THE ORDER DUAL OF THE SPACE OF RADON MEASURES(1)

BY

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Introduction. In [14] and [15] S. Kaplan studied the second dual of the Banach lattice of all continuous real-valued functions on a compact space. Then in [16] he initiated a study of the second dual of the lattice of continuous functions with compact support on a locally compact space. It is the purpose of this paper to continue the study of the locally compact case.

For a locally compact space X, let L_k denote the vector lattice of Radon measures on X. In §2 the basic properties of L_k are established. In §3 we devote our attention to the proof of the following theorem: *Every purely nonatomic measure defined on a σ-compact space can be extended to a measure on a countably compact space.*

Given a compact subset K of X, let $L(K)$ denote the set of Radon measures on K. Then $L(K)$ can be identified with an ideal in L_k. The set $\bigcup L(K)$, where the union is taken over all compact subsets of X, is the set of all measures with compact support. It appears that the order dual M of $\bigcup L(K)$ is an appropriate object of study as well as the order dual M_k of L_k and the order dual M_b of the space L_b of all finite measures. In particular, C (the set of all continuous real-valued functions on X) can be embedded in M while in general this is not possible for either M_k or M_b. The spaces M_k and M_b appear as ideals in M. Also, M can be characterized as the set of all multiplication operators on L_k.

In §5 we consider the question of whether M_k can be identified with a set of continuous functions with compact support. This is the question raised by Kaplan in [16, §§5,7]. After giving an example which shows that in general this is not possible, we state conditions which are sufficient to insure that M_k will be the set of continuous functions with compact support on some locally compact space. In §6 we turn our attention to the duality relations which exist between the ideals in L_k and those in M.

It will be assumed that the reader is familiar with Kaplan's papers, *On the second dual of the space of continuous functions* [14], [15], [16].

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1. **Preliminaries.** In this section E denotes a vector lattice whose positive cone is E_+. For elements $a \in E$, $b \in E$, the set $\{x \in E : a \leqq x \leqq b\}$ will be called an *interval*. A subset A of E is *bounded* if A is contained in an interval. A linear subspace I is an *ideal* if for $x \in I$, $y \in E$, $|y| \leqq |x|$ implies that $y \in I$. If x belongs to an ideal I, then $0 \leqq x^+ \leqq |x|$ implies that $x^+ \in I$. Hence an ideal is a subvector lattice. The ideal *generated* by a subset A of E is the intersection of all ideals containing A.

For a net $\{x_\alpha\}$ in E and an element $x \in E$, we define $x_\alpha \downarrow x$ to mean $x = \bigwedge_\alpha x_\alpha$ and that $x_\alpha \leqq x_\beta$ whenever $\alpha > \beta$. Similarly $x_\alpha \uparrow x$ means $x = \bigvee_\alpha x_\alpha$ and that $x_\alpha \geqq x_\beta$ whenever $\alpha > \beta$. The net $\{x_\alpha\}$ *converges* to x if there exists a net $\{y_\alpha\}$ in E such that $|x - x_\alpha| \leqq y_\alpha$ and $y_\alpha \downarrow 0$. We write $\lim x_\alpha = x$ whenever $\{x_\alpha\}$ converges to x. Since $\big||x| - |x_\alpha|\big| \leqq |x - x_\alpha|$, it follows that $\lim |x_\alpha| = |x|$ provided $\lim x_\alpha = x$. A subset A of E is *closed* (σ-*closed*) if it contains the limit of each convergent net (convergent sequence) in A. The *closure* (σ-*closure*) of A is the intersection of all closed (σ-closed) sets containing A. The closure of A will be denoted by $\bar{A}$ and the σ-closure by σA. The closed ideal *generated* by A is defined to be the intersection of all closed ideals containing A. Two elements x and y in E are *disjoint* if $|x| \wedge |y| = 0$. For $A \subset E$, define $A' = \{x \in E : |x| \wedge |a| = 0$ for each $a \in A\}$. Then A' is a closed ideal in E. If A and B are subsets of E with $A \subset B$, then A is said to be *dense* in B provided each point of B is the limit of a net in A.

THEOREM 1.1 (RIESZ). *Let E be a complete vector lattice. For any subset A of E, $(A')'$ is the closed ideal generated by A, and $E = A' \oplus (A')'$.*

For a proof of Theorem 1.1 see [3, Théorème 1, p. 25].

REMARK 1. A set may fail to be dense in its closure. However, an ideal is dense in its closure. In fact, $(\bar{I})_+$ may be obtained from I_+ by adjoining the suprema of subsets of I_+. Whence it follows that the closed ideal generated by a set A is the closure of the ideal generated by A.

REMARK 2. We shall use the unmodified terms: dense, closure, convergent, etc. in the sense defined above. Whenever we wish to use these terms in the usual topological sense with respect to a topology τ, we shall then modify them with τ, e.g., τ-dense, τ-closure, τ-convergent, etc.

REMARK 3. If E is complete, then $\lim x_\alpha = x$ is equivalent to the statement that $\{x_\alpha : \alpha > \beta\}$ is bounded for some index β and $x = \bigvee_\beta \bigwedge_{\alpha \geqq \beta} x_\alpha = \bigwedge_\beta \bigvee_{\alpha \geqq \beta} x_\alpha$.

Let E and F be vector lattices. A mapping of E into F is *bounded* if it transforms bounded sets into bounded sets. A linear mapping is said to be *positive* if it maps E_+ into F_+. If F is a complete vector lattice, then a linear mapping of E into F is bounded if and only if it is the difference of two positive linear mappings. In addition, the set of all bounded linear mappings of E into a complete vector lattice F is a complete vector lattice. The proofs of these statements can be patterned after those for linear functionals [3, Théorème 1, p. 35]. A mapping ϕ of E into F is *continuous* if for each convergent net $\{x_\alpha\}$ in E, the net $\{\phi(x_\alpha)\}$ converges in F and $\phi(\lim x_\alpha) = \lim \phi(x_\alpha)$. It is easy to show that when ϕ is a bounded linear mapping of E into a complete vector lattice F, ϕ is continuous if $\lim |\phi|(x_\alpha) = 0$ for each net in E such that $x_\alpha \downarrow 0$.

