NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

BY
ROBERT H. OEHMKE

1. A finite-dimensional power-associative algebra $\mathfrak{A}$ is said to be nodal [6] if every element of $\mathfrak{A}$ can be written as $\alpha 1 + z$ where $\alpha \in \mathfrak{F}$, 1 is the unity element of $\mathfrak{A}$ and z is nilpotent and if the set of all nilpotent elements is not a subalgebra of $\mathfrak{A}$.

In [3; 4], Kokoris has shown that every simple nodal noncommutative Jordan algebra of characteristic $p \neq 2$ has the form $\mathfrak{A} = \mathfrak{F}1 + \mathfrak{N}$ with $\mathfrak{N}^+ = \mathfrak{F}[x_1, \cdots, x_n]$ for some n where the generators are all nilpotent of index p and the multiplication is associative. If f and g are two elements of $\mathfrak{A}$ then the multiplication table of $\mathfrak{A}$ is given by

$$fg = f \circ g + \frac{1}{2} \sum_{i,j} \frac{\partial f}{\partial x_i} \circ \frac{\partial g}{\partial x_j} \circ c_{ij}$$

where the circle product is the product in $\mathfrak{A}^+$ and

$$c_{ij} = x_i x_j - x_j x_i.$$

In [7] Schafer considers nodal noncommutative Jordan algebras defined by a skew-symmetric bilinear form (i.e., $c_{ij} \in \mathfrak{F}$) and those with two generators. All of these algebras are Lie-admissible (i.e., $\mathfrak{A}^-$ is a Lie algebra). Schafer obtained the derivation algebras of these algebras defined by a skew-symmetric bilinear form.

Here, we examine all simple nodal noncommutative Jordan algebras that are Lie-admissible over a field $\mathfrak{F}$ of characteristic $p \neq 2$. First a set of generators is obtained having properties suitable for further study. This set of generators is then used to find the algebras $D(\mathfrak{A})$ of derivatives of $\mathfrak{A}$ and the algebras adj $\mathfrak{A}^-$ and (adj $\mathfrak{A}^-)'$. Schafer has shown that all of the simple Lie algebras defined by Block [1] can be realized as (adj $\mathfrak{A}^-)'$ for some $\mathfrak{A}$ that is simple, nodal noncommutative Jordan and Lie-admissible. Hence we have obtained a somewhat different formulation of these algebras. The question remains whether all of these algebras, (adj $\mathfrak{A}^-)'$, are in the class defined by Block. It is our intention to investigate this question in a subsequent paper.

2. We define the mapping $D_y = D(y)$ by

$$xD_y = xy - yx.$$

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Then $D_y = R_y - L_y$ where R_y and L_y are the right and left multiplications by y on $\mathfrak{A}$.

A derivation of an algebra $\mathfrak{B}$ is a linear transformation T on $\mathfrak{B}$ into $\mathfrak{B}$ such that for $x, y \in \mathfrak{B}$

$$(xy)T = (xT)y + x(yT).$$

Since $\mathfrak{A}^-$ is a Lie algebra D_y is the right multiplication by y of $\mathfrak{A}^-$ and is a derivation of $\mathfrak{A}^-$. By expanding

$$2(x \circ y)D_z - 2(xD_z) \circ y - 2x \circ (yD_z)$$

in terms of the multiplication of $\mathfrak{A}$ and using the flexible law

$$(xy)z + (zy)x = x(yz) + z(yx)$$

we see that D_z is also a derivation of $\mathfrak{A}^+$ and hence of $\mathfrak{A}$.

It is well known [2, p. 108] that any set of n elements of $\mathfrak{N}$ whose cosets form a basis of the n-dimensional space $\mathfrak{N} - \mathfrak{N} \circ \mathfrak{N}$ can serve as a set of generators of $\mathfrak{A}^+$. This result shall be our chief tool in the proof of the following theorem.

THEOREM 1. *Let $\mathfrak{A}$ be a simple, Lie-admissible, nodal noncommutative Jordan algebra over a base field $\mathfrak{F}$ of characteristic $p \neq 2$. If $\mathfrak{A}^+$ has an even number of generators then a set of generators $x_1, \cdots, x_{2r}$ can be chosen for $\mathfrak{A}^+$ so that*

$$\begin{aligned} x_i D(x_{i+r}) &= 1 + \alpha_i x_i^{p-1} \circ x_{i+r}^{p-1}, \quad i = 1, \cdots, r, \\ x_i D(x_j) &= 0, \qquad\qquad j \neq i + r, \end{aligned} \tag{1}$$

with $\alpha_i \in \mathfrak{F}$. If $\mathfrak{A}^+$ has an odd number of generators then a set of generators $x_1, \cdots, x_{2r+1}$ can be chosen for $\mathfrak{A}^+$ so that (1) *is satisfied and*

$$\begin{aligned} x_{2r+1} D(x_j) &= 0, \qquad j = 1, \cdots, 2r-2, \\ x_{2r+1} D(x_{2r}) &= x_{2r-1}^{p-1} \circ (1 + \beta x_{2r+1}^{p-1}), \\ x_{2r+1} D(x_{2r-1}) &= x_{2r}^{p-1} \circ \alpha(1 + \beta x_{2r+1}^{p-1}), \end{aligned} \tag{2}$$

with α and β in $\mathfrak{F}$.

Proof. Since $\mathfrak{A}$ is simple $\mathfrak{N}$ can not be an ideal of $\mathfrak{A}^-$. For if $\mathfrak{N}$ is an ideal of $\mathfrak{A}^-$ then since it is an ideal of $\mathfrak{A}^+$ it would be closed under both the operations $R_y - L_y$ and $R_y + L_y$ for $y \in \mathfrak{A}$. Therefore it would be also an ideal of $\mathfrak{A}$. Hence there must be a pair of generators x and y such that yD_x is nonsingular. Since y can be replaced by αy for any α in $\mathfrak{F}$ we assume

$$yD_x = 1 + m \circ y^k = b^{-1}. \tag{3}$$

We also assume y has been chosen so that k is a maximum. If $k < p - 1$ then letting $q = (y - 1/(k+1))y^{k+1} \circ m \circ b$ we have

$$qD_x = 1 - \frac{1}{k+1} y^{k+1} \circ (m \circ b) D_x$$
$$= 1 + q^{k+1} \circ m'$$

which contradicts the choice of k. Hence we can assume in (3) that $k = p - 1$.

