ON GAUSSIAN MEASURES EQUIVALENT TO WIENER MEASURE(1)

BY
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Introduction. The question of when two Gaussian measures are equivalent continues to receive attention (see, for example, [4]). In the present paper we nvestigate this question in the case where one of the measures is Wiener measure. We succeed in giving necessary and sufficient conditions for a wide class of Gaussian measures to be equivalent to Wiener measure (Theorem 5). Moreover, in the case of equivalence, we give an explicit formula for the Radon-Nikodym derivative of one measure with respect to the other. These results, which generalize earlier ones of the author [6], are based on a recent paper of Woodward [7] on linear transformations of the Wiener process. In particular, our sufficiency results are obtained by examining the question of when it is possible to represent a Gaussian process by means of a linear transformation of the Wiener process. In answering this question we are led to study a new class of Gaussian processes, namely those which have what we shall call *factorable* covariance functions. The covariance function r is factorable on $[0,b]$ if r may be written in the form

$$r(s,t) = \int_0^b R(s,u)\,R(t,u)du.$$

The class of processes with factorable covariance functions forms the subject of most of our study.

1. **Some preliminary definitions and lemmas.** By a Gaussian process (sometimes symbolized $\{x(t),\ a \leqq t \leqq b\}$), we shall mean a triple $\{X, \mathscr{B}, \lambda_{rm}\}$ where $X \equiv X(a,b)$ is a set of real-valued functions defined on an interval $[a,b]$, $\mathscr{B}$ is a Borel field of subsets of X containing all sets of the form

$$\{x \in X \mid x(t_0) \leqq c,\ t_0 \in [a,b]\}$$

and λ_{rm} is a Gaussian probability measure on $\mathscr{B}$ determined by a covariance function r and a mean function m. How r and m determine such a measure is well known [2, pp. 71–74]. In this paper we shall always take the interval $[a,b]$ to be

Received by the editors June 25, 1963.
(1) This research was supported by the Air Force Office of Scientific Research.

$I \equiv [0, b]$. Moreover, we shall assume that the mean function is identically zero; thus without confusion we may write λ_r in place of λ_{r0}. The most familiar of all Gaussian processes is the Wiener process $\{C, B, \lambda_w\}$. Here $C \equiv C(0, b)$ denotes the set of all continuous functions on I, B is the Borel field generated by sets of the form indicated above and λ_w is the measure on B determined by the covariance function $w(s,t) = \min(s,t)$. It is easy to show that

$$\lambda_w\{x \in C \mid x(0) = 0\} = 1.$$

Hence C is often replaced by C_0, the set of all continuous functions on $[0, b]$ vanishing at 0. In particular this is true of Woodward's paper [7], which we shall be using later.

Min(s,t) is an example of a large class of factorable covariance functions which are of major concern to us. We make the following definition.

DEFINITION. We say that $r \in G \equiv G(0, b)$ if

$$r(s,t) = \int_0^b R(s,u) R(t,u) du$$

for some R defined on $I \times I$ and satisfying:

(1.1) the total variation of $R(s,t)$ with respect to t is less than some constant which is independent of s,

(1.2) $R(s,b) = 0$ and $R(s,0)$ is continuous on I,

(1.3) $\int_0^t R(s,w)dw$ is continuous in s on I for each t in I.

We note immediately that (1.2) places no restriction on r. We require it merely as a matter of convenience. As a matter of fact, we impose conditions (1.1)–(1.3) so that the transformation

$$T(x(\cdot)) = \int_0^b x(u) d_u R(\cdot, u)$$

will be a bounded linear transformation of C into itself. More than this, we will want to use the fact that every bounded linear transformation of C into itself has a representation of this type where R satisfies (1.1)–(1.3) (see, for example, [5, pp. 219, 220] but note that we have chosen the normalization $R(t,b) = 0$ in place of $R(t,0) = 0$).

