$ is the collection of sets $U \subset S$ such that for each $x \in U$ there is a convex set $V \in T(C)$ with $x \in V \subset U$.

Clearly $T_c(C) \subset T(C)$; $T_c(C)$ is the strongest locally convex topology weaker than $T(C)$. Of course $T_c(C)$ may be the indiscrete topology containing only S and the empty set, for example if S is the metric space $\mathscr{L}^p$, $p < 1$, with the metric convergence $C : f_n \to_c f$ if $\int |f_n - f|^p \to 0$. To exclude such cases, we have

DEFINITION. If (S, C) is an L^*-linear space, C is a *convex* convergence if $C(T_c(C)) = C$.

DEFINITION. If (S, T) is a topological linear space and $T_c(C(T)) = T$, then T is a *convex-sequential* topology (*CS*-topology) and (S, T) is a *CS-space*.

If C is convex, $T_c(C)$ is a *CS*-topology, and if T is a *CS*-topology then $C(T)$ is convex. There is a 1-1 correspondence between convex convergences C and *CS*-topologies T set up by $T = T_c(C)$, $C = C(T)$. However, $T_c(C)$ may be *CS* without C convex (as in $\mathscr{L}^p$, $p < 1$) or $C(T')$ convex without T' being *CS* (see the discussion after Corollary 6.5 and let $T' = T(C(T))$).

THEOREM 6.1. *A linear mapping L from one CS-space (S, T) to another (S_1, T_1) is continuous (for T and T_1) if and only if it is sequentially continuous (for $C(T)$ and $C(T_1)$).*

Proof. If L is continuous for T and T_1, then it is continuous for $C(T)$ and $C(T_1)$ (Theorem 2.2).

Conversely, if L is sequentially continuous let $0 \in U \in T_1$, U convex. Then $L^{-1}(U)$ is a convex set in S and $L^{-1}(U) \in T(C(T))$ by Theorem 2.2. Thus $L^{-1}(U) \in T_c(C(T)) = T$, so L is continuous, q.e.d.

The difficulties with product topologies for two spaces with sequential topologies $T(C)$ do not arise for topologies $T_c(C)$, e.g., *CS*-topologies:

THEOREM 6.2. *Let (S_1, C_1) and (S_2, C_2) be L^*-linear spaces, $S = S_1 \times S_2$, and C the product convergence of C_1 and C_2 on S. Then $T_c(C)$ is the product topology T of $T_c(C_1)$ and $T_c(C_2)$.*

Proof. Clearly $T \subset T_c(C)$. For the converse, let $0 \in U \in T(C)$ with U convex and let

$$U_1 = U/2 \cap (S_1 \times \{0\}), \qquad U_2 = U/2 \cap (\{0\} \times S_2).$$

Then $U_1 = V_1 \times \{0\}$ and $U_2 = \{0\} \times V_2$ where

$$V_i \in T_c(C_i), \quad i = 1, 2, \text{ and } V_1 \times V_2 \subset U$$

since U is convex. Thus U is a neighborhood of 0 for T. Since C is translation-invariant, so are both topologies. Hence they are equal, q.e.d.

Before discussing infinite products we shall now show that every "bornological" topology is a *CS*-topology (this fact was pointed out to me by L. Bungart). Let

us recall some definitions: if (S, T) is a topological linear space, a set A *absorbs* a set B if $\lambda B \subset A$ for all small enough $\lambda > 0$. A set is *bounded* if it is absorbed by every neighborhood of 0, and (S, T) is *bornological* if every convex set A which absorbs every bounded set is a neighborhood of 0 (see [8; 13]).

THEOREM 6.3. *Every locally convex bornological space is a CS-space.*

Proof. Let (S, T) be bornological; we must prove that $T_c(C(T)) = T$. Since T is locally convex, clearly $T \subset T_c(C(T))$. Conversely, let $U \in T_c(C(T))$, $x \in U$. Then $V = U - x \in T_c(C(T))$, $0 \in V$, and V is convex. Let B be a bounded set. Then $B/n \subset V$ for some positive integer n, for if not there are $b_n \in B$ with $b_n/n \notin V$ for all n, but $b_n/n \to_{C(T)} 0$ since B is bounded, so that $b_n/n \in V$ for n large enough, a contradiction. Thus V absorbs every bounded set, so V is a neighborhood of 0, U is a neighborhood of x, and $U \in T$, q.e.d.

It is known that a product of bornological spaces is bornological if the number of factors is weakly accessible. This can be inferred from Theorem 7 of [4] about products of "boundedly closed" spaces. However, a complete proof based on the set-theoretic results of §4 is given below.

THEOREM 6.4. *Any product $S = \prod_{\alpha \in J} S_\alpha$ where (S_α, T_α) is bornological for each α, S has the product topology T, and J has weakly accessible cardinality, is bornological.*

Proof. Let K be a convex set in S which absorbs every bounded set. Let I be the class of subsets B of J such that if $x_\alpha = 0$ for $\alpha \notin B$, then $x = \{x_\alpha\} \in K$. Clearly $A \subset B \in I$ implies $A \in I$, and if $B_1, \cdots, B_n \in I$ any element $\{x_\alpha\}$ with $x_\alpha = 0$ for $\alpha \notin \bigcup_{m=1}^n B_m$ is a convex linear combination $\sum_{m=1}^n x^{(m)}/n$ of elements $\{x_\alpha^{(m)}\} \in K$. Thus I is an ideal.