We now write (3) as

$$yD_x = 1 + y^{p-1} \circ x^t \circ m' = b^{-1}$$

and assume that y and x have been chosen so that t is a maximum. If $t < p - 1$ then, as above, we can replace x by $x - 1/(t+1) \circ x^{t+1} \circ m' \circ y^{p-1} \circ b$ to obtain a contradiction to our choice of t. Hence we can assume x and y have been chosen so that

(4) $$yD_x = 1 + m_y \circ y^{p-1} \circ x^{p-1}.$$

If z is a third generator, in the same way that we altered the generator y, we can add an element q of $y \circ \mathfrak{A}$ to z to obtain the property

$$(z + q)D_x \in y^{p-1} \circ \mathfrak{A}.$$

Hence we assume that all generators z different from x and y have been chosen so that

(5) $$zD_x = y^{p-1} \circ m_z.$$

Since for any q in $\mathfrak{A}$ we have D_q a derivation of both $\mathfrak{A}$ and $\mathfrak{A}^-$ then

(6) $$zD_yD_x = zD_xD_y - yD_xD_z.$$

If (4) and (5) are substituted in (6) we have

(7) $$zD_yD_x = y^{p-1} \circ m_zD_y + y^{p-2} \circ x^{p-1} \circ m_y \circ yD_z - y^{p-1} \circ x^{p-1} \circ m_yD_z.$$

But the right-hand side of (7) is in $y^{p-2} \circ \mathfrak{A}$; so also is the left-hand side. From (4) and (5) the only possible way for this to happen is to have

$$zD_y = n_0 + y^{p-1} \circ n_1$$

in which n_i is independent of y. (i.e., n_i is a polynomial in which y does not appear.) In (7) this implies

$$n_0D_x - y^{p-2} \circ n_1 = y^{p-1} \circ m_zD_y - y^{p-2} \circ x^{p-1} \circ m_y \circ n_0 - y^{p-1} \circ x^{p-1} \circ m_yD_z$$

and

(8) $$n_1 = x^{p-1} \circ m_y \circ n_0.$$

Write $n_0 = x^k \circ t$. If $k < p - 1$ we can replace the generator z by the generator $z + 1/(k+1) \circ x^{k+1} \circ t = z'$ to get

$$\begin{aligned} z'D_y &= n_0 + y^{p-1} \circ m_y \circ n_0 \circ x^{p-1} + x^k \circ t \circ xD_y + \frac{1}{k+1} x^{k+1} \circ tD_y \\ &= y^{p-1} \circ x^{p-1} \circ m_y \circ n_0 + \frac{1}{k+1} x^{k+1} \circ tD_y \\ &= n_0' + y^{p-1} \circ x^{k+1} \circ m_y \circ n_1' ; \end{aligned}$$

in which n_i' is again independent of y. Note that if (5) holds and z is replaced by a generator $z + q$ in which q is independent of y then (5) will be retained.

Again arguing on the maximum value of k that can be obtained in the expression $n_0 = x^k \circ t$ we can conclude that $k = p - 1$, $n_1 = 0$ and

$$(9) \qquad \begin{aligned} zD_y &= x^{p-1} \circ n_z, \\ zD_x &= y^{p-1} \circ m_z \end{aligned}$$

in which n_z is independent of y.

Identity (7) can now be reduced to

$$(10) \qquad x^{p-1} \circ n_z D_x = y^{p-1} \circ m_z D_y - y^{p-1} \circ x^{p-1} \circ m_y D_z.$$

For a particular choice of a set of generators including x and y satisfying (4) assume there are two distinct generators w and z (both satisfying (9)). Write

$$(11) \qquad m_z = \sum x^i \circ m_i, \quad m_0 = \sum w^i \circ n_i.$$

(When obvious, we shall omit index and range of the summation.) Then

$$m_z D_y = - \sum i x^{i-1} \circ m_i + \sum x^i \circ m_i D_y - m_0 \circ y^{p-1} \circ x^{p-1}.$$

But from (10) $y^{p-1} \circ m_z D_y \in x \circ \mathfrak{A}$. Therefore

$$(12) \qquad \begin{aligned} &- y^{p-1} \circ m_1 + y^{p-1} \circ m_0 D_y \in x \circ \mathfrak{A}, \\ &- y^{p-1} \circ m_1 + y^{p-1} \circ \sum i w^{i-1} \circ n_i \circ wD_y + \sum w^i \circ n_i D_y \in x \circ \mathfrak{A}. \end{aligned}$$

If w is replaced as a generator by $w' = w - x$ then (9) still holds for z and hence so do the corresponding relationships (12). Note that if $P(w)$ is a polynomial in w then $P(w) - P(w + x) \in x \circ \mathfrak{A}$ and $wD_y - (w + x)D_y - 1 \in x \circ \mathfrak{A}$. If we write q' for $q = q(w)$ with w replaced by $w + x$ then $w'D_x = y^{p-1} \circ m_z'$; $m_z' = \sum x^i \circ m_i'$; $m_0' = \sum w^i \circ n_i'$ and from (11) we have

$$\begin{aligned} 0 &\equiv - y^{p-1} \circ m_1' + y^{p-1} \circ \sum i w'^{i-1} \circ n_i' \circ w'D_y + \sum w'^i \circ n_i' D_y \\ &\equiv - y^{p-1} \circ m_1 + y^{p-1} \circ \sum i w^{i-1} \circ n_i \circ w'D_y + \sum w^i \circ n_i D_y \\ &\equiv - y^{p-1} \circ m_1 + y^{p-1} \circ \sum i w^{i-1} \circ n_i \circ (wD_y - 1) + \sum w^i \circ n_i D_y \end{aligned}$$

modulo $x \circ \mathfrak{A}$.

But this implies $y^{p-1} \circ \sum i w^{i-1} \circ n_i \in x \circ \mathfrak{A}$. Therefore $y^{p-1} \circ n_i \in x \circ \mathfrak{A}$ for $i > 0$.

Now assume that in (11) we have chosen the m_i to be independent of x. Then since m_z is independent of y and m_0 is independent of x we have $n_i = 0$ for $i > 0$. Hence m_0 is independent of w. Since w was arbitrary we must have m_0 a polynomial in the single generator z. But then $y^{p-1} \circ m_0 D_y \in x \circ \mathfrak{A}$ by (9) and $y^{p-1} \circ m_1 \in x \circ \mathfrak{A}$ by (12). However m_1 is independent of x and y. Hence $m_1 = 0$.

Once again looking at (12) we have

$$y^{p-1} \circ m_z D_y \equiv - y^{p-1} \circ \sum i x^{i-1} \circ m_i + y^{p-1} \circ \sum x^i \circ m_i D_y \equiv 0$$

modulo $x^{p-1} \circ \mathfrak{A}$. With $m_0 D_y$ in $x^{p-1} \circ \mathfrak{A}$ and $m_1 = 0$ we see that $m_2 = \cdots = m_{p-1} = 0$ and $m_z = m_0$ is a polynomial in z with coefficients in $\mathfrak{F}$. Similarly we obtain n_z as a polynomial in z with coefficients in $\mathfrak{F}$. Therefore if the number of generators is greater than or equal to 4 and they have been picked so that (4) and (9) hold then m_z and n_z in (9) are polynomials in the single generator z.