PROPOSITION 1.2. *If ϕ is a continuous mapping of E into F, then $\phi(\bar{A}) \subset \overline{\phi(A)}$ for each subset of E.*

Since the vector lattice operations are continuous mappings of $E \times E$ into E we have:

PROPOSITION 1.3. *The closure of a subvector lattice is a subvector lattice.*

A linear mapping of E into F is a *lattice homomorphism* if it preserves the lattice operations. A one-to-one lattice homomorphism is called a *lattice isomorphism.*

PROPOSITION 1.4. *Let ϕ be a linear transformation of E into F which maps E_+ onto F_+. If ϕ is one-to-one on E_+ and if its inverse maps F_+ into E_+, then ϕ is a lattice isomorphism of E onto F.*

In this paper we shall be mainly concerned with linear functionals. The set of all bounded linear functionals defined on the vector lattice E will be denoted by $\Omega(E)$ while $\tilde{\Omega}(E)$ will signify the set of elements of $\Omega(E)$ which are continuous. If the set of positive linear functionals is taken as the positive cone, then $\Omega(E)$ and $\tilde{\Omega}(E)$ are complete vector lattices and $\tilde{\Omega}(E)$ is a closed ideal in $\Omega(E)$ [15, (3.1)]. An element ϕ of $\Omega(E)$ is also an element of $\tilde{\Omega}(E)$ if and only if $\lim |\phi|(x_\alpha) = 0$ for each net in E such that $x_\alpha \downarrow 0$.

Given $a \in E_+$, define $\|\phi\|_a = |\phi|(a)$ for each $\phi \in \Omega(E)$. Then $\| \quad \|_a$ is a pseudonorm on $\Omega(E)$. The family of all such pseudonorms defines a locally convex topology on $\Omega(E)$ which is compatible with the order in $\Omega(E)$ and with respect to which $\Omega(E)$ is complete [3, Exercise 9, p. 40]. This topology will be denoted by $|w|(\Omega(E), E)$. In a similar way, the $|w|(E, \Omega(E))$-topology on E can be defined.

For a subset A of E, let $A^\perp$ denote the orthogonal complement of A in $\Omega(E)$, i.e., $A^\perp = \{\phi \in \Omega(E) : \phi(x) = 0 \text{ for all } x \in A\}$. Similarly, define $B^\perp = \{x \in E: \phi(x) = 0 \text{ for all } \phi \in B\}$ for each subset B of $\Omega(E)$. The *null ideal* in $\Omega(E)$ of a subset A of E is the set $\{\phi \in \Omega(E) : |\phi|(|x|) = 0 \text{ for each } x \in A\}$. The null ideal of A is the largest ideal contained in $A^\perp$. Furthermore, if I is the ideal in E generated by A, then $I^\perp$ is the null ideal of A. The null ideal in E of a subset of $\Omega(E)$ is defined in a similar manner. Some of the properties of null ideals are listed in [15, §§2,3].

2. **The spaces L_k and L_b.** Throughout this paper X will be a locally compact space. The symbol $C = C(X)$ will be used to represent the vector lattice of real continuous functions on X. The set of bounded functions in C will be denoted

by $C_b = C_b(X)$. The set of real continuous functions which vanish at infinity will be denoted by $C_\infty = C_\infty(X)$ while $C_k = C_k(X)$ will be the set of functions in C which have compact support. C_k and C_∞ appear as lattice ideals in C.

Let $L_k = L_k(X)$ be the order dual $\Omega(C_k)$ of C_k. Similarly define $L_b = L_b(X) = \Omega(C_\infty)$. Then L_b is also the Banach space dual of C_∞ [14, (3.8)] while L_k is the space of all Radon measures [3, p. 54]. For $\mu \in L_k$ define $\|\mu\| = \sup\{|\mu(h)| : h \in C_k, \|h\| \leqq 1\}$ ($\|\mu\|$ is taken to be ∞ if the defining set is unbounded). The space L_b consists precisely of those elements of L_k for which $\|\mu\| < \infty$. The elements of L_b will be called *finite* measures and those in the σ-closure σL_b of L_b will be called *σ-finite.* The spaces L_b and σL_b are dense ideals in L_k [16,(3.5)].

We shall write $C(X)$, $L_k(X)$, etc. only when we wish to emphasize the underlying space X. At other times we will write merely C, L_k, etc.

PROPOSITION 2.1. *The mapping $\mu \to \|\mu\|$ is continuous on L_b.*

Proof. Let $\{\mu_\alpha\}$ converge to μ in L_b. Then there is a net $\{\lambda_\alpha\}$ in L_b such that $|\mu - \mu_\alpha| \leqq \lambda_\alpha$ while $\lambda_\alpha \downarrow 0$. Thus the $\lim \|\lambda_\alpha\| = 0$ [14, (4.2)] and [16, (3.1)]; therefore $\lim \|\mu_\alpha\| = \|\mu\|$.

The *support* of $\mu \in L_k$ is defined to be the intersection of all closed sets $F \subset X$ such that $\mu(f) = 0$ for each $f \in C_k$ which vanishes on F. It follows that if $f \in C_k$ fails to vanish on the support of μ, then $|\mu|(|f|) > 0$ [3, Proposition 9, p. 72].

For each $x \in X$ define $\varepsilon_x(f) = f(x)$ where $f \in C$. Then $\varepsilon_x \in L_b$, $\varepsilon_x > 0$ and $\|\varepsilon_x\| = 1$. The support of ε_x is $\{x\}$.

PROPOSITION 2.2. *Let $\mu \in L_k$. If the support of μ consists of a single point x, then $\mu = r\varepsilon_x$ for some real number r.*

Proof. Let $h \in C_k$ be such that $h(x) = 1$ and then set $r = \mu(h)$. Since $f - [f(x)]h$ vanishes on the support of μ, $\mu(f) = f(x)\mu(h) = r\varepsilon_x(f)$ for each $f \in C_k$. Hence $\mu = r\varepsilon_x$.