Suppose A_n converges to the empty set (see §4). If $A_n \notin I$ for arbitrarily large n, we may assume $A_n \notin I$ for all n. For each n, choose an element $x^{(n)} = \{x_\alpha^{(n)}\}$ of S such that $x_\alpha^{(n)} = 0$ for $\alpha \notin A_n$ but $\{x_\alpha^{(n)}\} \notin K$. Then the sum $\sum_n nx^{(n)}$ converges for T (since $A_n \to \phi$, at most finitely many $x_\alpha^{(n)}$ are nonzero for a given α). Thus the sequence $nx^{(n)}$ is bounded in S, so for some $\lambda > 0$, $\lambda nx^{(n)} \in K$ for all n; but this is a contradiction if $n > 1/\lambda$. Thus $A_n \in I$ for large enough n, so by Theorem 4.2 I is open and by Theorem 4.4 I consists of all subsets of $J \sim F$ where F has k members for some finite k. Thus if $x_\alpha = 0$ for $\alpha \in F$, $\{x_\alpha\} \in K$. For each $\gamma \in F$ there is a convex neighborhood U_γ of 0 in S_γ such that if $x_\gamma \in U_\gamma$ and $x_\beta = 0$ for $\beta \neq \gamma$, then $\{x_\beta\} \in K$. Thus by convexity the neighborhood of 0 in S consisting of all $\{x_\alpha\}$ such that $x_\alpha \in U_\alpha/(k+1)$, $\alpha \in F$, is included in K, q.e.d.

COROLLARY 6.5. *If the spaces $(S_\alpha, T_\alpha)_{\alpha \in J}$ are bornological and J has weakly accessible cardinality, then the product topology T is equal to $T_c(C(T))$.*

For example, let each S_α be a copy of the real line with its usual topology and J also the real line, so that the product space is the set of all real functions of a real

variable. Now, $T(C(T))$ is a rather nasty topology; for example, if M is the set of all functions f such that $\{x \in J : |f(x)| < 1\}$ is of second category, then M is an open set. Thus $T(C(T))$ is strictly larger than $T_c(C(T))$ and is not locally convex. We also have

THEOREM 6.6. *Assuming the continuum hypothesis, addition of functions is not continuous for $T(C(T))$.*

Proof. A result of Banach and Kuratowski [1, Théorème II] asserts that, assuming the continuum hypothesis, if J has the cardinality of the continuum there exist subsets $A_i^j, i, j = 1, 2, \cdots$, of J, such that for each i the sets A_i^j are disjoint with union J and such that for any sequence $\{k_i\}$ of positive integers, $\bigcap_{i=1}^{\infty} \bigcup_{j=1}^{k_i} A_i^j$ is countable.

Now, suppose that addition is continuous for $T(C(T))$. Let U_0 be the set of $f \in S$ such that $\{x : |f(x)| < 1\}$ is infinite. Then U_0 is a neighborhood of 0 for $T(C(T))$. For each $n \geq 1$, there must then be a neighborhood U_n of 0 such that $U_n + U_n + U_n \subset U_{n-1}$.

For each $n = 1, 2, \cdots$, there is a $k = k_n$ such that every function equal to 0 on $\bigcup_{j=1}^{k} A_n^j$ belongs to U_n (since a sequence of such functions for $k = 1, 2, \cdots$, converges to 0 for $C(T)$). Let

$$\{x_r\}_{r=1}^{\infty} = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{k_n} A_n^j.$$

Then

$$\bigcup_{n=1}^{\infty} \left\{ \{x_n\} \cup \bigcup_{j=k_n+1}^{\infty} A_n^j \right\} = J.$$

Thus for some N, every function equal to 0 on $\bigcup_{n=1}^{N} \{\{x_n\} \cup \bigcup_{j=k_n+1}^{\infty} A_n^j\}$ belongs to U_1. Hence any function equal to 0 on $\{x_1, \cdots, x_N\}$ belongs to

$$U_1 + U_1 + U_2 + U_3 + \cdots + U_N \subset U_1 + U_1 + U_1 \subset U_0.$$

But this is a contradiction, so our assertion is proved.

Other L^*-linear spaces (S, C) such that $(S, T(C))$ is not a topological linear space are discussed in Theorems 7.4 and 8.5; for example, $\mathscr{D}$ is such a space.

An interesting subspace of the space of all real-real functions is the set B of Borel functions. B is $T(C)$-closed, so in the notation of §3 $T(C_B) = T(C)_B$. Of course B is not $T_c(C)$-closed. Theorem 4.3 and the proof of Theorem 6.4 show that $(B, T_c(C))$ is bornological, since the class of Borel sets is σ-atomic, and as in Corollary 6.5 we obtain that $T_c(C)$ is the product topology.

7. **Quasi-metric linear spaces.** If C is convergence with respect to a metric and $T(C)$ is locally convex (the completion of S for $T(C)$ is an "F-space" or "Fréchet space") then C is convex and $T(C) = T_c(C)$ is a CS-topology. We next consider simple quasi-metric linear spaces. On the dual space S^* of a linear

space S there is always at least one natural L^*-convergence, namely pointwise convergence on S ("weak* convergence").

If (S, ρ, f) is a simple quasi-metric linear space (i.e., $(S, C(\rho, f))$ is an L^*-linear space) we shall use the following notations:

$$S_n = S(n) = \{x \in S : f(x) \leqq n\},$$

$$S(n, \varepsilon) = \{x \in S(n) : \rho(0, x) < \varepsilon\},$$

$$\bar{S}(n, \varepsilon) = \text{closure of } S(n, \varepsilon),$$

$$Co(n, \varepsilon) = \text{convex hull of } S(n, \varepsilon),$$

$$\overline{Co}(n, \varepsilon) = \text{closure of } Co(n, \varepsilon).$$

The "convexity" of a sequential convergence C was defined in §6 in terms of the topology $T_c(C)$. Here is a related purely sequential condition:

DEFINITION. An L^*-linear space (S, C) is L^*-convex if whenever $x_n \to_C 0$ and for each n, y_n is a convex combination of the x_m for $m \geqq n$, $y_n \to_C 0$.

A convex L^*-convergence, being of the form $C(T)$ where T is locally convex, is obviously L^*-convex. In Theorem 7.6 we shall prove the converse for certain spaces defined as follows:

DEFINITION. A function f on an L^*-linear space (S, C) is an *LS-function* if it satisfies (a)–(c) below:

(a) $f(x + y) \leqq f(x) + f(y)$ for all x, $y \in S$, and $f(0) = 0$.

(b) For each $n > 0$, S_n is convex and symmetric: $f(x) = f(-x)$.