However if z and w are two generators distinct from x and y then z can be replaced as a generator by $z + w$. Indentity (9) still holds, i.e.,

$$(z + w)D_x = y^{p-1} \circ m_{z+w},$$

$$(z + w)D_y = x^{p-1} \circ n_{z+w}$$

in which m_{z+w} and n_{z+w} are polynomials in the single generator $(z + w)$. But $m_{z+w} = m_z + m_w$ and $n_{z+w} = n_z + n_w$. For these sums to be polynomials in $(z + w)$, m_z and n_z must be of degree at most 1. If z is replaced by $z + z^2$ then (9) still holds for the generator $(z + z^2)$. In particular $m_z + 2z \circ m_z$ is of degree at most 1 in $(z + z^2)$. Write m_z as $\alpha + \beta z$ and $m_z + 2m_z \circ m_z$ as $\gamma + \delta(z + z^2)$. Then $\beta = 2\alpha$. Since z was arbitrary we must also have $\delta = 2\gamma$. But the same relationships that gave us $\beta = 2\alpha$ also give us $\delta = 4\gamma$, i.e., $\delta = \gamma = \alpha = \beta = 0$. Hence $m_z = 0$ and in the same manner $n_z = 0$.

We still assume we have at least two generators z and w distinct from x and y. We also assume that they have been chosen so that

$$zD_x = zD_y = wD_x = wD_y = 0. \tag{13}$$

We must have

$$wD_z D_x = wD_x D_z - zD_x D_w$$

and therefore $(wD_z)D_x = 0$. This implies that wD_z is independent of y. Similarly $(wD_z)D_y = 0$ and wD_z is independent of x. Then if we assume that all the generators distinct from x and y have been chosen so that their product in $\mathfrak{A}^-$ by either x or y is 0, we can assume that the polynomials over $\mathfrak{F}$ in these generators is an ideal $\mathfrak{J}$ of $\mathfrak{A}^-$. But then $\mathfrak{J} \circ \mathfrak{A}$ is an ideal in both $\mathfrak{A}^-$ and $\mathfrak{A}^+$ and hence in $\mathfrak{A}$. Therefore $\mathfrak{J}$ must contain a nonsingular element. This means that there are two generators w and z, distinct from x and y, such that wD_z is nonsingular.

At this point we reconsider the polynomial m_y obtained in (4). If the generators x, y, z, w have been chosen so that (4) and (13) hold and z and w are such that wD_z is nonsingular then (7) reduces to

$$y^{p-1} \circ x^{p-1} \circ m_y D_z = 0.$$

Therefore $m_y D_z$ is 0 since it is independent of both x and y. But this implies that m_y is independent of w and by symmetry m_y is independent of z. If t is a fifth generator then either tD_z or $(w+t)D_z$ is nonsingular. In either case we see that m_y is also independent of t. Hence $m_y \in \mathfrak{F}$.

We can now proceed in $\mathfrak{I}$ (defined above) with the same argument as above to obtain the result of the theorem for the even-dimensional case.

In the odd dimensional case we can proceed with the above argument until we are presented with an $\mathfrak{I}$ which is the set of polynomials over $\mathfrak{F}$ in three generators, say x, y and z. Again by the previous arguments we can assume that x, y and z have been chosen so that

$$\begin{aligned} yD_x &= 1 + y^{p-1} \circ x^{p-1} \circ m_y, \\ zD_x &= y^{p-1} \circ m_z, \\ zD_y &= x^{p-1} \circ n_z. \end{aligned}$$

Consider (7). We have

$$x^{p-1} \circ n_z D_x = y^{p-1} \circ m_z D_y - y^{p-1} \circ x^{p-1} \circ m_y D_z. \tag{14}$$

Since m_y is a polynomial in x, y and z then $y^{p-1} \circ x^{p-1} \circ m_y D_z = 0$. Also since m_z is independent of y and by (9) $m_z D_y$ is independent of y we must have $m_z D_y \in x^{p-1} \circ \mathfrak{A}$. This implies that m_z is independent of x. Hence $y^{p-1} \circ m_z D_y = \partial m_z / \partial z \circ x^{p-1} \circ y^{p-1}$ and $x^{p-1} \circ m_z D_x = \partial n_z / \partial z \circ x^{p-1} \circ y^{p-1}$. From (14) we have

$$\frac{\partial n_z}{\partial z} = \frac{\partial m_z}{\partial z}. \tag{15}$$

If both n_z and m_z are singular then zD_x and zD_y are in $z \circ \mathfrak{A}$. Hence $z \circ \mathfrak{A}$ is an ideal of $\mathfrak{A}^-$ and $\mathfrak{A}^+$. Since this denies the simplicity of $\mathfrak{A}$ we must have either m_z or n_z nonsingular. Assume $m_z = 1 + q$ in which $q \in \mathfrak{N}$. Then if l is a polynomial in z over $\mathfrak{F}$ we have

$$(z + l)D_x = y^{p-1} \circ (1 + q) + y^{p-1} \circ \frac{\partial l}{\partial z} \circ (1 + q).$$

Clearly, l can be chosen so that $\partial l / \partial z \circ (1 + q) \equiv w$ modulo $z^{p-1} \circ \mathfrak{A}$. Hence we can assume

$$zD_x = y^{p-1} + \beta y^{p-1} \circ z^{p-1}, \tag{16}$$

in which $\beta \in \mathfrak{F}$. Now since m_z is nonsingular the solutions of (15) are of the form $n_z = \alpha m_z$. Hence

$$zD_y = \alpha x^{p-1} \circ (1 + \beta z^{p-1}). \tag{17}$$

Again, let l be a polynomial in z over $\mathfrak{F}$. Then

$$(y + x^{p-1} \circ l)D_x = 1 + y^{p-1} \circ x^{p-1} \circ \left[m_y + \frac{\partial l}{\partial z} \circ (1 + \beta z^{p-1}) \right].$$

Write $m_y = \gamma + z \circ t$ and $l = \delta z + z^2 \circ l'$ in which

$$\frac{\partial(z^2 \circ l')}{\partial z} + z \circ t$$

is a multiple of z^{p-1}. Then

$$\frac{\partial(z^2 \circ l')}{\partial z} \circ (1 + \beta z^{p-1}) + z \circ t$$

is also a multiple of z^{p-1}. Now choose δ so that

$$\delta(1 + \beta_z^{p-1}) + \frac{\partial(z^2 \circ l')}{\partial z} \circ (1 + \beta z^{p-1}) + z \circ t$$

is a constant. We now have

$$(y + x^{p-1} \circ l)D_x = 1 + \gamma \circ x^{p-1} \circ y^{p-1}.$$

Since $(y + x^{p-1} \circ l)^{p-1} \circ x^{p-1} = y^{p-1} \circ x^{p-1}$ we can assume the generator y can be chosen so that

$$yD_x = 1 + \gamma x^{p-1} \circ y^{p-1}.$$

We can now repeat the construction of the z in (9) to obtain

$$zD_x = y^{p-1} \circ m_z,$$

$$zD_y = x^{p-1} \circ n_z.$$

From these we can obtain (16) and (17). Hence we have concluded the proof of the theorem.