It is a simple matter to demonstrate that every r in G is a covariance function. Whether every covariance function defined on $I \times I$ belongs to G is not as simple to determine. The analagous question for matrices is whether every positive definite matrix Q can be represented in the form $P'P$ where P' is the transpose of P. Here the answer is yes and in fact P can be chosen in a variety of ways, for example, triangular or symmetric. Apparently, the situation is similar though not as transparent in the case of a covariance function. Certainly the class G is very large and contains all triangular covariance functions (see §4). The author conjectures and hopes to show in a future paper that appropriate regularity conditions on r will guarantee that r is in G.

Covariance functions in G arise naturally from the study of bounded linear transformations of the Wiener process. We make this precise in our first lemma.

LEMMA 1. *Let $\{x(t),\ 0 \leqq t \leqq b\}$ be the Wiener process and let $r \in G$ with factor R. The Gaussian process $\{y(t),\ 0 \leqq t \leqq b\}$ given by*

$$y(t) = \int_0^b x(u)\, d_u R(t,u)$$

has covariance function r and hence r determines a measure λ_r on $\{C, B\}$.

Proof.

$$\begin{aligned} E\{y(s)\,y(t)\} &= E\left\{\int_0^b \int_0^b x(u)\,x(v)\,d_u R(s,u)\,d_v R(t,v)\right\} \\ &= \int_0^b \int_0^b \min(u,v)\,d_u R(s,u)\,d_v R(t,v) \\ &= \int_0^b \left[\int_0^v u\,d_u R(s,u) + v\int_v^b d_u R(s,u)\right] d_v R(t,v) \\ &= \int_0^b \left[-\int_0^v R(s,u)du\right] d_v R(t,v) \\ &= \int_0^b R(s,v)\,R(t,v)dv = r(s,t). \end{aligned}$$

Now $y(t) = \int_0^b x(u)\,d_u R(t,u)$, being a transformation satisfying (1.1)–(1.3), maps C onto C' (where C' is a subset of C) and hence r determines a measure λ_r on $\{C', B\}$. This can be extended to $\{C, B\}$ by defining $\lambda_r(C - C') = 0$.

Our proof shows incidentally that any process arising from a bounded linear transformation of the Wiener process has covariance function r in G. Even more is true, as we see in the following lemma.

LEMMA 2. *Let $\{x(t),\ 0 \leqq t \leqq b\}$ be a Gaussian process determined by a covariance function r in G. Let T be a bounded linear transformation of C into itself. Then the process $\{y(t),\ 0 \leqq t \leqq b\}$ determined by $y(\cdot) = T(x(\cdot))$ is a Gaussian process with covariance function in G.*

Proof. We may represent T in the form

$$T(x(\cdot)) = \int_0^b x(u)\,d_u K(\cdot, u)$$

where K satisfies (1.1)–(1.3). Thus

$$
\begin{aligned}
E\{y(s)y(t)\} &= \int_0^b \int_0^b r(u,v)\, d_uK(s,u)\, d_vK(t,v) \\
&= \int_0^b \int_0^b \int_0^b R(u,w)\, R(v,w)\, dw d_uK(s,u)\, d_vK(t,v) \\
&= \int_0^b \left[\int_0^b R(u,w)\, d_uK(s,u) \right] \left[\int_0^b R(v,w)\, d_vK(t,v) \right] dw
\end{aligned}
$$

where the Stieltjes integrals may have to be interpreted as Lebesgue Stieltjes integrals.

Now let $P(s,t) = \int_0^b R(u,t)\, d_uK(s,u)$. We must show that P satisfies (1.1)–(1.3). (1.1) follows by using standard estimates and the fact that both R and K satisfy(1.1). (1.2) follows immediately from the corresponding fact for R. Finally

$$
\begin{aligned}
\int_0^t P(s,w)dw &= \int_0^t \int_0^b R(u,w)\, d_uK(s,u)dw \\
&= \int_0^b \int_0^t R(u,w)\, dw d_uK(s,u)
\end{aligned}
$$

and since $\int_0^t R(u,w)dw$ is continuous in u, it is mapped into a continuous function by a transformation of C into C. Hence (1.3) is satisfied.