Recall that $\tilde{\Omega}(E)$ denotes the set of all bounded continuous linear functionals on the vector lattice E.

PROPOSITION 2.3. *Let $x \in X$. Then $\varepsilon_x \in \tilde{\Omega}(C_k)$ if and only if x is an isolated point of X.*

Proof. Set $f_x = \bigwedge\{g : g \in (C_k)_+, g(x) = 1\}$. Clearly f_x exists in C_k and $f_x(y) = 0$ for all $y \in X$, $y \neq x$. If $\varepsilon_x \in \tilde{\Omega}(C_k)$, then $f_x(x) = \varepsilon_x(f_x) = 1$. Thus x is isolated. Conversely, suppose x is an isolated point. Then $[\lim f_\alpha](x) = \lim f_\alpha(x)$ for any convergent net in C_k. Hence $\varepsilon_x \in \tilde{\Omega}(C_k)$.

PROPOSITION 2.4. *The following are equivalent:*

(i) $L_k = \tilde{\Omega}(C_k)$.

(ii) $\varepsilon_x \in \tilde{\Omega}(C_k)$ *for all* $x \in X$.

(iii) *X is a discrete space.*

Proof. Clearly (i) implies (ii). That (ii) and (iii) are equivalent follows from Proposition 2.3. We show that (iii) implies (i). Let $f_\alpha \downarrow 0$ in C_k. Since X is discrete $\lim f_\alpha(x) = 0$ for each $x \in X$. Thus $\{f_\alpha\}$ converges uniformly to 0 on each compact set. In addition f_α vanishes outside the support of f_β for each $\alpha > \beta$; therefore $\lim \|f_\alpha\| = 0$. Since the elements of L_k are Radon measures $\lim |\mu|(f_\alpha) = 0$ for each $\mu \in L_k$. Thus $L_k = \tilde{\Omega}(C_k)$.

Let $(L_k)_0$ denote the closed ideal in L_k generated by $\{\varepsilon_x : x \in X\}$.

PROPOSITION 2.5. *$(L_k)_0$ is the $|w|(L_k, C_k)$-closure of the linear subspace of L_k generated by $\{\varepsilon_x : x \in X\}$.*

Proof. Let I be the linear subspace generated by $\{\varepsilon_x : x \in X\}$. We shall prove that I is an ideal. It suffices to show that the linear space genrated by each ε_x is an ideal [14, (1.2)]. If $\mu \in L_k$, $|\mu| \leqq s\varepsilon_x$ for some s, then the support of μ is $\{x\}$. Hence $\mu = r\varepsilon_x$ [Proposition 2.2]. Thus I is an ideal. Since an ideal in L_k is closed if and only if it is $|w|(L_k, C_k)$-closed [14, (11.5)], the $|w|(L_k, C_k)$-closure of I is $(L_k)_0$.

Let us identify the element $\sum r_i \varepsilon_{x_i}$, $i = 1, 2, \cdots n$, with the function on X defined by $\mu(x_i) = r_i$ and $\mu(x) = 0$ for $x \neq x_i$, $i = 1, 2, \cdots, n$. Since L_k is $|w|(L_k, C_k)$-complete [3, Exercise 9, p. 40], it follows that $(L_k)_0$ is the $|w|(L_k, C_k)$-completion of the linear space generated by $\{\varepsilon_x : x \in X\}$. Therefore we have

PROPOSITION 2.6. *$(L_k)_0$ is lattice isomorphic with the lattice of real-valued functions μ on X such that $\sum_{x \in K} |\mu(x)| < \infty$ for each compact subset K of X. (The order is given by: $\mu \geqq 0$ if $\mu(x) \geqq 0$ for all $x \in X$.)*

Define $(L_k)_1 = (L_k)_0'$. Then $L_k = (L_k)_0 \oplus (L_k)_1$. For $\mu \in L_k$, let μ_0 and μ_1 denote the components of μ in $(L_k)_0$ and $(L_k)_1$ respectively. For $A \subset L_k$ define $A_0 = \{\mu_0 : \mu \in A\}$ and $A_1 = \{\mu_1 : \mu \in A\}$. In particular $(L_b)_0$ and $(L_b)_1$ denote the projections of L_b into $(L_k)_0$ and $(L_k)_1$. We can now state the following:

PROPOSITION 2.7. *$(L_b)_0$ is isometric and lattice isomorphic with the Banach lattice of real functions μ on X for which $\sum_{x \in X} |\mu(x)| < \infty$. (The order is given by: $\mu \geqq 0$ if $\mu(x) \geqq 0$ for all $x \in X$ and the norm by: $\|\mu\| = \sum_{x \in X} |\mu(x)|$.)*

The space $(L_k)_0$ consists of the purely atomic Radon measures and $(L_k)_1$ contains the nonatomic Radon measures.

The following is an easy consequence of Proposition 2.6.

PROPOSITION 2.8. *If every atomic measure is finite, then X is countably compact.*

The converse of Proposition 2.8 is false. This will be shown in §3.

For $\phi \in C$ and $\mu \in L_k$ define $\phi^t\mu(f) = \mu(f\phi)$, $f \in C_k$. This defines ϕ^t as a continuous operator on L_k [16, (6.6)]. Observe that $\phi^t L_k \subset L_b$ for each $\phi \in C_k$.

PROPOSITION 2.9. *If $\mu \in (L_k)_+$, the mapping: $\phi \to \phi^t\mu$ is a lattice homomorphism of C into L_k.*

Proof. Let $\phi \in C$. It is enough to show that $(\phi^t\mu)^+ = (\phi^+)^t\mu$. According to [16, (8.2)] $(\phi^+)^t\mu$ and $(\phi^-)^t\mu$ are disjoint. Thus $(\phi^t\mu)^+ = [(\phi^+)^t\mu - (\phi^-)^t\mu]^+ = (\phi^+)^t\mu$ [2, Lemma 2, p. 220].