3. Let $\mathfrak{A}$ and $\mathfrak{A}^*$ be two simple nodal algebras that are equal as vector spaces and have the same + algebras. Let there be an even number of generators $x_1, \cdots, x_{2r}$ with the multiplication in $\mathfrak{A}$ given by the $c_{ij} = x_i D(x_j)$ obtained in Theorem 1 and the multiplication in $\mathfrak{A}^*$ given by

$$c'_{i\,i+r} = 2,$$

$$c'_{ij} = 0$$

for $i = 1, \cdots, r$ and $j \neq i + r$. The algebra $\mathfrak{A}^*$ then falls into the class of simple nodal algebras defined by a skew-symmetric bilinear form and studied by Schafer [7].

Every derivation of $\mathfrak{A}$ must be a derivation of $\mathfrak{A}^+$. The derivations of $\mathfrak{A}^+$ have been given by Jacobson [2, p. 107] as

$$f \to \sum_1^{2r} \frac{\partial f}{\partial x_k} \circ a_k. \tag{18}$$

We shall denote this derivation by $(a_1, \cdots, a_{2r})$. Assume $(a_1, \cdots, a_{2r})$ is a derivation of $\mathfrak{A}$ and consider the possibility that $(b_1, \cdots, b_{2r})$ is a derivation of $\mathfrak{A}^*$ in which

$$b_i = c_{is}^{-1} \circ c'_{is} \circ a_i \tag{19}$$

and s is $i + r$ if $i \leqq r$ and is $i - r$ if $i > r$. In the same way we choose t so $t = j + r$ or $j - r$ and $t \leqq 2r$.

Consider the expression

$$\sum_{k=1}^{2r} \left(\frac{\partial c'_{ij}}{\partial x_k} \circ b_k + \frac{\partial b_i}{\partial x_k} \circ c'_{jk} + \frac{\partial b_j}{\partial x_k} \circ c'_{ki} \right) \tag{20}$$

obtained from Schafer's criteria [7, p. 312] that $(b_1, \cdots, b_{2r})$ be a derivative of $\mathfrak{A}^*$. We want to show that for all i and j (20) is 0. By the choice of the c'_{ij}'s (20) can be reduced to

$$\frac{\partial b_i}{\partial x_t} \circ c'_{jt} + \frac{\partial b_j}{\partial x_s} \circ c'_{si}$$

and by substituting the expressions (19) we have

$$c'_{jt} \circ c'_{is} \circ \left(\frac{\partial a_i}{\partial x_t} \circ c_{is}^{-1} - \frac{\partial a_j}{\partial x_s} \circ c_{jt}^{-1} \right)$$
$$+ c'_{jt} \circ c'_{is} \circ \left(\frac{\partial c_{is}^{-1}}{\partial x_t} \circ a_i - \frac{\partial c_{jt}^{-1}}{\partial x_s} \circ a_j \right).$$

For our purposes we can drop the factor $c'_{jt} \circ c'_{is}$, use the fact that if $q \in N$ then $(1 + q^{p-1})^{-1} = 1 - q^{p-1}$, and

$$c_{is} \circ c_{jt} \circ \frac{\partial c_{jt}}{\partial x_s} = c_{is} \circ c_{jt} \circ \frac{\partial c_{is}}{\partial x_t} = 0$$

to further reduce (20) to

$$\frac{\partial a_i}{\partial x_t} \circ c_{jt} - \frac{\partial a_j}{\partial x_s} \circ c_{is} + \frac{\partial c_{jt}}{\partial x_s} \circ a_j - \frac{\partial c_{is}}{\partial x_t} \circ a_i. \tag{21}$$

But the criteria that must be satisfied for $(a_1, \cdots, a_{2r})$ to be a derivation of $\mathfrak{A}$ is that (21) be zero. Hence $(b_1, \cdots, b_{2r})$ is a derivation of $\mathfrak{A}^*$. From identities (14) of Schafer [7] we can now conclude that there is a g such that

$$b_i = \left(\frac{\partial g}{\partial x_s} + \sigma_i \circ x_s^{p-1}\right) \circ c'_{is}$$

in which σ_i is in $\mathfrak{F}$. Therefore

$$(22) \qquad a_i = \left(\frac{\partial g}{\partial x_s} + \sigma_i \circ x_s^{p-1}\right) \circ c_{is}.$$

Schafer has already proved [7, Theorem 8] that if the a's are defined as in (22) then they define a derivation.

We summarize as follows.

THEOREM 2. *If $\mathfrak{A}$ is a simple, nodal, Lie-admissible noncommutative Jordan algebra of characteristic $p \neq 2$ such that $\mathfrak{A}^+$ has an even number n of generators then the derivation algebra $\mathfrak{D}(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all mappings*

$$f \to \sum_1^n \frac{\partial f}{\partial x_i} \circ a_i$$

in which the a_i are defined as in (22). The dimension of $\mathfrak{D}(\mathfrak{A})$ is $p^n + n - 1$.

We now investigate the algebras adj $\mathfrak{A}^-$, (adj $\mathfrak{A}^-)'$ and (adj $\mathfrak{A}^-)''$.

Using Schafer's result [7, Theorem 7] we have $\mathfrak{A}^- / \mathfrak{F}1 \cong$ adj $\mathfrak{A}^-$ is of dimension $p^{2r} - 1$.

Since $D_n D_m - D_m D_n = D(nD_m)$ we can consider (adj $\mathfrak{A}^-)'$ as the set of all $D_x, x \in \mathfrak{A}^-$ such that there are y and z in $\mathfrak{A}^-$ with $x \equiv yD_z$ modulo $\mathfrak{F}1$. Also $x_i^2 D(x_{i+r}) = 2x_i$ implies $D(x_i) \in$ (adj $\mathfrak{A})'$.

Before examining the dimension of (adj $\mathfrak{A}^-)'$ we consider a slightly more general situation.