It will be convenient at various times to impose some or all of the following conditions on the factor R of the covariance function r of G.

(1.4) $R(0,t)$ is continuous in t on $(0,b]$.

(1.5) $q(t) = R(t^+,t) - R(t^-,t)$ exists and is continuous for each $t \in (0, b)$ and $q(0) = R(0,0)$.

(1.6) If

$$
Q(s,t) = \begin{cases} q(t), & s > t \text{ or } s = t = 0, \\ 0, & \text{otherwise,} \end{cases}
$$

and if $L(s,t) = R(s,t) - Q(s,t)$, then $L(s,t)$ is absolutely continuous in s on I for each $t \in I$.

(1.7) $(\partial/\partial s)L(s,t) \equiv L_1(s,t)$, which would exist for almost all $s \in I$ for each $t \in I$ by (1.6), satisfies

$$
\int_0^b \int_0^b [L_1(s,t)]^2 ds dt < \infty.
$$

(1.8) There exist functions $M(s,t)$ and $J(s)$ such that $L_1(s,t) = \bar{M}(s,t)$ for almost all $s \in I$ for all $t \in I$ where

$$
\bar{M}(s,t) = \begin{cases} M(s,t) + J(s), & s > t, \\ M(s,t) + J(s)/2, & s = t, \\ M(s,t), & s < t. \end{cases}
$$

(1.9) $M(s,t)$ is of B.V. on $I \times I$ (see [7, p. 459]) and $J(s)$ is of B.V. on I.

$$D(\bar{M}) \equiv 1 + \sum_1^\infty \frac{1}{n!} \int_0^b \cdots \int_0^b \begin{vmatrix} \bar{M}(s_1,s_1) & \cdots & \bar{M}(s_1,s_n) \\ \vdots & & \vdots \\ \bar{M}(s_n,s_1) & \cdots & \bar{M}(s_n,s_n) \end{vmatrix} ds_1 \cdots ds_n, \tag{1.10}$$

the Fredholm determinant of $\bar{M}(s,t)$ evaluated at -1, is not equal to zero.

Note that (1.8) and (1.9) imply (1.7) since L_1 is bounded on $I \times I$ if (1.8) and (1.9) hold.

2. An extension of a theorem of Baxter [1].

THEOREM 1. *Let $\{C,B,\lambda_r\}$ be a Gaussian process with mean function identically zero and covariance function $r \in G$. Suppose that the factor R of r satisfies* (1.5)–(1.7). *Then with probability one,*

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} [x(t_k) - x(t_{k-1})]^2 = \int_0^T q^2(u)du$$

where $t_k = kT/2^n$, $k = 0,1,\cdots,2^n$, $T \in I$ and q is defined as in (1.5).

Proof. Following Baxter, we let

$$a_{jk} = E\{[x(t_k) - x(t_{k-1})][x(t_j) - x(t_{j-1})]\}, \; b_n = 2\sum_{j,k=1}^{2^n} (a_{jk})^2$$

and note that (see [1, pp. 523, 524]) the theorem will be proved if we can show

$$2^n b_n \text{ is bounded,} \tag{2.1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} a_{kk} = \int_0^T q^2(u)du. \tag{2.2}$$

We introduce the following notation:

$$\begin{aligned} R_k(u) &= R(t_k,u) - R(t_{k-1},u), \\ Q_k(u) &= Q(t_k,u) - Q(t_{k-1},u) = \begin{cases} q(u), & t_{k-1} \leqq u \leqq t_k,\ u \neq 0, \\ 0, & \text{otherwise,} \end{cases} \\ L_k(u) &= L(t_k,u) - L(t_{k-1},u) = \int_{t_{k-1}}^{t_k} L_1(s,u)ds, \end{aligned}$$

$$N = \int_0^b \int_0 [L_1(s,t)]^2 dsdt.$$

Now

$$b_n/2 = \sum_{j,k} \left[\int_0^b R_k(u) R_j(u) du \right]^2$$

$$= \int_0^b \int_0^b \left[\sum_k R_k(u) R_k(v) \right]^2 dudv$$

$$= \int_0^b \int_0^b \left[\sum_k \{L_k(u) L_k(v) + L_k(u) Q_k(v) + L_k(v) Q_k(u) + Q_k(u) Q_k(v)\} \right]^2 dudv.$$