We now consider measures on subspaces of X.

PROPOSITION 2.10. *If W is a locally compact subspace of X, then $L_k(X) \cap L_k(W)$ is a closed ideal in $L_k(X)$.*

Proof. Let $I = \{f \in C_k(X) : f(x) = 0 \text{ on } W\}$. Then $C_k(X)/I$ is isomorphic with $C_k(W^-)$ where W^- denotes the closure of W with respect to the topology on X. Thus $L_k(X) = L_k(W^-) \oplus I^{\perp\prime}$ [16, §2]. Since $C_k(W)$ can be identified with an ideal in $C_k(W^-)$, we have $L_k(W^-) = L_k(W) \cap L_k(W^-) \oplus C_k(W)^{\perp}$. Thus $L_k(W) \cap L_k(X) = L_k(W) \cap L_k(W^-)$ is a closed ideal in $L_k(X)$.

COROLLARY 2.11. *If W is a closed subspace of X, then $L_k(W)$ is a closed ideal in $L_k(X)$. Furthermore $L_k(W)$ consists of those elements of $L_k(X)$ whose support is contained in W.*

Let $\mu \in L_k(X)$. The component of μ in $L_k(W) \cap L_k(X)$ will be called the *restriction* of μ to W.

If K is a compact subset of X, we shall write $L(K)$ rather than $L_k(K)$. Our primary interest will lie with the set $\bigcup L(K)$ where the union is taken over all of the compact subsets of X. The set $\bigcup L(K)$ consists of those elements of L_k whose support is compact.

PROPOSITION 2.12. $\bigcup L(K) = C_k L_k$.

Proof. Let $\phi \in C_k$ and $\mu \in L_k$. Then the support of $\phi^t\mu$ is contained in that of ϕ [3, Proposition 10, p. 73]. Thus $C_k L_k \subset \bigcup L(K)$. Clearly $\bigcup L(K) \subset C_k L_k$.

The following is due to S. Kaplan [16, (4.2)]:

PROPOSITION 2.13. *The set $\bigcup L(K)$ is an ideal in L_b and is dense in L_k.*

COROLLARY 2.14. *If X is discrete, then $L_k = (L_k)_0$.*

Proof. If X is discrete, then $\bigcup L(K)$ is the set of measures with finite support. Hence $\bigcup L(K) \subset (L_k)_0$. The corollary now follows from Proposition 2.13.

EXAMPLE. Let N^* be the one-point compactification of N. Then every measure on N^* is atomic. Thus the converse of Corollary 2.14 is false.

3. **Extensions of measures.** Let $\mu \in L_k(X)$. We shall say that μ is *extendable* to a locally compact space T if X is a topological subspace of T and μ is the restriction to X of some element of $L_k(T)$. Any element of $L_k(T)$ whose restriction to X is μ will be called an *extension* of μ.

In this section we shall be concerned primarily with the proof of the following:

THEOREM 3.1. *Let μ be a Radon measure on a σ-compact space X. If the atomic part of μ is finite, then μ can be extended to a locally compact space T on which every atomic measure is finite. In particular, T is countably compact.*

This theorem was motivated by a question posed by S. Kaplan [16, §3]: *If every atomic measure on a space is finite, does it follow that every measure on the space is finite*? To answer this question we need only to let X be the real line, μ be the Lebesgue measure and then consider the space T mentioned in Theorem 3.1. While every atomic measure on T is finite, the space admits measures which are not finite. In particular, no extension of the Lebesgue measure is finite.

In proving Theorem 3.1 we shall construct a space T for each $\mu \in L_k$ and then show that T has the required properties when the atomic part of μ is finite. Let μ be an arbitrary element of L_k. Denote by A the set of $f \in C_b$ for which $f^t\mu \in L_b$; then set $A^\beta = \{f^\beta : f \in A\}$. Here f^β denotes the continuous extension of f to the Stone-Čech compactification βX [8, Theorem 6.5]. Now let T be the complement in βX of the null set of A^β, i.e., $T = \{p \in \beta X : f^\beta(p) \neq 0 \text{ for some } f \in A\}$. Since T is open in the compact space βX, it is locally compact. It follows from the fact that $C_k(X) \subset A$ that $X \subset T \subset \beta X$. Thus $C_b(X)$ and $C_b(T)$ are isomorphic with $C(\beta X)$ [8, Theorem 6.7]. For each $f \in C_b(X)$ we shall identify f^β and $f^\beta \mid T$ with f and use the symbol f to denote all three functions. We are now in a position to prove the following:

LEMMA 3.2. (i) *A is an ideal in C_b.*
(ii) *$C_k(T) \subset A \subset C_\infty(T)$.*
(iii) *μ is extendable to T, i.e., $\mu \in L_k(X) \cap L_k(T)$.*

Proof. (i) Since $|\phi|^t|\mu| = |\phi^t\mu|$ for each $\phi \in C$ [Proposition 2.9 and 16, (6.6)], $f \in A$, $g \in C_b$, $|g| \leqq |f|$ implies $|g^t\mu| \leqq |f|^t|\mu| = |f^t\mu|$. Thus $g \in A$.

(ii) It is clear from the manner in which T was constructed that $A \subset C_\infty(T)$. Next let $g \in C_k(T)$ and denote the support of g by K. The family $\{U_f\}_{f \in A}$ where $U_f = \{x \in T : f(x) > 1\}$ forms an open cover for K. Since K is compact there are finitely many elements $f_1, f_2, \cdots, f_n$ in A such that $\bigvee_{i=1}^n f_i$ dominates the characteristic function of K. Thus $|g| \leqq \|g\| \bigvee_{i=1}^n |f_i|$. Since A is an ideal, $g \in A$. Thus $C_k(T) \subset A$.