(c) If $x_n \to_C x$, $f(x) \leqq \limsup f(x_n)$.

(S, C) is an *LS-space* (by (ρ, f)) if it is L^*-convex and $C = C(\rho, f)$ where ρ is an invariant metric $(\rho(x, y) = \rho(x - y, 0))$ and f is an LS-function on S for C.

(a) and (b) together may seem close to the assertion that f is a pseudo-norm, but the function f on $\mathscr{D}, f(\phi) = \sup\{|x| : \phi(x) \neq 0\}$, is an LS-function and cannot be replaced by a pseudo-norm without changing the convergence $C(\rho, f)$ (discussed just before Proposition 5.1).

(c) says that f is sequentially lower semi-continuous; if $\limsup$ is replaced by $\liminf$, an equivalent condition is obtained (consider subsequences).

All the specific locally convex simple quasi-metric linear spaces to be mentioned in this paper are actually LS-spaces.

THEOREM 7.1. *If (S, ρ) is a complete separable metric linear space, S^* is its dual space, and C is weak* sequential convergence in S^*, then (S^*, C) is an LS-space.*

Proof. We may assume ρ is a translation-invariant metric on S [10, N and O, pp. 209–210]. Let σ be a metric on S^* such that convergence for σ is equivalent to pointwise convergence on a countable dense set D in S, and for $x \in S^*$ let

$$f(x) = \sup\{|x(s)|/\rho(0,s): |x(s)| \leqq 1, \ \rho(0,s) > 0\}.$$

It is easy to verify that f is an LS-function for C.

A C-convergent sequence is also convergent for $C(\sigma,f)$ since by the Banach-Steinhaus theorem [6, p. 52] it is equicontinuous.

Conversely, if $x_n \to_{C(\sigma,f)} x$, then for some k, $f(x_n - x) \leqq k$ for all n. Let $s \in S$. Then for some $s_j \in D, \rho(s,s_j) \to 0$, and $\tau(s - s_j) \to 0$, where $\tau(s') = \sup_{0 \leqq \lambda \leqq 1} \rho(0, \lambda s')$, by joint continuity and compactness. The set of $s' \in S$ with $\tau(s') \leqq 1/kr$ is connected, and $f(x') \leqq k$ implies $|x'(s')| > 1$ or $|x'(s')| \leqq 1/r$ on this set, hence $|x'(s')| \leqq 1/r$ for $r > 1$. Thus if $r > 1$, $|(x_n - x)(s - s_j)| \leqq 1/r$ for all n if j is large enough. Since $|(x_n - x)(s_j)| \to 0$ for each j, $|(x_n - x)(s)| \to 0$, q.e.d.

It is easy to infer the following from the proof of Proposition 5.2:

PROPOSITION 7.2. *The product of two LS-spaces with the product convergence is an LS-space.*

We can explicitly describe the topology $T_c(C)$ if C is simple quasi-metric and L^*-convex:

THEOREM 7.3. *Let* $(S, C(\rho,f))$ *be an* L^*-*convex* L^*-*linear space. For each sequence* $\{\varepsilon_n\}_{n=1}^{\infty}$ *of positive numbers let* $U_{\{\varepsilon_n\}}$ *be the set of all finite sums* $\sum_{n=1}^{N} w_n$ *where* $w_n \in Co(n, \varepsilon_n)$ *for* $n = 1, \cdots, N$ *and* N *is arbitrary. Then the collection of all sets* $U_{\{\varepsilon_n\}}$ *is a base for the neighborhoods of* 0 *for* $T_c(C(\rho,f))$.

Proof. Each $U_{\{\varepsilon_n\}}$ is obviously convex; let us show that it belongs to $T(C)$ where $C = C(\rho,f)$. If $y_m \to_C y \in U_{\{\varepsilon_n\}}$, then $y_m - y \to_C 0$. Thus for some $K, f(y_m - y) \leqq K$ for all m, and $\rho(0, y_m - y) \to 0$. Let $y = \sum_{n=1}^{N} w_n$ as described, and $r = \max(N, K) + 1$. Then $\rho(0, y_m - y) < \varepsilon_r$ for m large enough, so that $y_m = (\sum_{n=1}^{N} w_n) + w_r(m)$ where $w_r(m) = y_m - y$ satisfies the required conditions so that $y_m \in U_{\{\varepsilon_n\}}$ for m large enough. Thus $U_{\{\varepsilon_n\}} \in T_c(C)$.

Now let U be an arbitrary convex neighborhood of 0 for $T(C)$ (and hence for $T_c(C)$). For each n, there is an $\varepsilon_n > 0$ such that $Co(n, \varepsilon_n) \subset U/2^n$. For, if not, a sequence $x_{11}, \cdots, x_{1k_1}, x_{21}, \cdots, x_{2k_2}, x_{31}, \cdots$ could be constructed, convergent to 0 for C $(f(x_{ij}) \leqq n, \rho(0, x_{ij}) \leqq 1/i$ for all $i, j \leqq k_i)$ such that L^*-convexity would be contradicted.

If ε_n is so chosen for each n, then

$$U_{\{\varepsilon_n\}} \subset \bigcup_{N=1}^{\infty} \sum_{n=1}^{N} U/2^n \subset U,$$

so that the set of all $U_{\{\varepsilon_n\}}$ is a base at 0 for $T_c(C)$, q.e.d.

A similar argument yields

THEOREM 7.4. *If* $(S, C(\rho,f))$ *is* L^*-*convex,* $T_c(C)$ *is the strongest topology* T *weaker than* $T(C)$ *such that* (S, T) *is a topological linear space.*

Proof. It suffices to show that if U_0 is a neighborhood of 0 for $T(C)$ such that for all $n \geq 1$ there is a $U_n \in T(C)$ with $U_n + U_n \subset U_{n-1}$, then U_0 is a neighborhood of 0 for $T_c(C)$.