Let $\mathfrak{I}$ be an ideal of $\mathfrak{A}^-$ containing all of the generators $x_1, \cdots . x_{2r}$. Let m be a monomial of $\mathfrak{A}^-$ that is not in $\mathfrak{F}1$, and in which the exponent of x_1 is i and $0 \leqq i < p-1$. Write $m = x_1^i \circ n$. Then

$$(23) \qquad \left(\frac{1}{i+1} x_1^{i+1} \circ n\right) D(x_{1+r}) = x_1^i \circ n \circ c_{11+r}.$$

If $i > 0$, $c_{11+r} \in \mathfrak{F}$, or x_{1+r} appears in m with nonzero exponent then $x_1^i \circ n \circ c_{11+r} = x_1^i \circ n = m \in \mathfrak{I}$. Arguing on the arbitrariness of the choice of x_1 we see that all terms of degree greater than 0 are in $\mathfrak{I}$ except possibly those in which:

(1) every generator appears to either the 0 or $p-1$ power,

(2) x_i has exponent $p-1$ if and only if x_{i+r} has exponent $p-1$ for $i = 1, \cdots, r$ and

(3) x_i and x_{i+r} have exponent $p-1$ if $c_{ii+r} \in \mathfrak{F}$.

However, assume such a term is m, and assume x_1 has exponent 0 in m and $c_{11+r} \notin \mathfrak{F}$. Then from (23) we see that $m \equiv -\alpha_1 m \circ x_1^{p-1} \circ x_{1+r}^{p-1}$ modulo $\mathfrak{I}$.

This leaves us with *at most* two residue classes modulo $\mathfrak{J}$; the class containing 1 and the class containing $x_{i_1}^{p-1} \circ x_{i_1+r}^{p-1} \circ \cdots \circ x_{i_t}^{p-1} \circ x_{i_t+r}^{p-1}$ in which $\mathfrak{S} = \{i_1, \cdots, i_t\}$ is the set of all $i \leqq r$ such that $c_{ii+r} \in \mathfrak{F}$. If $\mathfrak{S}$ is empty then since

$$x_i D(x_{i+r}) = 1 + \alpha_i x_i^{p-1} \circ x_{i+r}^{p-1}$$

and $\alpha_i \neq 0$ there is at most one residue class, that one containing 1.

We now let $\mathfrak{J}$ be the ideal in $\mathfrak{A}^-$ such that $\mathfrak{J} \cong (\text{adj } \mathfrak{A}^-)'$. If $\mathfrak{S} = \varnothing$ by the above result we have $\mathfrak{J} = \mathfrak{A}^-$ and $(\text{adj } \mathfrak{A}^-)' = \text{adj } \mathfrak{A}$.

In case $\mathfrak{S} \neq \varnothing$ we first note that we have shown that $\mathfrak{J}$ contains all monomials and binomials of the form

$$n \circ c_{ii+r}, \tag{24}$$

$i = 1, \cdots, r$, and n is a monomial without the factor $x_i^{p-1} \circ x_{i+r}^{p-1}$. To show that these are the only terms in $\mathfrak{J}$ we consider two monomials $n = x \circ x_i^u \circ x_{i+r}^v$ and $m = y \circ x_i^k \circ x_{i+r}^j$ in which x and y are independent of x_i and x_{i+r}. Every element of $\mathfrak{J}$ is a sum of terms of the form nD_m and every nD_m is a sum of terms of the form

$$\begin{aligned} &(x_i^u \circ x_{i+r}^v) D(x_i^k \circ x_{i+r}^j) \circ y \circ x \\ &\qquad = y \circ x \circ x_i^{u+k-1} \circ x_{i+r}^{v+j-1} \circ (vk - uj) \circ c_{ii+r}. \end{aligned} \tag{25}$$

If $u + k - 1 = v + j - 1 = p - 1$ then $vk - uj = 0$. Hence every element of $\mathfrak{J}$ is a sum of terms of the form (24).

Now let q be the product of all x_i^{p-1} such that $i \in \mathfrak{S}$. If q is in $\mathfrak{J}$ then it must be a sum of terms of the form (24). In fact we must have

$$q = \sum q \circ n_i \circ c_{ii+r}$$

in which $i \notin \mathfrak{S}$, n_i is a polynomial independent of any of the generators in q. But this is a polynomial identity that holds in any scalar extension of $\mathfrak{F}$. Hence we can substitute field elements δ_i, δ_{i+r} of some scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ for x_i and $x_{i+r}, i \notin \mathfrak{S}$, so that $1 + \alpha_i \delta_i^{p-1} \circ \delta_{i+r}^{p-1} = 0$. But then the polynomial identity $q = 0$ holds over $\mathfrak{K}$. Hence $q \notin \mathfrak{J}$.

We now show that $(\text{adj } \mathfrak{A}^-)'$ is simple. Let $\mathfrak{J}$ be an ideal of $(\text{adj } \mathfrak{U}^-)'$. To simplify the notation we will again actually work with an ideal in $\mathfrak{U}^-$ and assume everything is reduced modulo $\mathfrak{F}1$.

Let $\mathfrak{T}$ be the set of all polynomials in $\mathfrak{J}$ with a minimal number of terms in them. If the generator x_1 appears in any of these polynomials in $\mathfrak{T}$ choose one such polynomial m in which x_1 appears to the minimal positive degree. Consider $mD(x_{1+r}^2)$ which is in $\mathfrak{J}$ and has fewer terms than m unless x_1 appears with positive exponent in every term of m. Also, if any term is of degree greater than 1 in x_1 then we have a contradiction to our choice of m to be of minimal degree in x_1. Hence we can assume $m = x_1 \circ n$ in which n is independent of x_1. By choosing n

to be of minimal positive degree in some second generator and avoiding the use of derivations D_y for which x_{1+r} appear in y we can repeat the above argument finally obtaining a monomial m in $\mathfrak{J}$ which is the product of distinct generators. If both x_i and x_{i+r} are in m we can replace m by mD_{x_i}. Hence we can assume in addition that the subscripts i of the generator in m satisfy $i \leqq r$. Write

$$m = x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_t}$$

and apply successively the derivations

$$D(x_{i_1+r}), D(x_{i_2+r} \circ c_{i_1 i_1+r}^{-1}), \cdots, D(x_{i_t+r}^2 \circ c_{i_{t-1} i_{t-1}+r}^{-1})$$

obtaining $x_{i_t+r} \in \mathfrak{J}$ and $x_{i_t+r} D(x_i^2) = 2x_{i_t} \in \mathfrak{J}$. Hence we can conclude that any generator that appears in a monomial of $\mathfrak{J}$ is in $\mathfrak{J}$. If x_i is one such generator then for $i \neq j$, $x_i D(x_{i+r}^2 \circ x_j) = 2x_{i+2} \circ x_j$ is in $\mathfrak{J}$ and $x_j \in \mathfrak{J}$. Therefore $\mathfrak{J}$ contains all generators. By the results above $\mathfrak{J}$ must be all of $(\text{adj } \mathfrak{A}^-)'$ and $(\text{adj } \mathfrak{A}^-)'$ is simple. We summarize in the following theorem.