(iii) For $f \in C_k(T)$ define $\bar{\mu}(f) = \|(f^t\mu)^+\| - \|(f^t\mu)^-\|$. It follows from (ii) and the L-space property of L_b [2, p. 256] that $\bar{\mu}$ is a bounded linear functional on $C_k(T)$. Thus $\bar{\mu} \in L_k(T)$. We show that $\bar{\mu}$ is an extension of μ. For $g \in C_k(X)$ let $\phi \in C_k(X)$ be such that $g\phi = g$. Then $\phi(x) = 1$ on the support of $g^t\mu$. Hence $\|(g^t\mu)^+\| = (g^t\mu)^+(\phi)$ and $\|(g^t\mu)^-\| = (g^t\mu)^-(\phi)$. Thus $\bar{\mu}(g) = g^t\mu(\phi) = \mu(g\phi) = \mu(g)$ for each $g \in C_k(X)$.

LEMMA 3.3. *Let $\lambda \in L_k(T)$. If $|\lambda| \wedge |\mu| = 0$, then $\lambda \in L_b(T)$.*

Proof. Since μ and $|\mu|$ give rise to the same space T and since L_b is an ideal, we may assume that λ and μ are positive. First we verify the following: (I) If $\lambda \wedge \mu = 0$ and $g \in C_k$, $0 \leqq g \leqq \mathbf{1}$ ($\mathbf{1}(x)=1$ for $x \in T$) with $\lambda(g) > 0$, then for each $\varepsilon > 0$ there exists $f \in C_k$, $0 \leqq f \leqq \mathbf{1}$ whose support is contained in that of g such that $\lambda((2f - \mathbf{1})^+) \geqq (1 - 3\varepsilon)\lambda(g)$ while $\mu(f) < \varepsilon$. To prove this statement let $\phi \in (C_k)_+$ be such that $g\phi = g$ and choose $\phi_1 \in C_k$ so that $0 \leqq \phi_1 \leqq \phi$ and $\mu(\phi_1) + \lambda(\phi - \phi_1) < \min\{\varepsilon, \varepsilon\lambda(g)\}$. Direct computations will show that $f = (rg \wedge \mathbf{1})\phi_1$ is the required function when $r = \lambda(\phi)[\varepsilon\lambda(g)]^{-1}$.

Since X is σ-compact, there is an increasing sequence of open relatively compact sets $\{U_n\}$ such that $X = \bigcup U_n$. Suppose that $\mu \wedge \lambda = 0$ while $\|\lambda\| = \infty$. By using (I) in the first part of this proof, it can be shown that there exists a sequence $\{f_n\}$ in $C_k(T)$ such that $0 \leqq f_n \leqq \mathbf{1}$, f_n vanishes on U_n and $\lambda((2f_n - \mathbf{1})^+) \geqq n$ while $\mu(f_n) \leqq 2^{-n}$. Since all but finitely many of the f_n vanish on each compact set in X, $f = \bigvee f_n$ exists in $C_b(X)$. Also, $f\phi$ is a finite supremum of the $f_n\phi$ for each $\phi \in (C_k(X))_+$. Thus $f^t\mu(\phi) = \mu(f\phi) \leqq \sum \mu(f_n\phi) \leqq \|\phi\|$; whence $f^t\mu \in L_b$ and $f \in A$. This implies that $(2f - \mathbf{1})^+ \in C_k(T)$ [Lemma 3.2, (ii)].

On the other hand $\lambda((2f - \mathbf{1})^+) \geqq \lambda((2f_n - \mathbf{1})^+) \geqq n$ for each n. This contradiction completes the proof of Lemma 3.3.

Observe that if the atomic part of μ is finite, then μ and the nonatomic part of μ give rise to the same space. If μ is purely nonatomic, then it follows from Lemma 3.3 that every atomic measure on T is finite. The proof of Theorem 3.1 is now complete.

REMARK. The finiteness of the atomic part of μ is used only in the last paragraph of the proof of Theorem 3.1. Hence Lemmas 3.2 and 3.3 are valid for any $\mu \in L_k$.

THEOREM 3.4. *Let μ be a Radon measure on a σ-compact space X. Then μ can be extended to a countably compact space T if and only if the atomic part of μ (considered as a function on X) vanishes at infinity.*

Proof. *Sufficiency.* As noted in the proof of Lemma 3.3, μ may be taken to be positive. Assume that μ_0 vanishes at infinity and let T be constructed as in the proof of Theorem 3.1. Then $\mu \in L_k(X) \cap L_k(T)$ [Lemma 3.2]. Observe that $\mu_0(x) = 0$ for all $x \in T - X$. Suppose that T is not countably compact. Then there is an infinite set $F \subset T$ which has no limit points in T. Given an increasing sequence $\{U_n\}$ of open relatively compact sets in X such that $X = \bigcup U_n$, there exists a sequence $\{x_n\}$ of distinct points of F such that $x_n \notin \mathrm{cl}_X U_n$. Furthermore, since μ_0 vanishes at infinity, the sequence can be chosen so that $\sum \mu_0(x_n)$ converges. Since ε_x and $\mu - [\mu_0(x)]\varepsilon_x$ are disjoint for each $x \in T$, there exists $f_n \in C_k(T)$, $0 \leqq f_n \leqq \mathbf{1}$ such that f_n vanishes on U_n, $f_n(x_n) = 1$ and $\mu(f_n) < \mu_0(x_n) + 2^{-n}$ (see the proof of (6.8) in [14]). As in the proof of Lemma 3.3, we see that $f = \bigvee f_n$ exists in $C_b(X)$ and that $f \in A \subset C_\infty(T)$. Since $f(x_n) = 1$ for each n, the sequence $\{x_n\}$ is contained in a compact set. This gives a contradiction. Therefore T is countably compact.