For each $n \geq 1$, there is an $\varepsilon_n > 0$ such that $Co(n, \varepsilon_n) \subset U_n$. Then

$$U_{\{\varepsilon_n\}} \subset \bigcup_{N=1}^{\infty} \sum_{n=1}^{N} U_n \subset U_0, \quad \text{q.e.d.}$$

COROLLARY. *If* $(S, C(\rho, f))$ *is* L^*-*convex and* $(S, T(C))$ *is a topological linear space then* $T_c(C) = T(C)$.

Both here and in §6 we have seen that the assertions that $(S, T(C))$ is a topological linear space and that $T(C)$ is locally convex are closely related; the latter always implies the former since if $U \in T(C)$ and $b > 0$, $bU \in T(C)$. Of course there are metrizable nonlocally convex linear spaces.

The proofs of Theorems 7.3 and 7.4 are taken from an outlined problem solution of N. Bourbaki [3, Chapitre II, §2, Problème 10, p. 68] on inductive limits.

THEOREM 7.5. *If* $(S, C(\rho, f))$ *is* L^*-*convex then* $T_c(C)$ *is stronger than the topology of* ρ *and hence Hausdorff.*

Proof. Given $\varepsilon > 0$, choose ε_n for all n such that $Co(n, \varepsilon_n) \subset \bigcup_{m=1}^{\infty} S(m, \varepsilon/2^n)$. Then $U_{\{\varepsilon_n\}} \subset \{x \in S : \rho(0, x) < \varepsilon\}$, so that by Theorem 7.3 $T_c(C)$ is stronger than the ρ topology, q.e.d.

THEOREM 7.6. *If* (S, C) *is an LS-space by* (ρ, f) *then* C *is convex.*

Proof. We must show that $B \subset C$ where $B = C(T_c(C))$ (the other inclusion is immediate).

Let $x_n \to_B x$. Suppose $f(x_n)$ is unbounded: then for some $y_k = x_{n_k} - x$, $f(y_k) > k(k+1)/2$, $k = 1, 2, \cdots$. Of course $y_k \to_B 0$. By lower semi-continuity of f there is a $\delta_k > 0$ for each k such that if $f(y) \leq k(k+1)/2$, then $\rho(y, y_k) \geq \delta_k$. Let

$$\gamma_m = \min_{k < m} \delta_k / 3^{m-k}.$$

Choose $\varepsilon_m > 0$ for each m so that $Co(m, \varepsilon_m) \subset S(m, \gamma_m)$; this is possible by L^*-convexity and the fact that $S(m)$ is convex.

Now for k large enough, $y_k \in U_{\{\varepsilon_m\}}$ so that for some $M(k)$,

$$y_k = \sum_{m=1}^{M(k)} w_m(k)$$

where for each k and m, $w_m(k) \in Co(m, \varepsilon_m)$. Thus $\rho(0, w_m(k)) \leq \gamma_m$ and $f(w_m(k)) \leq m$.

But $f(\sum_{m=1}^{k} w_m(k)) \leq k(k+1)/2$, so

$$\rho\left(0, \sum_{m=k+1}^{M(k)} w_m(k)\right) = \rho\left(y_k, \sum_{m=1}^{k} w_m(k)\right) \leqq \delta_k$$

(and $M(k) > k$). On the other hand,

$$\rho\left(0 \sum_{m=k+1}^{M(k)} w_m(k)\right) \leqq \sum_{m=k+1}^{M(k)} \rho(0, w_m(k))$$

$$\leqq \sum_{m=k+1}^{M(k)} \gamma_m \leqq \sum_{m=k+1}^{M(k)} \delta_k/3^{m-k} < \delta_k/2,$$

a contradiction. Thus for some K, $f(x_n - x) \leqq K$ for all n, $\rho(x_n, x) \to 0$ by Theorem 7.5, so $x_n \to_C x$, q.e.d.

We now consider LF-spaces. For $n = 0, 1, \cdots$, let (S_n, ρ_n) be a locally convex complete metric linear space, with $S_1 \subset S_2 \subset \cdots$ and $\rho_n(x_m, x) \to 0$ if and only if $\rho_k(x_m, x) \to 0$ for $k > n$, if $x_m \in S_n$ for all m. Let S be the union of all the S_n. The inductive limit topology T on S is the finest locally convex topology coarser than the ρ_n topology on S_n for each n. Let C be the sequential convergence defined by $x_m \to_C x$ if and only if for some n, $x_m \in S_n$ for all m and $\rho_n(x_m, x) \to 0$. Then it is clear that $T = T_c(C)$. Let $\sigma_n(x, y) = \rho_n(x, y)$ for $x, y \in S_n$, otherwise $\sigma_n(x, y) = +\infty$ if $x \neq y$, and $\sigma_n(x, x) = 0$. Let

$$\rho(x, y) = \sum_n \arctan \sigma_n(x, y)/2^n,$$

where $\arctan(+\infty) = \pi/2$. Each σ_n satisfies the triangle inequality (even where it is infinite), so since $\arctan(u + v) \leqq \arctan u + \arctan v$ for any nonnegative u and v, ρ is a metric. Let $f(x)$ be the least n such that $x \in S_n$. Then clearly $C = C(\rho, f)$. Since f is an LS-function for C, (S, C) is an LS-space, and by Theorem 7.6 $C(T_c(C)) = C$. Thus we have proved

THEOREM 7.7. *If (S, T) is an LF-space then it is a CS-space and $(S, C(T))$ is an LS-space.*

Conversely, LS-spaces are a generalization of LF-spaces in which linear subspaces are replaced by convex subsets.