Making use of fairly standard inequalities on this expression, one may show that

$$b_n/2 \leqq 4 \left\{ \left[\int_0^b \int_0^T L_1^2(s,u) ds du (T/2^n) \right]^2 + 2\int_0^T q^2(v) dv \int_0^b \int_0^T L_1^2(s,u) ds du (T/2^n) + \iint_S q^2(u) q^2(v) dudv \right\}$$

where

$$S = \bigcup_{j=1}^{2^n} \{(x,y) \mid t_{j-1} \leqq x \leqq t_j,\ t_{j-1} \leqq y \leqq t_j\}.$$

We conclude that

$$b_n/2 \leqq 4b^2 N^2/2^{2n} + \left[8bN \int_0^T q^2(v) dv \right] \Big/ 2^n + 4 \iint_S q^2(u) q^2(v) dudv,$$

from which it follows that $2^n b_n$ is bounded.

To establish (2.2), we observe that

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} a_{kk} = \lim_{n\to\infty} \int_0^b \sum_k R_k^2(u) du$$

$$= \lim_{n\to\infty} \int_0^b \sum_k [L_k(u) + Q_k(u)]^2 du$$

$$= \lim_{n\to\infty} \left\{ \int_0^b \sum_k L_k^2(u) du + 2\int_0^b \sum_k L_k(u) Q_k(u) du + \int_0^b \sum_k Q_k^2(u) du \right\}$$

$$= 0 + 0 + \int_0^T q^2(u) du.$$

Corollary. *Let λ_r be a Gaussian probability measure determined by a covariance function $r \in G$ with factor R satisfying* (1.4)–(1.7). *Then if λ_r is absolutely continuous with respect to Wiener measure λ_w on $\{C,B\}$, it follows that*

$$q^2(t) \equiv [R(t^+,t) - R(t^-,t)]^2 = 1 \text{ on } (0,b), \tag{2.3}$$

$$R(0,t) = 0 \text{ for all } t \in (0,b]. \tag{2.4}$$

Proof. Noting that the factor W of $w(s,t) = \min(s,t)$ is given by

$$W(s,t) = \begin{cases} 1, & s < t \text{ or } s = t = 0, \\ 0, & \text{otherwise,} \end{cases}$$

so that $[W(t^+,t) - W(t^-,t)]^2 = 1$ on $(0,b)$, we may easily show that

$$[R(t^+,t) - R(t^-,t)]^2 = 1$$

just as in [6, p. 753]. To show (2.4) we note that since

$$\lambda_w\{x \in C \mid x(0) = 0\} = 1$$

it must be true that the λ_r measure of this set is also 1. Hence $r(0,0) = 0$ but since $r(0,0) = \int_0^b R^2(0,t)dt$, it follows that $R(0,t) = 0$ for almost all $t \in I$. Finally, using the hypothesis that $R(0,t)$ is continuous on $(0,b]$, we have (2.4).

3. **Relation to a theorem of Woodward.** Cameron, Martin and others have developed an extensive transformation theory for the Wiener process. Of special interest to us is a recent paper of Woodward [7] which gives a very general linear transformation theorem.