Necessity. Suppose μ is extendable to a countably compact space T. For each $\varepsilon > 0$, consider the set $E = \{x \in T : |\mu_0(x)| \geqq \varepsilon\}$. If E is infinite, then it has a limit point x_0. Each neighborhood of x_0 meets E in an infinite set. Hence the $|\mu|$-measure of each neighborhood of x_0 is infinite. This is impossible since T is locally compact. Thus the set E is finite for each $\varepsilon > 0$. Therefore μ_0 vanishes at infinity.

EXAMPLE 1. In Theorem 3.4, it is essential that X be normal. To see this, let X be the Tychonoff plank [8, §8.20]. Any countably compact, locally compact space containing the plank also contains the one-point compactification of the plank. Thus the finite measures are the only ones that can be extended to a countably compact space. Define $\mu \in (L_k)_0$ as follows: $\mu(\omega_1, n) = 1/n$, $\mu(\sigma, \tau) = 0$ if $\sigma < \omega_1$. Then $\mu = \mu_0$ vanishes at infinity but it is not extendable to a countably compact space.

EXAMPLE 2. Let X be the real line, μ be the Lebesgue measure and T be the corresponding space given by Theorem 3.1. Let $h_n \in C_k(T)$ be given by $h_n(x) = 1$ for $|x| \leqq n$, $h_n(x) = 0$ for $|x| \geqq n + 1$ or $x \in T - X$, and by straight line segments elsewhere. It is not difficult to show that if $\lambda \in L_k(X)$, then λ is also in $L_k(T)$ only if the sequence $\{n^{-1}\lambda(h_n)\}$ is bounded. Then, in view of Lemma 3.3, this sequence is bounded for each $\lambda \in L_k(T)$. Observe that $\lim n^{-1}\lambda(h_n) = 0$ for each $\lambda \in L_b$ while $\lim n^{-1}\mu(h_n) = 2$. These facts will be needed for later examples.

4. **The spaces M, M_b and M_k.** In considering Radon measures on locally compact spaces we found in §2 that each of the sets $\bigcup L(K)$, L_b, σL_b and L_k is of interest. In this section we shall study the duals of each of these spaces. Define

$$M = M(X) = \Omega(\bigcup L(K)),\ M_b = M_b(X) = \Omega(L_b) \text{ and } M_k = M_k(X) = \tilde{\Omega}(L_k).$$

We take $\tilde{\Omega}(L_k)$ for M_k rather than $\Omega(L_k)$ since $\Omega(L_k)$ is not in general a subset of M and since the elements of $\Omega(L_k)$ which are not in $\tilde{\Omega}(L_k)$ are only remotely connected to the space C_k (the closure of C_k in $\Omega(L_k)$ is $\tilde{\Omega}(L_k)$ [15, (3.10)]). The various relations existing among M_k, $\Omega(\sigma L_b)$ and $\Omega(L_k)$ will be discussed at the end of this section.

Observe that for compact spaces $M = M_b = M_k$. Let K be a compact subset of X. Since $L(K)$ is a closed ideal in $\bigcup_{\text{all } K} L(K)$, $M = \Omega(L(K)) \oplus L(K)^{\perp}$ [16, (2.4)]. Therefore $M(K) = \Omega(L(K))$ is a closed ideal in M, and $\bigcup M(K)$, where the union is taken over all compact subsets K of X, is a subspace of M.

PROPOSITION 4.1. (i) $M = \tilde{\Omega}(\bigcup L(K))$.
(ii) $M_b = \tilde{\Omega}(L_b)$.
(iii) $M_k = \tilde{\Omega}(\sigma L_b)$.
(iv) $\bigcup M(K) \subset M_k \subset M_b \subset M$. *Furthermore, each set is a dense ideal in M.*

Proof. (i) Let $\mu_\alpha \downarrow 0$ in $\bigcup L(K)$. We must show that $\lim |f|(\mu_\alpha) = 0$ for each $f \in M$. We may assume that $\{u_\alpha\}$ has an initial element μ_β. If K is the support of μ_β, then $\{\mu_\alpha\}$ is contained in $L(K)$. Since $M = M(K) \oplus L(K)^{\perp}$, it suffices to show that $\lim |f|(\mu_\alpha) = 0$ for each $f \in M(K)$. This follows from the relation $M(K) = \tilde{\Omega}(L(K))$ [15, (4.1)].

For proofs of (ii) and (iii) see [16, (3.1) and (4.5)]. Consider (iv). Since $\bigcup L(K)$ is dense in L_b, it follows from [16, (2.8)] that $M_b = \tilde{\Omega}(L_b)$ is a dense ideal in $M = \tilde{\Omega}(\bigcup L(K))$. The remaining parts of (iv) are taken from [16, (4.3) and (3.6)].

We shall find the concept of multiplication operator a useful tool in characterizing the space M. A bounded operator on a vector lattice E is called a *multiplication operator* if each closed ideal in E is invariant with respect to the operator. The multiplication operators on a complete vector lattice constitute a closed ideal in the vector lattice of all bounded operators on E [16, §6]. The multiplication operators also form a commutative ring.

PROPOSITION 4.2. *Each multiplication operator on L_k is continuous.*

Proof. Since L_k is a complete vector lattice, each bounded operator on L_k is a difference of positive operators. Thus we can restrict our attention to positive operators. Let T be a positive multiplication operator and let $\mu_\alpha \downarrow 0$ in L_k. We must show that $\mu = \bigwedge T\mu_\alpha = 0$. Suppose $\mu > 0$. Since $\bigcup L(K)$ is dense in L_k, there exists a compact $K \subset X$ such that the component λ of μ in $L(K)$ is nonzero. Let λ_α be the component of μ_α in $L(K)$. Then $\lambda_\alpha \downarrow 0$. Since T is a multiplication operator $T\lambda_\alpha$ is the component of $T\mu_\alpha$ in the closed ideal $L(K)$. Thus $\lambda = \bigwedge T\lambda_\alpha$. Now $L(K)$ is a Banach lattice; hence T is norm bounded on $L(K)$. (Cf. [2, Theorem 10, p. 248].) It follows from Proposition 2.1 that $\lim_\alpha \| T\lambda_\alpha \| = 0$. This implies that $\lambda = 0$ [14, (3.4)]. This is a contradiction.