If (S, ρ) is a Banach space, then the function f in the proof of Theorem 7.1 is the norm in S^*. This norm also defines a standard topology. To compare the topology $T(C(\sigma, f))$ with the norm and weak* topologies, let us review some definitions and known facts. If S is a topological linear space and X is a linear set of linear functionals on S, the "X topology" $T(X)$ on S is the weakest topology such that each member of X is continuous, and the "bounded X topology" $T_b(X)$ is the strongest topology on S yielding the same relative topology as $T(X)$ on each bounded set (see [6, pp. 425–430]). If X is a Banach space and $S = X^*$ is its dual space, a neighborhood base for $T_b(X)$ at the origin in S is given by all sets of the form

$$\{s \in S: |s(x_n)| < 1, \ n = 1, 2, \cdots\}$$

where $\|x_n\| \to 0$, $x_n \in X$. Thus if X is infinite-dimensional and T is the norm topology on X^*,

$$T(X) \subset T_b(X) \subset T$$

and both inclusions are strict. From the present viewpoint we can add the following:

THEOREM 7.8.   *If X is a separable, infinite-dimensional Banach space, then*

$$T(C(T(X))) = T_c(C(T(X))) = T_b(X)$$

on X^*, so that $T_b(X)$ is both sequential and a CS-topology but $T(X)$ is neither.

Proof. Since X is separable, $T(X)$ yields a metrizable relative topology on each bounded set in X^* (see [6, p. 426]). By the Banach-Steinhaus theorem a sequence convergent for $T(X)$ is bounded. Thus $T(C(T(X)))$ is the strongest topology equal to $T(X)$ on bounded sets, i.e., $T_b(X)$. Since $T_b(X)$ is locally convex, it is also equal to $T_c(C(T(X)))$. Thus $T_b(X)$ is the weakest sequential or CS-topology finer than $T(X)$, and $T(X)$ has neither property, q.e.d.

A base for $T_b(X)$ at 0 was mentioned above, while Theorem 7.3 also furnishes a base according to Theorems 7.1 and 7.6. It can be proved directly (without 7.1 and 7.6) that these bases define the same topology.

A topology $T(C)$ on a linear space, not defined by a metric, is seldom locally convex (equal to $T_c(C)$). Theorem 8.5 below shows when this occurs for complete LS-spaces. Here $T_b(X)$ was locally convex since each $S(n)$ is ρ-compact.

Let X and Y be Banach spaces, X separable, and let $\mathscr{B} = \mathscr{B}(X, Y)$ be the linear space of bounded linear operators from X to Y. Let T be the "strong" topology on $\mathscr{B}$ for which a base at 0 is given by all sets of the form

$$\{B \in \mathscr{B}: \max(\|Bx_1\|, \cdots, \|Bx_n\|) < \varepsilon\},$$

where $\varepsilon > 0$ and $\{x_1, \cdots, x_n\}$ is any finite subset of X. Then $C(T)$ is convergence of $B(x)$ in the norm of Y for each $x \in X$. By the Banach-Steinhaus theorem, a $C(T)$-convergent sequence in $\mathscr{B}$ is uniformly bounded in the operator norm. Thus $C(T) = C(\rho, f)$ where convergence for ρ is equivalent to convergence of $B(x)$ in the norm of Y for each x in a countable dense set in X, and f is the operator norm:

$$f(B) = \sup_{\|x\| \neq 0} \|B(x)\| / \|x\|.$$

It is easy to see that $(\mathscr{B}, C(T))$ is an LS-space by (ρ, f). We can then infer from Theorem 7.3 that $T_c(C(T))$ is strictly stronger than T if X is infinite-dimensional.

There is an important difference between this situation and that of Theorem 7.8 if X and Y are both infinite-dimensional in that $\{B \in \mathscr{B} : f(B) \leq n\}$ is not T-compact. In fact, Theorem 8.5 below implies that $T(C(T))$ is strictly stronger than $T_c(C(T))$.

8. Completeness and category arguments.

DEFINITION. A sequence $\{x_n\}$ in an L^*-linear space (S, C) is a C-*Cauchy sequence* if $m(n) \geq n$ for all n implies $x_n - x_{m(n)} \to_C 0$. If every such sequence is C-convergent, (S, C) is *complete* (S is C-*complete*).

If S is the space of Borel functions on the unit interval $[0,1]$ and C is pointwise convergence, then S is C-complete but is not complete for the $T_c(C)$ uniformity. However, we have

THEOREM 8.1. *A complete LS-space (S, C) by (ρ, f) is complete for $T_c(C)$.*

Proof. Again, we use an adaptation of a method indicated by N. Bourbaki [3, Chapitre II, §2, Problème 9, p. 68].

Let $\mathscr{F}$ be a Cauchy filter in S for $T_c(C)$. Let $\mathscr{S}$ be the filter with a base of all sets of the form $F + U$, where $F \in \mathscr{F}$ and U is a neighborhood of 0 in S for $T_c(C)$. Then $\mathscr{S}$ is also a Cauchy filter, and $\mathscr{S}$ is convergent if and only if there is a $p \in S$ with $p \in V$ for all $V \in \mathscr{S}$.

If for some n, $V \cap S_n$ is nonempty for all $V \in \mathscr{S}$, then the set $\mathscr{S}_n$ of all $V \cap S_n$, $V \in \mathscr{S}$, is a filter in S_n. $\mathscr{S}_n$ is a ρ-Cauchy filter and S_n is complete for ρ by lower semi-continuity of f and L^*-completeness. Thus $\mathscr{S}_n$ converges for ρ to some point $x \in S_n$, i.e., for any $F \in \mathscr{F}$ and neighborhood U of 0 in S, x belongs to the ρ-closure of $(F + U/2) \cap S_n$, so $x \in F + U$. Thus $\mathscr{S}$ converges to x and $\mathscr{F}$ does also.

Now suppose that for each $n = 1, 2, \cdots$, there is a $V_n \in \mathscr{S}$ with $V_n \cap S_n$ empty. We may assume $V_n = F_n + U_n$ where $F_n \in \mathscr{F}$, U_n is a convex, symmetric neighborhood of 0 in S, and the U_n form a decreasing sequence of sets. Then $Y_n = \frac{1}{2}(U_{2n} \cap S_n)$ is a convex symmetric neighborhood of 0 in S_n. Let W_n be the convex hull of $\frac{1}{2}U_{2n}$ and all the Y_k for $k < n$; then the W_n form a decreasing sequence of sets all including W, the convex hull of all the Y_k. W is a convex symmetric neighborhood of 0 in S.