THEOREM 3. *If $\mathfrak{A}$ is a simple, Lie-admissible nodal noncommutative Jordan algebra of characteristic $p \neq 2$ with $2r$ generators then* $(\text{adj } \mathfrak{A}^-)'$ *is a simple Lie algebra of dimension either $p^{2r} - 1$ or $p^{2r} - 2$ in the cases $\mathfrak{S} = \varnothing$ or $\mathfrak{S} \neq \varnothing$ respectively.*

4. Let $\mathfrak{A}$ and $\mathfrak{A}^*$ be two nodal algebras that are equal as vector spaces and have the same + algebra. Let there be an odd number $n = 2r + 1$, of generators $x_1, \cdots, x_n$ with the multiplication in $\mathfrak{A}$ given by $c'_{ii+r} = 2$ for $i = 1, \cdots, r$ and all other $c_{ij} = 0$.

Let $(a_1, \cdots, a_n)$ be a derivation of $\mathfrak{A}$. Just as in the previous section we can show $(b_1, \cdots, b_{2r-2}, 0,0,0)$ is a derivation of $\mathfrak{A}^*$ if

$$b_i = c_{is}^{-1} \circ c'_{is} \circ a_i$$

for $i = 1, \cdots, r - 2$. Therefore we must have

$$a_i = \left(\frac{\partial g}{\partial x_s} + \sigma_i \circ x_s^{p-1} \right) \circ c_{is}$$

for $i = 1, \cdots, r - 1$. Here though, σ_i can apparently be any polynomial in $\mathfrak{F}[x_{2r-1}, x_{2r}, x_{2r+1}]$. To obtain further restrictions on the σ_i we examine derivations of the form

$$(\sigma_1 \circ x_{1+r}^{p-1} \circ c_{11+r}, \cdots, \sigma_{2r-2} \circ x_{r-1}^{p-1} \circ c_{2r-2\, r-2}, a_{2r-1}, a_{2r}, a_{2r+1}).$$

We now use identity (5) of Schafer [7] with $i \leqq r - 2$ and $j \geqq 2r - 1$ to obtain

$$\sum_{2r-1}^{n} \frac{\partial \sigma_i}{\partial x_k} \circ x_{i+r}^{p-1} \circ c_{jk} + \frac{\partial a_j}{\partial x_{i+r}} \circ c_{i+r\, i} = 0. \tag{26}$$

Since $c_{i+r\, i}$ is nonsingular and $\partial a_j / \partial x_{i+r}$ is of degree at most $p - 2$ in x_{i+r} we must have $\partial a_j/\partial x_{i+r} = 0$ and a_j independent of x_{i+r}. Interchanging i and $i + r$ in (26) we see a_j is also independent of x_i. Hence a_j is a polynomial in $\mathfrak{F}[x_{2r-1}, x_{2r}, x_{2r+1}]$.

We now select j in (26) to be $2r$. Then

$$\frac{\partial \sigma_i}{\partial x_{2r-1}} x_{i+r}^{p-1} c_{2r\,2r-1} + \frac{\partial \sigma_i}{\partial x_{2r+1}} \circ x_{i+r}^{p-1} \circ c_{2r\,2r+1} = 0.$$

Since x_{2r}^{p-1} is a factor of $c_{2r\,2r+1}$ and σ_i is independent of x_{i+r} we must have

$$\begin{aligned} &\frac{\partial \sigma_i}{\partial x_{2r+1}} \circ c_{2r\,2r+1} = 0, \\ (27) \qquad &\frac{\partial \sigma_i}{\partial x_{2r-1}} = 0. \end{aligned}$$

Hence σ_i is independent of x_{2r-1}. In the same way we see that σ_i is independent of x_{2r}. Now by the first relationship in (27) we have σ_i independent of x_{2r+1} and $\sigma_i \in \mathfrak{F}$.

We can now confine our attention to finding the derivations of an algebra $\mathfrak{A}$ with three generators x, y, z in which multiplication is defined by

$$\begin{aligned} yD_x &= 1 + \gamma x^{p-1} \circ y^{p-1} = d_{12}, \\ zD_x &= y^{p-1} \circ (1 + \beta z^{p-1}) = d_{13}, \\ zD_y &= \alpha x^{p-1} \circ (1 + \beta z^{p-1}) = d_{23}. \end{aligned}$$

Let (a, a_2, a_3) be a derivation of $\mathfrak{A}$. Since there are derivations of the form

$$b_i = \frac{\partial g}{\partial x} \circ d_{i1} + \frac{\partial g}{\partial y} \circ d_{i2} + \frac{\partial g}{\partial z} \circ d_{i3}$$

[7, Theorem 8] and $a_1 \circ d_{12}^{-1} = \partial g/\partial y$ can be solved to within a multiple o y^{p-1} [7, Lemma 1], we can subtract off the derivation induced by g and assume $a_1 = \delta \circ y^{p-1}$ in which δ is a polynomial in x and z. Using the same lemma we can solve $-\mu^{-1} \circ \delta = \partial g/\partial z$ to within a multiple of z^{p-1} and such that g is in $\mathfrak{F}[x, z]$. Subtracting off the derivation corresponding to this y leaves us with $a_1 = \delta_0 \circ z^{p-1} \circ y^{p-1}$ in which δ_0 is a polynomial in x.

The three conditions [7] that (a_1, a_2, a_3) be a derivation can be written in the form

$$(28) \qquad -\frac{\partial(d_{12}^{-1} a_2)}{\partial y} \circ d_{12}^2 + \frac{\partial a_1}{\partial x} \circ d_{21} + \frac{\partial a_2}{\partial z} \circ d_{31} + \frac{\partial a_1}{\partial z} \circ d_{23} = 0,$$

$$(29) \qquad -y^{p-1} \circ \frac{\partial(\mu^{-1} a_3)}{\partial z} \circ \mu^2 + \frac{\partial d_{13}}{\partial y} \circ a_2 + \frac{\partial a_1}{\partial y} \circ d_{32} + \frac{\partial a_3}{\partial y} \circ d_{21} = 0,$$

$$\begin{aligned}(30)\qquad & -\alpha x^{p-1}\circ\frac{\partial(\mu^{-1}a_3)}{\partial z}\circ\mu^2+\frac{\partial d_{23}}{\partial x}\circ a_1+\frac{\partial a_2}{\partial y}\circ d_{32}\\ & \qquad +\frac{\partial a_2}{\partial x}\circ d_{31}+\frac{\partial a_3}{\partial x}\circ d_{12}=0\end{aligned}$$

in which $\mu = 1+\beta z^{p-1}$.