THEOREM 2 (WOODWARD). *Let*

$$T(x(\cdot)) = x(\cdot) + \int_0^b \int_0^{\cdot} \bar{M}(u,v)dudx(v)$$

be a transformation defined on C_0 *(see introduction for definition of* C_0*) and suppose that there exist functions* M *and* J *such that*

$$\bar{M}(s,t) = \begin{cases} M(s,t), & s < t, \\ M(s,t) + J(s)/2, & s = t, \\ M(s,t) + J(s), & s > t, \end{cases} \tag{3.1}$$

where M *is of B.V. on* $I \times I$ *and* J *is of B.V. on* I. *Suppose further (see* (1.10)*) that*

$$D(\bar{M}) \neq 0. \tag{3.2}$$

Then the transformation T *carries* C_0 *onto* C_0 *in a one-to-one manner and if* F *is a* B *measurable function such that either side of the following equation exists, both sides exist and are equal.*

$$E^w\{F(y)\} = |D(\bar{M})| E^w\{F(Tx)\exp[-\Psi(x)/2]\} \tag{3.3}$$

where

$$\Psi(x) = \int_0^b \int_0^b \left[\bar{M}(s,t) + \bar{M}(t,s) + \int_0^b \bar{M}(u,s)\bar{M}(u,t)du\right] dx(s)dx(t). \tag{3.4}$$

Some minor differences between this theorem and the one in Woodward's original paper should be noted. We have stated the result for the interval $[0,b]$ rather than $[0,1]$. Also we use $E^w\{\cdots\}$ in place of Woodward's $\int_{C_0}^w \cdots d_w$. Finally there is an extra factor of 1/2 in the exponential in our formula (3.3) due to the fact that we use $\min(s,t)$ rather than $(1/2)\min(s,t)$ as the covariance function for the Wiener process.

For our purposes the following form of Woodward's theorem is more useful.

THEOREM 3. *Let*

$$T(x(\cdot)) = x(\cdot) + \int_0^b \int_0^{\cdot} \bar{M}(u,v)\,du\,dx(v)$$

satisfy the conditions of Theorem 2. Then the transformation T carries C_0 onto C_0 in a one-to-one manner and if G is a B measurable function such that either side of the following equation exists, then both sides exist and are equal.

$$E^w\{G(Tx)\} = |D(\bar{M})|^{-1} E^w\{G(x)\exp[-\Phi(x)/2]\} \tag{3.5}$$

where

$$\Phi(x) = \int_0^b \int_0^b \left[\bar{M}^{-1}(s,t) + \bar{M}^{-1}(t,s) + \int_0^b \bar{M}^{-1}(u,s)\bar{M}^{-1}(u,t)\,du\right] dx(s)\,dx(t), \tag{3.6}$$

$\bar{M}^{-1}$ being the Volterra reciprocal kernel of $\bar{M}$.

Proof. In Woodward's paper it is shown that T^{-1} exists and has the form

$$T^{-1}(y(\cdot)) = y(\cdot) + \int_0^b \int_0^{\cdot} \bar{M}^{-1}(u,v)\,du\,dy(v).$$

It is not difficult to show that if $\bar{M}$ satisfies (3.1) and (3.2) then so does $\bar{M}^{-1}$. In fact $D(\bar{M}^{-1}) = [D(\bar{M})]^{-1}$. Hence applying Theorem 2 to the transformation T^{-1}, we have

$$E^w\{F(x)\} = |D(\bar{M})|^{-1} E^w\{F(T^{-1}x)\exp[-\Phi(x)/2]\}.$$

Letting $F(x) = G(Tx)$ we obtain (3.5).

It may appear that Theorem 3 covers only a very limited class of bounded linear transformations on C_0, namely those of the form

$$T(x(\cdot)) = x(\cdot) + \int_0^b \int_0^{\cdot} \bar{M}(u,v)\,du\,dx(v).$$

In actuality, however, it is only transformations of this type for which a result like (3.5) can hold. Before making this statement precise, we remark that every bounded linear transformation of C_0 into itself has a representation of the form

$$T(x(\cdot)) = \int_0^b x(u)\,d_u R(\cdot,u)$$

where R satisfies (1.1)–(1.3) and in addition $R(0,t)=0$ for $t \in (0,b]$.