Now, if $q \in W_n$, then $q = r + s$ where $r \in S_n$ and $s \in \frac{1}{2}U_{2n}$. If

$$p \in P_n = F_{2n} + \tfrac{1}{2}U_{2n}, \quad f(p + q) \geq f(p + s) - f(r) > 2n - n = n,$$

i.e., $p + q \notin S_n$.

Thus for all n, $P_n + W$ does not intersect S_n. Since W is a neighborhood of 0, there are $u \in S$ and $Q \in \mathscr{S}$ such that $Q \subset u + W$. Suppose $f(u) \leq n$; then $Q \subset S_n + W = S_n - W$, so that Q does not intersect P_n, contradicting the fact that $\mathscr{S}$ is a filter. Thus the proof is complete.

We now prove two results using only the formally weaker condition of sequential completeness; the first is a "Banach-Steinhaus theorem" or "principle of uniform boundedness."

THEOREM 8.2. *Let (S,C) be a complete LS-space by (ρ,f) and $\mathcal{N}$ a collection of pseudo-norms on S, each continuous for C, such that for every $x \in S$,*

$$M(x) = \sup_{N \in \mathcal{N}} N(x) < +\infty.$$

Then M is a continuous pseudo-norm on S for $T_c(C)$.

Proof. First, if N is a sequentially continuous pseudo-norm and $\varepsilon > 0$, then $V_\varepsilon = \{x : N(x) < \varepsilon\}$ is convex and belongs to $T(C)$, hence to $T_c(C)$, while for any $A \geq 0, \{x : N(X) > A\}$ is a union of sets $y + V_\varepsilon$ where $N(y) > A + \varepsilon$. Thus N is continuous for $T_c(C)$.

Each $S_n, n = 1, 2, \cdots$, is ρ-complete, so by the Baire category theorem there is a positive integer m such that the ρ-closed set

$$A_{n,m} = \{x \in S_n : M(x) \leq m\}$$

has a nonempty interior. Thus $A_{n,m}$ is a neighborhood of 0 in S_n. Clearly M is a pseudo-norm, and $A_{n,m}/m$ is included in $A_{n,1}$ which is hence also a neighborhood of 0 in S_n, so that for some $\delta_n > 0$, $Co(n, \delta_n) \subset A_{n,1}$. If $x \in Co(n, \delta_n/2^n)$, then $2^n x \in Co(n, \delta_n)$ so that $M(x) \leq 1/2^n$. Thus

$$U_{\{\delta_n/2^n\}} \subset \{x \in S : M(x) \leq 1\}$$

so that M is $T_c(C)$-continuous, q.e.d.

Using Theorem 8.2 we can make a further step along the line begun in Theorem 7.1 by proving the following:

THEOREM 8.3. *If (S,C) is a complete LS-space by (ρ,f) with a countable dense subset, then in its dual space S^*, weak* sequential convergence C_w is countably quasi-metric.*

Proof. Since $(S, T(C(\rho,f)))$ has a countable dense subset so does the metric space (S, ρ) and each subspace (S_n, ρ). Let X be a union of countable sets dense in (S_n, ρ) for $n = 1, 2, \cdots$.

Let σ be a metric on S^* such that convergence for σ is equivalent to pointwise convergence on X. For $n = 1, 2, \cdots, A \in S^*$, let

$$g_n(A) = \inf\{\lambda > 0 : |A(x)| < 1/n \text{ for } x \in S(n, 1/\lambda)\}.$$

Continuity of A implies $g_n(A)$ finite for all n. (Incidentally, we do not claim that the g_n are LS-functions, and leave open the question of whether they can be so chosen.) Let G be the set of all $g_n, n = 1, 2, \cdots$.

First, if $A_k \to_{C(\sigma,G)} A$, let $y \in S$, $\varepsilon > 0$, and $n > \max(f(y), 1/\varepsilon)$. Then there are $x_j \in S_n \cap X$ with $\rho(x_j, y) \to 0$, so that since $g_{2n}(A_k - A)$ is bounded and $f(x_j - y) \leqq 2n$ for all j, $|(A_k - A)(x_j - y)| < 1/2n < \varepsilon/2$ for all k if j is large enough, so since $(A_k - A)(x_j) \to 0$ for each j, $|(A_k - A)(y)| < \varepsilon$ for large k, so $(A_k - A)(y) \to 0$, $A_k \to_{C_w} A$.

Conversely, if $A_k \to_{C_w} A$ clearly $\sigma(A_k, A) \to 0$. By Theorem 8.2 there is a $\delta_n > 0$ such that if $x \in S(n, \delta_n)$, $|A_k(x)| < 1/n$, so that $g_n(A_k) \leqq 1/\delta_n$ for all k, and $A_k \to_{C(\sigma,G)} A$, q.e.d.

We now give a necessary and sufficient condition that $T(C) = T_c(C)$ for complete LS-spaces, using

LEMMA 8.4. *If (S,C) is a complete LS-space by (ρ,f) and for some n and $\varepsilon > 0$ $\bar{S}(n,\varepsilon)$ is (ρ-) compact, then $\overline{Co}(n,\varepsilon)$ is (ρ-) compact.*

Proof. Both sets $\bar{S}(n,\varepsilon)$ and $\overline{Co}(n,\varepsilon)$ are included in $S(n)$ and hence their relative $T_c(C)$ topologies are both the ρ topology by Theorem 7.5 and definition of quasi-metric convergence. Thus ρ-compactness for them is equivalent to $T_c(C)$ compactness. Also $(S, T_c(C))$ is complete by Theorem 8.1 and Hausdorff by Theorem 7.5. Thus we need only apply the fact that the closed convex hull of a compact set in a complete locally convex Hausdorff linear space is compact [3, Chapitre II, §4, 1, Problème 2, p. 80, Corollaire, p. 81].