The last three terms of (28) are in $y^{p-1}\circ\mathfrak{A}$ since a_1, and d_{31} are. Hence both $-d_{12}^2\circ\partial(d_{12}^{-1}\circ a_2)/\partial y$ and $\partial(d_{12}^{-1}\circ a_2)/\partial y$ are in $y^{p-1}\circ\mathfrak{A}$. But the second polynomial is of degree at most $p-2$ in y and hence is 0. Therefore there is a polynomial δ_1 independent of y and such that $a_2=\delta_1\circ d_{12}$.

Identity (28) now reduces to

$$\begin{aligned}(31)\qquad & \mu^{-1}\circ\frac{\partial\delta_0}{\partial x}\circ y^{p-1}\circ z^{p-1}+\frac{\partial\delta_1}{\partial z}\circ y^{p-1}\\ & \qquad -\alpha y^{p-1}\circ x^{p-1}\circ\frac{\partial(\delta_0 z^{p-1})}{\partial z}=0.\end{aligned}$$

Arguing on the degree of z in each term of (31) we can conclude $\partial\delta_0/\partial x=0$ and δ_0 is independent of x. But $\delta z^{p-1}\circ y^{p-1}=\delta z^{p-1}\circ d_{13}$ and

$$(\delta z^{p-1}\circ d_{13},\delta z^{p-1}\circ d_{23},0)$$

is a derivation of A. Subtracting off this derivation we can assume $a_1=\delta_0=0$.

From (31), since δ_1 is independent of y, we also get δ_1 independent of z, i.e., δ_1 is a polynomial in $\mathfrak{F}[x]$. Therefore we can find a polynomial g in $\mathfrak{F}[x]$ that is a solution of $\delta_1\circ d_{12}=d_{21}\circ\partial g/\partial x$ to within a constant multiple of x^{p-1}, say ηx^{p-1}.

Subtracting off the derivation

$$\left(0,d_{21}\circ\frac{\partial g}{\partial x}-\eta x^{p-1}\circ d_{21},d_{31}\circ\frac{\partial g}{\partial x}-\eta x^{p-1}\circ d_{31}\right)$$

we can assume $a_1=a_2=0$. Equations (29) and (30) now reduce to

$$\begin{aligned}(32)\qquad & -y^{p-1}\circ\frac{\partial(\mu^{-1}\circ a_3)}{\partial z}\circ\mu^2+\frac{\partial a_3}{\partial y}\circ d_{21}=0,\\ & -\alpha x^{p-1}\circ\frac{\partial(\mu^{-1}\circ a_3)}{\partial z}\circ\mu^2+\frac{\partial a_3}{\partial x}\circ d_{12}=0.\end{aligned}$$

Since d_{21} is nonsingular we can argue on the degree of y to get $\partial a_3/\partial y=0$ and a_3 is independent of y. In the same manner a_3 is independent of x. But then $\mu^{-1}a_3$ is independent of z. Hence $a_3=\eta\mu$ for $\eta\in\mathfrak{F}$.

By direct substitution in (32) it can be seen that $(0,0,\eta\mu)$ is a derivation of $\mathfrak{A}$. We investigate to see if it is of the form (a_1,a_2,a_3) in which

$$
\begin{aligned}
a_1 &= \left(\frac{\partial g}{\partial y} + \alpha_2 y^{p-1}\right) \circ d_{12} + \left(\frac{\partial g}{\partial z} + \alpha_3 z^{p-1}\right) \circ d_{13}, \\
a_2 &= \left(\frac{\partial g}{\partial x} + \alpha x^{p-1}\right) \circ d_{21} + \left(\frac{\partial g}{\partial z} + \alpha_3 z^{p-1}\right) \circ d_{23}, \qquad (33)\\
a_3 &= \left(\frac{\partial g}{\partial x} + \alpha_1 x^{p-1}\right) \circ d_{31} + \left(\frac{\partial g}{\partial y} + \alpha_2 y^{p-1}\right) \circ d_{32}.
\end{aligned}
$$

If $a_1 = a_2 = 0$ then $a_1 \equiv 0$ modulo y^{p-1} and $\partial g/\partial y \in y^{p-1} \circ \mathfrak{A}$. Hence g is independent of y. In the same way g is independent of x. Therefore $a_3 = -x^{p-1} \circ y^{p-1} \circ \mu \circ (\alpha_1 + \alpha\alpha_2)$ which is not of the form $\eta\mu$ for $\eta \in \mathfrak{F}$.

We can now conclude:

THEOREM 4. *Let $\mathfrak{A}$ be a simple, nodal, Lie-admissible noncommutative Jordan algebra of characteristic $p \neq 2$ with $2r+1$ generators; then the derivation algebra $\mathfrak{D}(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all mappings*

$$f \to \sum_1^n \frac{\partial f}{\partial x_i} \circ a_i$$

in which

$$
\begin{aligned}
a_i &= \sum_{j=1}^n \left(\frac{\partial g}{\partial x_j} + \alpha_j x_j^{p-1}\right) \circ c_{ij}, \\
a_{2r+1} &= \sum_{2r-1}^{2N} \left(\frac{\partial g}{\partial x_i} + \alpha_i x_i^{p-1}\right) \circ c_{2r+1\,i} + \eta\mu
\end{aligned}
$$

for $i = 1, \cdots, 2r$. (In case $i < 2r-1$ then a_i reduces to a single summand.) The dimension of $\mathfrak{D}(\mathfrak{A})$ is $p^{2r+1} + 2r + 1$.

To determine the dimension of $(\operatorname{adj} \mathfrak{A}^-)'$ we proceed as in the even-dimensional case. Let $\mathfrak{J}$ be an ideal of $\mathfrak{A}^-$ containing all of the generators $x_1, \cdots, x_{2r}$. Using only the generators $x_1, \cdots, x_{2r-2}$ we have the result from the even-dimensional case that the only possible residue classes modulo $\mathfrak{J}$ are the classes determined by 1 and the polynomials of the form $q\ m$ in which $q = x_1^{p-1} \circ \cdots \circ x_{2r-2}^{p-1}$ and m is a polynomial in x_{2r-1}, x_{2r} and x_{2r+1}. We adopt the notation above using x, y and z, x_{2r-1}, x_{2r}, and x_{2r+1} respectively. Assume m is a monomial and $m = x^i \circ n$, n independent of x and $i < p-1$; then

$$\left(\frac{1}{i+1} q \circ x^{i+1} \circ n\right) D_y = -q \circ m \circ d_{12}.$$

Also if $m = y^i \circ n$, n independent of y and $i < p-1$ then

$$\left(\frac{1}{i+1} q \circ y^{i+1} \circ n\right) D_x = q \circ m \circ d_{12}.$$