Now we pose the question. For what bounded linear transformations T of C_0 into itself does there exist a function J such that

$$E^w\{G(Tx)\} = E^w\{G(x)J(x)\} \tag{3.7}$$

for all B measurable functions G, equality being interpreted in the sense of Theorem 3? We answer in

THEOREM 4. *Let* $T(x(\cdot)) = \int_0^b x(u)\,d_uR(\cdot,u)$ *be an arbitrary bounded linear transformation of* C_0 *into* C_0 *(see remarks above). Suppose in addition that R satisfies* (1.5)–(1.7). *Then if there exists a function J such that* (3.7) *holds, it follows that T may be written in the form*

$$T(x(\cdot)) = \pm x(\cdot) + \int_0^b \int_0^{\cdot} K(u,v)\,du\,dx(v)$$

for some function K.

Proof. $y(t) = T(x(t))$ determines a Gaussian process $\{C_0, B, \lambda_r\}$ with covariance function r given by (see Lemma 1)

$$r(s,t) = \int_0^b R(s,u)R(t,u)du.$$

Now for all B measurable functions G,

$$E^r\{G(y)\} = E^w\{G(Tx)\} = E^w\{G(x)J(x)\},$$

from which it follows that λ_r is absolutely continuous with respect to λ_w. Hence by the corollary to Theorem 1, $q^2(u) = 1$ on $(0,b)$. Since q is continuous, it follows that either $q(u) = 1$ or $q(u) = -1$ on $(0,b)$. Thus

$$\begin{aligned}
\int_0^b x(u)\,d_uR(t,u) &= -\int_0^b R(t,u)\,dx(u) \\
&= -\int_0^b [Q(t,u) + L(t,u)]\,dx(u) \qquad \text{(see (1.6))} \\
&= \pm x(t) - \int_0^b L(t,u)\,dx(u).
\end{aligned}$$

But by (1.6), $L(t,u)$ is absolutely continuous in t. Also it is easy to show that $L(0,u) = 0$ for all $u \in I$ since $R(0,u) = 0$ for all $u \in (0,b]$. Hence there exists a function K such that

$$-L(t,u) = \int_0^t K(w,u)dw.$$

The result follows.

This brings us to our main theorem on Gaussian measures equivalent to Wiener measure.

THEOREM 5. *Let λ_r be a Gaussian probability measure determined by a covariance function $r \in G$ with factor R satisfying* (1.4)–(1.10). *Then λ_r is equivalent to Wiener measure λ_w on $\{C,B\}$ if and only if*

$$[R(t^+,t) - R(t^-,t)]^2 = 1 \text{ on } (0,b) \tag{3.8}$$

and

$$R(0,t) = 0 \text{ for all } t \in (0,b]. \tag{3.9}$$

Moreover when these two conditions hold, the Radon-Nikodym derivative of λ_r with respect to λ_w is given by

$$(d\lambda_r/d\lambda_w)(x) = |D(\bar{M})|^{-1}\exp[-\Phi(x)/2] \tag{3.10}$$

where $\bar{M}$ and Φ are defined as in (1.8) *and* (3.6) *respectively.*

Proof. The "only if" part of the theorem is the corollary to Theorem 1. To show the "if" part we suppose that (3.8) and (3.9) hold. We may as well assume that $R(t^+,t) - R(t^-,t) = 1$ for if not we may replace R by $-R$ without changing r.

Now consider the process $\{y(t), 0 \leqq t \leqq b\}$ determined by

$$y(t) = T(x(t)) = -\int_0^b x(u)\, d_u R(t,u)$$

where $\{x(t), 0 \leqq t \leqq b\}$ is the Wiener process. By Lemma 1, $E\{y(s)\,y(t)\} = r(s,t)$ so that

$$E^r\{F(y)\} = E^w\{F(Tx)\} \tag{3.11}$$

for all B measurable functions F for which either side exists. Making use of (1.5), (1.6), (1.8) and (3.9), we may write (for all $x \in C_0$ and hence for almost all $x \in C$)

$$T(x(\cdot)) = x(\cdot) + \int_0^b L(\cdot,u)\,dx(u) = x(\cdot) + \int_0^b \int_0^{\cdot} \bar{M}(w,u)\,dw\,dx(u)$$

where moreover $\bar{M}$ satisfies the hypotheses of Theorem 3. Hence applying that theorem to (3.11), we obtain

$$E^r\{F(y)\} = |D(\bar{M})|^{-1}E^w\{F(x)\exp[-\Phi(x)/2]\}. \tag{3.12}$$

But this implies that λ_r is absolutely continuous with respect to λ_w and that (3.10) holds.