THEOREM 8.5. *If (S,C) is a complete LS-space by (ρ,f) then $T(C) \neq T_c(C)$ if and only if both the following conditions hold:*

(a) *There is an n such that $\bar{S}(n,\varepsilon)$ is not compact for any $\varepsilon > 0$.*

(b) *There are arbitrarily large values of m such that 0 is the limit of a sequence in $S_m \sim S_{m-1}$.*

Proof. First suppose (a) is false. Then for each $n = 1, 2, \cdots$, there is an $\varepsilon_n > 0$ such that $\bar{S}(n, \varepsilon_n)$ is compact (for ρ). Let $0 \in U \in T(C)$. There is a δ_1, $0 < \delta_1 \leqq \varepsilon_1$, such that $\overline{Co}(1, \delta_1) \subset U$. Given $\delta_1, \cdots, \delta_n$, $0 < \delta_j \leqq \varepsilon_j$, $j = 1, \cdots, n$, such that

$$A_n = B_{n-1} + Co(n, \delta_n) \subset U,$$

where $B_n = \sum_{j=1}^n \overline{Co}(j, \delta_j/2)$, let

$$\delta_{n+1}(x) = \sup\{\delta : x + Co(n+1, \delta) \subset U\}.$$

$\delta_{n+1}(x) > 0$ for each $x \in A_n$ by L^*-convexity. Since B_n is ρ-compact by Lemma 8.4, there is a $\delta_{n+1} > 0$ such that $\delta_{n+1}(x) \geqq \delta_{n+1}$ for all $x \in B_n$, $\delta_{n+1} \leqq \varepsilon_{n+1}$. Thus by induction we obtain a sequence $\{\delta_n\}$ such that $U_{\{\delta_n/2\}} \subset U$, so that U is a $T_c(C)$-neighborhood of 0 and $T(C) = T_c(C)$.

Next, suppose (b) is false. Then there is an m_0 such that for $m > m_0$ there is a $\delta_m > 0$ such that $S(m, \delta_m) \subset S(m_0, \delta_m)$. If $0 \in U \in T(C)$, then there are δ_m

for $1 \leqq m \leqq m_0$ such that $\sum_{m=1}^{m_0} Co(m, \delta_m) \subset U$, and so $U_{\{\delta_m/2^m\}} \subset U$, so $T(C) = T_c(C)$.

Now suppose (a) and (b) both hold. Let n_1 be such that $\bar{S}(n_1, 1/m)$ is not compact for any $m = 1, 2, \cdots$. Since each $\bar{S}(n_1, 1/m)$ is ρ-complete, there exist for each m an $\varepsilon_m > 0$ and an infinite sequence y_{mj}, $j = 1, 2, \cdots$, of elements of $\bar{S}(n_1, 1/m)$ such that $\rho(y_{mj}, y_{mk}) \geqq \varepsilon_m$ for $j \neq k$.

Also, according to (b) there are $n_2, n_3, \cdots$, $n_1 < n_2 < n_3 \cdots$, and infinite sequences $\{z_{mr}\}$ such that $z_{mr} \in S(n_m) \sim S(n_m - 1)$ and $\rho(0, z_{mr}) < 1/r$ for all m and r. We shall assume $n_{m+1} > n_m + 2n_1 + 1$ for all m.

Since f is lower semi-continuous there is an $\varepsilon_{mr} > 0$ for each m and r such that if $u \in S(n_{m-1} + 2n_1)$, $\rho(u, z_{mr}) > \varepsilon_{mr}$.

Let $U_1 = S(2n_1, 2)$. Given $U_1, \cdots, U_{m-1}$, let $x \in U_m$ if and only if $x \in S(n_1 + n_m)$ and there is a $y \in U_{m-1}$ such that $\rho(x, y) < \delta(m, y)$, where

$$\delta(m, y) = \min(\varepsilon(m, y), \gamma(m, y), \varepsilon_m/4)$$

and ε and γ are defined as follows:

$$\varepsilon(m, y) = \inf\{\rho(y, w): w \in S(n_{m-1} + n_1) \sim U_{m-1}\},$$

and

$$\gamma(m, y) = \inf_j[\rho(y, y_{mj}) - \varepsilon_m/3]$$

if this is positive; if not, then $\rho(y, y_{mj}) < \varepsilon_m/2$ for exactly one value of j, with $\rho(y, y_{mj}) \leqq \varepsilon_m/3$, and we define $\gamma(m, y) = \varepsilon_{mj}$.

It is clear that U_m is relatively open in $S(n_m + n_1)$ for all m and that $U_{m+1} \cap S(n_m + n_1) = U_m$ for all m since $\delta(m, y) \leqq \varepsilon(m, y)$. Thus the union U of all the U_m belongs to $T(C)$, and $U \cap S(n_m + n_1) = U_m$ for all m.

Now suppose $U \in T_c(C)$. Then since $0 \in U$ there is a sequence $\{\delta_i\}$ of positive numbers such that $U_{\{\delta_i\}} \subset U$. Fix m with $1/m < \delta_{n_1}$. Then if $1/r < \delta_{n_m}$, $y_{mj} + z_{mr}$ belongs to U for all j, and since $y_{mj} + z_{mr} \in S(n_m + n_1)$ there is for all j a $v_{jr} \in S(n_{m-1} + n_1)$ such that

$$\rho(v_{jr}, z_{mr} + y_{mj}) < \delta(m, v_{jr}).$$

Since $\rho(y_{mj}, z_{mr} + y_{mj}) = \rho(0, z_{mr}) < 1/r$,

$$\rho(v_{jr}, y_{mj}) < 1/r + \delta(m, v_{jr}) \leqq 1/r + \varepsilon_m/4.$$

Thus for r large enough, $\rho(v_{jr}, y_{mj}) < \varepsilon_m/3$ for all j so that $\gamma(m, v_{jr}) = \varepsilon_{mj}$. Thus

$$\rho(z_{mr}, v_{jr} - y_{mj}) = \rho(v_{jr}, z_{mr} + y_{mj}) < \varepsilon_{mj}$$

for all large r and all j, but $v_{jr} - y_{mj} \in S(n_{m-1} + 2n_1)$, so this contradicts the definition of ε_{mr} if we take $j = r$ for r sufficiently large, and the proof is complete.