Hence the only remaining residue classes of $\mathfrak{I}$ to examine are those determined by $q \circ x^{p-1} \circ y^{p-1} \circ n$ in which n is a polynomial in z. However the equation

$$
\begin{aligned}
(q \circ x^{p-1} \circ t)D_x &= q \circ x^{p-1} \circ y^{p-1} \circ \frac{\partial t}{\partial y} \circ \mu \\
&= q \circ x^{p-1} \circ y^{p-1} \circ n
\end{aligned}
$$

can be solved for t, a polynomial in $\mathfrak{F}[z]$, to within a scalar multiple of $q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1}$. Hence the only possible residue class of $\mathfrak{I}$ is that containing $q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1}$. If $\mathfrak{S} = \varnothing$ (the set of all $i = 1, \cdots, r$ such that $c_{ii+r} \in \mathfrak{F}$) and $\beta \neq 0$ then as we have seen in the even-dimensional case $q \circ x^{p-1} \circ y^{p-1} \in \mathfrak{I}$ and $(q \circ x^{p-1} \circ z)D_x = (q \circ x^{p-1} \circ y^{p-1} \circ \mu) \in \mathfrak{I}$. Therefore $q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1} \in \mathfrak{I}$.

If $\mathfrak{I}$ is the ideal in $\mathfrak{A}^-$ such that $\mathfrak{A}^- / \mathfrak{F}1$ is isomorphic to (adj $\mathfrak{A}^-$)$'$ then we can show, exactly as in the even-dimensional case, that $q \circ x^{p-1} \circ y^{p-1} \circ z^{p-1}$ is not in $\mathfrak{I}$ if either $\mathfrak{S} \neq \varnothing$ or $\beta = 0$. Hence (adj $\mathfrak{A}^-$)$'$ is of dimension $p^{2r-1} - 1$ or $p^{2r+1} - 2$.

We now examine the ideals of (adj $\mathfrak{A}^-$)$'$. Let $\mathfrak{I}$ be an ideal of (adj $\mathfrak{A}^-$)$'$. (We again use the notation of $\mathfrak{A}^-$.) As in the even-dimensional case we can assume there are polynomials of the form $x_i \circ m$ for any $i \leqq 2r - 1$ and in which m is a polynomial in $\mathfrak{F}[x, y, z]$.

Consider those polynomials $x \circ m$. If m is in $\mathfrak{F}[x]$ we choose a k so that

$$(x_1 \circ m)D(x_1 \circ x_{1+r} \circ x^k) = x_1 \circ x^{p-1}.$$

If $m \notin \mathfrak{F}[x]$, write

$$m = m_1 + \sum_{k}^{p-1} x^i \circ n_i$$

in which m_1 is a polynomial in x, every term of every nonzero n_i has either a y or z in it and some $n_i \neq 0$. If $k \neq 0$ then

$$(x_1 \circ m)D(x^{p-k}) = - kx^{p-1} \circ n_k D_x \circ x_1 \neq 0$$

is in $\mathfrak{I}$. If $k = 0$ then

$$(x_1 \circ m)D(x^{p-1}) = (- n_0 D_x \circ x^{p-2} - x^{p-1} \circ n_1 D_x) \circ x_1 \neq 0$$

is in $\mathfrak{I}$. If $n_0 D_x$ and $n_1 D_x$ are in $\mathfrak{F}$ then as above we can conclude $x_1 \circ x^{p-1} \in \mathfrak{I}$. If $n_0 D_x$ is in $\mathfrak{F}$ but $n_1 D_x$ is not then

$$(x_1 \circ m)D(x^{p-1})D_x = - x^{p-1} \circ x_1 \circ n_1 D_x D_x \neq 0$$

is in $\mathfrak{I}$. If $n_0 D_x \notin \mathfrak{F}$ then

$$(x_1 \circ m)D(x^{p-1})D(x^2) = - 2n_0 D_x^2 \circ x^{p-1} \circ x_1 \neq 0$$

is in $\mathfrak{I}$. In any case, there is a polynomial $x_1 \circ x^{p-1} \circ m$ in $\mathfrak{I}$ in which $m \in \mathfrak{F}[y, z]$. If m is in $\mathfrak{F}$ we can proceed as in the even-dimensional case to show that $x_1, \cdots, x_{2r}$ are in $\mathfrak{I}$.

If m is independent of y then assume m is such a polynomial of minimal degree in z. We have

$$(x_1 \circ x^{p-1} \circ m)D_x^p = x_1 \circ x^{p-1} \circ \frac{\partial m}{\partial z}.$$

By the minimality of the degree of z in m we have $\partial m / \partial z = 0$, $m \in \mathfrak{F}$ and $x_1 \circ x^{p-1} \in \mathfrak{I}$.

If m is not independent of y then

$$(x_1 \circ x^{p-1} \circ m)D_y^{p-2} = x_1 \circ x \circ m$$

is in $\mathfrak{I}$. Let k be the smallest exponent of y in m. If $k = 0$ then $(x_1 \circ x \circ m)D_z = \alpha x_1 \circ y^{p-1} \circ m \circ \mu = x_1 \circ y^{p-1} \circ n$ is in $\mathfrak{I}$ for some polynomial n in $\mathfrak{F}[z]$. If $k \neq 0$ then $(x_1 \circ x \circ m)D(y^{p-k}) = kx_1 \circ y^{p-1} \circ n$ is in $\mathfrak{I}$ for some polynomial n in $\mathfrak{F}[z]$. Choose n to be of minimal degree in z. Then as above we can show n is in $\mathfrak{F}$ and $x_1 \circ y^{p-1} \in \mathfrak{I}$.

Thus either $x_1 \circ x^{p-1}$ or $x_1 \circ y^{p-1}$ is in $\mathfrak{I}$. As in the even-dimensional case this implies $x_1, \cdots, x_{2r}$ are in $\mathfrak{I}$. Hence from our conclusion above on such ideals $\mathfrak{I}$ we have (adj $\mathfrak{A}^-)'$ is simple. Thus

THEOREM 5. *If $\mathfrak{U}$ is a simple, Lie-admissible nodal noncommutative Jordan algebra of characteristic $p \neq 2$ with $2r+1$ generators then* (adj $\mathfrak{A}^-)'$ *is a simple Lie-algebra. The dimension of* (adj $\mathfrak{A}^-)'$ *is $p^{2r-1} - 1$ if $\mathfrak{S} = \varnothing$ and $\beta \neq 0$ and is $p^{2r+1} - 2$ if either $\mathfrak{S} \neq 0$ or $\beta = 0$.*

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INSTITUTE FOR DEFENSE ANALYSES,
PRINCETON, NEW JERSEY
MICHIGAN STATE UNIVERSITY,
EAST LANSING, MICHIGAN