To see that λ_w is absolutely continuous with respect to λ_r, let χ_A be the set characteristic function of a set of λ_r measure zero. By (3.12)

$$0 = E^r\{\chi_A(y)\} = |D(\bar{M})|^{-1}E^w\{\chi_A(x)\exp[-\Phi(x)/2]\}.$$

But since the exponential function is positive it follows that A is a set of λ_w measure zero. The conclusion follows.

4. **Application to triangular covariance functions.** In an earlier paper [6], the author studied the class of Gaussian measures determined by triangular covariance functions, that is, by functions r satisfying

$$r(s,t) = \begin{cases} u(s)v(t), & s \leqq t, \\ u(t)v(s), & s \geqq t, \end{cases}$$

where

$$u(0) \geqq 0, \tag{4.1}$$

$$v(t) > 0 \text{ on } I, \tag{4.2}$$

$$u'' \text{ and } v'' \text{ exist and are continuous on } I, \tag{4.3}$$

$$v(t)u'(t) - u(t)v'(t) > 0 \text{ on } I. \tag{4.4}$$

One of the results in that paper was the following theorem (see [6, p. 752]).

THEOREM 6. *Let λ_r be a Gaussian probability measure determined by a triangular covariance function satisfying* (4.1)–(4.4). *Then λ_r is equivalent to Wiener measure λ_w on $\{C, B\}$ if and only if*

$$v(t)u'(t) - u(t)v'(t) = 1 \textit{ on } I, \tag{4.5}$$

$$u(0) = 0. \tag{4.6}$$

Moreover if (4.5) *and* (4.6) *hold,*

$$(d\lambda_r/d\lambda_w)(x) = [v(0)/v(b)]^{1/2}\exp(1/2)\int_0^b v'(t)d[x^2(t)/v(t)]. \tag{4.7}$$

We wish to demonstrate that this result is a special case of Theorem 5. First we note that the class of triangular covariance functions satisfying (4.1)–(4.4) belongs to G. To see this, let

$$R(s,t) = \begin{cases} v(s)\sqrt{([d/dt][u(t)/v(t)])}, & s > t \text{ or } s = t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then R satisfies (1.1)–(1.10) and moreover

$$r(s,t) = \int_0^b R(s,w)R(t,w)dw.$$

Also in this case,

$$[R(t^+,t)-R(t^-,t)]^2 = v(t)u'(t)-u(t)v'(t)$$

so that (3.8) is equivalent to (4.5). Similarly $R(0,t)=0$ on $(0,b]$ is equivalent to $r(0,0)=0$ which in our case means that $u(0)=0$ so that (3.9) becomes (4.6).

Finally assuming that (4.5) and (4.6) hold, one may readily check that

$$R(s,t) = \begin{cases} v(s)/v(t), & s>t \text{ or } s=t=0,\\ 0, & \text{otherwise,}\end{cases}$$

$$\bar{M}(s,t) = \begin{cases} v'(s)/v(t), & s>t,\\ v'(s)/2v(s), & s=t,\\ 0, & s<t,\end{cases}$$

$$\bar{M}^{-1}(s,t) = \begin{cases} -v'(s)/v(s), & s>t,\\ -v'(s)/2v(s), & s=t,\\ 0, & s<t,\end{cases}$$

$$\begin{aligned} D(\bar{M}) &= \exp\int_0^b \bar{M}(s,s)ds = \exp(1/2)\int_0^b [v'(s)/v(s)]ds \\ &= [v(b)/v(0)]^{1/2}, \\ \Phi(x) &= -\int_0^b v'(t)d[x^2(t)/v(t)]. \end{aligned}$$

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