9. **Topologies on test functions and distributions.** Let $\mathcal{D}'$ be the space of distributions on a Euclidean space, T_w its standard weak* topology as dual of $\mathcal{D}$, and T the "strong" topology of uniform convergence (of nets or filters) on bounded sets in $\mathcal{D}$ (see Schwartz [12, Tome I, Chapitre III, §3]). Now $C(T) = C(T_w)$ (ibid., Théorème XIII, p. 74) and T is locally convex, so the topology $T_s = T_c(C(T_w))$ is at least as strong as T:

$$T_w \subset T \subseteq T_s.$$

$\mathcal{D}'$, as well as the spaces $\mathcal{E}'$ and $\mathcal{S}'$ to be discussed later, are bornological with their strong topologies (see Schwartz [13, I, p. 44] and Grothendieck [8, Théorème 10, p. 85]). Thus T is a CS-topology by Theorem 6.3, so

$$T_s = T_c(C(T)) = T.$$

Thus by Theorem 6.1, sequential continuity is equivalent to continuity for linear maps from $(\mathcal{D}', T)$ to other CS-spaces.

The convergence $C(T)$ on $\mathcal{D}'$, being equal to weak* convergence, is countably quasi-metric by Theorem 8.3. A direct proof of this, including an explicit form of the metric σ and functions g_n, can be obtained from [12, Chapitre III, §6, Théorème XXIII, p. 86].

An alternate method of proving that $T = T_s$ is to show that $\mathcal{D}$ is the dual of $\mathcal{D}'$ with the topology T_s, and to apply Mackey's theorem [3, Chapitre IV, §2, Théorème 2, p. 68, Corollaire, p. 69].

Now let us consider other spaces of test functions and distributions, first the space $\mathcal{E}$ of all C^∞ fnctions on R^k and its dual $\mathcal{E}'$ [12, Chapitre III, §7]. The usual topology on $\mathcal{E}'$ is that of uniform convergence of each partial derivative on each compact set. Since this defines a metrizable topology T on $\mathcal{E}$, $T = (C)$ where C is sequential convergence in the sense described above, and $T(C) = T_c(C)$ since T is locally convex. Thus the dual space $\mathcal{E}'$ of $\mathcal{E}$ in the usual sense is the set of sequentially continuous complex linear functions on $\mathcal{E}$. Since $\mathcal{D}$ is a dense subset of $\mathcal{E}$ and has a finer topology, $\mathcal{E}'$ may be identified with a subset of $\mathcal{D}'$, namely the distributions with compact support (see [12, Chapitre III, §7, Théorème XXV, p. 89]).

A sequence of members of $\mathcal{E}'$ convergent pointwise on $\mathcal{E}$ is equicontinuous by the Banach-Steinhaus theorem and hence uniformly convergent on compact sets. Also, $\mathcal{E}$ is a Montel space: a bounded closed set in $\mathcal{E}$ is compact [12, Chapitre III, §7, p. 89] and so $C(T) = C(T^*)$ on $\mathcal{E}'$ where T^* is the weak* and T the strong topology. Thus $T_c(C(T^*)) \supset T$.

Since $\mathcal{E}'$ is bornological, $T_c(C(T^*)) = T_c(C(T)) = T$ in this case also. Again, the alternate method of proof based on Mackey's theorem is available. $C(T)$ is simple quasi-metric by Theorem 7.1.

The situation described above for $\mathcal{E}$ and $\mathcal{E}'$ is the same for $\mathcal{S}$, the space of

rapidly decreasing C^∞ functions, and its dual $\mathscr{S}'$ (the tempered distributions), as defined in [12, Tome II, Chapitre VII].

Returning to $\mathscr{D}$ with its usual convergence $C = C(\rho, f)$, it is easy to verify that hypotheses (a) and (b) of Theorem 8.5 are satisfied, so that $T(C)$ is not locally convex. Hence by Theorem 7.4, $(\mathscr{D}, T(C))$ is not a topological linear space, and $T(C)$ is pathological as compared with the usual strong topology $T_c(C)$ on $\mathscr{D}$. The same is true in the LS-spaces $\mathscr{E}'$ and $\mathscr{S}'$.

$(\mathscr{D}, C)$ and its dual $\mathscr{D}'$ with weak* convergence C (which is actually strong convergence) provide an example of two L^*-spaces on whose product the topology $T(C'')$ of the product convergence C'' of C and C' is not the product topology of $T(C)$ and $T(C')$. The function $(\phi, L) \to L(\phi)$ on $\mathscr{D} \times \mathscr{D}'$ to the complex numbers is continuous for C'' [12, Tome II, Chapitre III, §3, Théorème XI, p. 73] and hence for $T(C'')$ by Theorem 2.2, but it is not continuous for the product topology of $T(C)$ and $T(C')$ since a neighborhood of 0 for $T(C')$ contains all distributions which vanish on a sufficiently large open set, and a neighborhood of 0 for $T(C)$ contains some functions whose support is not included in the closure of this open set. Thus the topologies are different.

$T_c(C'')$ and the product topology of $T_c(C)$ and $T_c(C')$ are equal by Theorem 6.2. $(\phi, L) \to L(\phi)$ is not continuous for this topology since it is weaker than the product of $T(C)$ and $T(C')$. (Note that this is a bilinear, not a linear function and Theorem 6.1 does not apply.)

Let us sum up the information we have obtained on sequential convergence and the theory of distributions. It appears that all the spaces of test functions and distributions (including those not specifically discussed here) are bornological, and hence CS-spaces, so that continuity is always equivalent to sequential continuity for linear maps. The types of sequential convergence arising are metric, simple quasi-metric, and countably quasi-metric; if a space of test functions has one type of convergence, its dual space will have the next more complicated type (although this is not true for spaces of distributions, whose duals are the corresponding spaces of test functions). Weak and strong sequential convergence will coincide since the spaces are Montel spaces. Finally, the topologies $T(C)$ are not locally convex nor compatible with vector space structures unless they are metrizable, so that $T_c(C)$, the usual strong topology, must be used in each case.

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