For each multiplication operator T on L_k and each $\mu \in L_k$ such that $T\mu \in L_b$, define $T^s(\mu) = \|(T\mu)^+\| - \|(T\mu)^-\|$. Since L_b is an abstract L-space, T^s is a bounded linear functional on the ideal $I = \{\mu : \mu \in L_k,\ T\mu \in L_b\}$. The mapping $\mu \to T^s(\mu)$ is a composition of the continuous mappings: $\mu \to T\mu$, $\lambda \to \|\lambda\|$ and the lattice operations; hence T^s is a continuous linear functional on I. Now $L(K)$ is a closed ideal; hence $TL(K) \subset L(K)$ and $T \cup L(K) \subset \bigcup L(K) \subset L_b$. Thus $\bigcup L(K)$ is a dense ideal in I. Whence $T^s \in \tilde{\Omega}(\bigcup L(K)) = M$ [16, (2.7)]. Also, observe that if $T \geqq 0$, then $T = 0$ if and only if $T^s = 0$.

Next we consider the inverse of $T \to T^s$. For $f \in M$ and $\mu \in L_k$ let $f^t\mu$ denote the element of L_k defined by $f^t\mu(h) = f(h^t\mu)$, $h \in C_k$. Then f^t is a bounded linear operator on L_k. Arguing as in [16, (6.2)] it is easy to show that f^t is a continuous operator.

PROPOSITION 4.3. (i) *For each multiplication operator T on L_k, $T^s \in M$.*
(ii) *For each $f \in M$, f^t is a continuous multiplication operator on L_k.*

Proof. Part (i) is proved above. To prove (ii) let $f \in M$ and let I be a closed ideal in L_k. For $\mu \in I \cap \bigcup L(K)$ let K be the support of μ; then $h^t\mu \in L(K)$ for all $h \in C_k$ [3, p. 73]. The component of f in $M(K)$ is norm bounded; hence there is a number r such that $|f(h^t\mu)| \leqq r \| h^t\mu \|$ for all $h \in C_k$. Since $\| h^t\mu \| \leqq |\mu|(|h|)$, it follows that $|f^t\mu| \leqq r|\mu|$. This proves that f^t maps $I \cap \bigcup L(K)$ into I. The operator f^t is continuous and $I \cap \bigcup L(K)$ is dense in I; hence $f^t I \subset I$ [Proposition 1.2].

THEOREM 4.4. *M is lattice isomorphic with the vector lattice of all multiplication operators on L_k.*

Proof. We shall show that the isomorphism is given by the mapping $T \to T^s$. This mapping is one-to-one on the set of positive multiplication operators. Clearly, the mapping $f \to f^t$ is order preserving. In view of Proposition 1.4, it now suffices to show $f \to f^t$ is the inverse of $T \to T^s$. For $f \in M_+$, $\mu \in (\bigcup L(K))_+$ we have $f^{ts}(\mu) = \|f^t\mu\|$. Since f^t is a multiplication operator the support of $f^t\mu$ is contained in that of μ. If $\phi \in (C_k)_+$ is such that $\phi(x) = 1$ on the support of μ, then $\phi^t\mu = \mu$. Also, ϕ has the constant value 1 on the support of $f^t\mu$; hence $\|f^t\mu\| = f^t\mu(\phi)$ [14, (4.1)]. Thus $f^{ts}(\mu) = f^t\mu(\phi) = f(\phi^t\mu) = f(\mu)$ for all $\mu \in \bigcup L(K)$. This completes the proof of Theorem 4.4.

REMARK. For a given $f \in M$, f^{ts} is the extension of f to the ideal $\{\mu : \mu \in L_k, f^t\mu \in L_b\}$. We shall in general identify f with its extension f^{ts}.

THEOREM 4.5. (i) $M_b = \{f : f \in M, f^t L_b \subset L_b\}$,
(ii) $M_k = \{f : f \in M, f^t L_k \subset L_b\}$,
(iii) $M_k = \{f : f \in M, f^t(\sigma L_b) \subset L_b\}$.

Proof. We shall prove (ii). If $f \in M_k$, then $\|f^t\mu\| \leqq |f|(|\mu|)$ for each $\mu \in L_k$. Therefore $f^t L_k \subset L_b$. Conversely, if $f^t L_k \subset L_b$, then f^{ts} extends f to L_k. Whence $f \in \tilde{\Omega}(L_k) = M_k$. The proof of (i) is similar to that of (ii). Part (iii) follows from (ii) and [16, (4.5)].

For $f \in M$, $g \in M$ define $fg = (f^t g^t)^s$ where $f^t g^t$ denotes the composition of the operators f^t and g^t. Since the composition of multiplication operators is commutative [16, §6], the space M becomes a commutative ring under the multiplication defined above.

An element u of a vector lattice is called a *weak order unit* if $u \geqq 0$ and $u \wedge |x| = 0$ implies $x = 0$. In a normed vector lattice an element $\mathbf{1}$ is called a *strong order unit* if $\mathbf{1} \geqq 0$, $\|\mathbf{1}\| = 1$ and $\|x\| \leqq 1$ implies $|x| \leqq \mathbf{1}$.

If $\phi \in C$, then ϕ^t is a multiplication operator on L_k [16, (6.6)]. Thus ϕ^{ts} is an element of M and the mapping: $\phi \to \phi^{ts}$ is a natural embedding of C in M. It follows from Theorem 4.5 that under this embedding C_b is mapped into M_b and C_k goes into M_k.

PROPOSITION 4.6. *The embedding $\phi \to \phi^{ts}$ of C in M preserves the lattice and ring operations in C. Furthermore $\mathbf{1}^{ts}$ ($\mathbf{1}(x) = 1$ for $x \in X$) is the ring identity for M, a strong order unit for M_b, and a weak order unit for M.*

Proof. Since $\bigcup L(K)$ can be identified with an ideal in $\Omega(C)$, it follows from [18, Theorem 7.9] that the lattice operations are preserved. Let $\phi \in C$, $\psi \in C$;

then $(\phi\psi)^t = \phi^t\psi^t$ and hence $(\phi\psi)^{ts} = \phi^{ts}\psi^{ts}$. This shows that the ring operations are preserved. Now $\mathbf{1}^t$ is the identity operator on L_k; whence it follows that $\mathbf{1}^{ts}$is the ring identity of M. Also $\mathbf{1}^{ts}(\mu) = \|\mu^+\| - \|\mu^-\|$ for each $\mu \in L_b$. If $\|f\| \leqq 1$, $f \in M_b$, then $|f|(\mu) \leqq \|f\| \, \|\mu\| \leqq \|\mu\| = \mathbf{1}^{ts}(\mu)$ for each $\mu \in (L_b)_+$. Thus $|f| \leqq \mathbf{1}^{ts}$. To show that $\mathbf{1}^{ts}$ is a weak order unit for M, let $|g| \wedge \mathbf{1}^{ts} = 0$, $g \in M$. For any compact set K, the component g_K of g in $M(K)$ is in M_b. Since $\mathbf{1}^{ts}$ is a strong order unit for M_b, $|g_K| \wedge \mathbf{1}^{ts} = 0$ implies that $g_K = 0$. Hence g vanishes on $\bigcup L(K)$, i.e., $g = 0$.

Henceforth we shall identify $\phi \in C$ with $\phi^{ts}$and consider C to be a subspace of M.

Proposition 4.7. (i) *The closure of C_k in M is M itself.*
(ii) *C_k is $|w|(M, \bigcup L(K))$-dense in M.*

Proof. According to [15, (3.10)] the closure of C_k contains M_k. Since M_k is dense in M, the proof of (i) is complete. Part (ii) follows from the fact that a convergent net in M also $|w|(M, \bigcup L(K))$-converges to the same limit [14, (11.7)].

Proposition 4.8. *For each $\mu \in L_k$, the mapping $f \to f^t\mu$ from M into L_k is a continuous linear transformation.*

Proof. The mapping is clearly linear. If $|f| \leqq g$, $g \in M$, then $|f^t\mu| \leqq |f|^t|\mu| \leqq g^t|\mu|$; hence the transformation is bounded. Let $f_\alpha \downarrow 0$ in M. Then $f_\alpha^t|\mu|$ is directed downward. For any $h \in (C_k)_+$ $[\bigwedge_\alpha f_\alpha^t|\mu|](h) = \inf_\alpha f_\alpha^t|\mu|(h)$ [14, (2.2)] $= \inf_\alpha f_\alpha(h^t|\mu|) = 0$. Thus $\bigwedge_\alpha f_\alpha^t|\mu| = 0$; this shows that the transformation is continuous.

In the remaining part of this section we shall consider the relation between $M_k = \tilde{\Omega}(L_k)$ and $\Omega(L_k)$. It follows from [16, (2.4)] that $\Omega(L_k) = \Omega(\sigma L_b) \oplus \sigma L_b^\perp$. Also $\Omega(L_k) = M_k \oplus L_b^\perp$ [16, (3.6)]; hence $\Omega(L_k) = M_k \oplus \Omega(\sigma L_b) \cap L_b^\perp \oplus \sigma L_b^\perp$. Since $M_k = \tilde{\Omega}(\sigma L_b)$, this can be written as $\Omega(L_k) = \tilde{\Omega}(\sigma L_b) \oplus \Omega(\sigma L_b) \cap L_b^\perp \oplus \sigma L_b^\perp$. Thus the problem of deciding whether $\Omega(L_k) = \tilde{\Omega}(L_k)$ can be broken into two parts:

(i) *Under what circumstances will $\Omega(\sigma L_b) = \tilde{\Omega}(\sigma L_b)$?*
(ii) *Is $\Omega(L_k) = \Omega(\sigma L_b)$?*

The latter question is related to the existence of measurable cardinals [8, Chapter 12]. If X is a discrete space, then $L_k = (L_k)_0$ is the set of all real-valued functions on X while σL_b consists of those functions on X which vanish outside a countable set. It follows that for discrete spaces question (ii) is equivalent to Mackey's formulation of Ulam's problem concerning the existence of measurable cardinals [17].

We now turn our attention to question (i).

Example. Let X be the real line, μ the Lebesgue measure and T the corresponding space constructed in §3. Let $h_n \in C_k(T)$ be defined as in Example 2 at the end of §3. Then $\{\lambda(n^{-1}h_n)\}$ is a bounded sequence for each $\lambda \in L_k(T)$. Given a bounded sequence $\{r_n\}$, let r^β denote the extension of the function r $(r(n)=r_n)$ to βN, the Stone-Čech compactification of the space of natural numbers. Next choose $p \in \beta N - N$ and set $\varepsilon_p(\{r_n\}) = r^\beta(p)$. Finally define $\phi(\lambda) = \varepsilon_p(\{\lambda(n^{-1}h_n)\})$ for each $\lambda \in L_k(T)$. Clearly ϕ is a positive linear functional on $L_k(T)$. If $\lambda \in L_b$, then $|\lambda(n^{-1}h_n)| \leqq n^{-1}|\lambda|(h_n) \leqq n^{-1}\|\lambda\|$ and thus $\lim_n \lambda(n^{-1}h_n) = 0$. This implies that $\phi \in L_b^\perp$. On the other hand $\mu(n^{-1}h_n) = 2 + n^{-1}$; whence $\phi(\mu) = 2$. This shows that $\phi \in \Omega(\sigma L_b)$ $(L_k = \sigma L_b$ for the space $T)$ while $\phi \notin M_k = \tilde{\Omega}(\sigma L_b)$.

Next we shall consider conditions which will insure that $\Omega(\sigma L_b) = \tilde{\Omega}(\sigma L_b)$. For this we will need the concept of realcompact space. Let βT denote the Stone-Čech compactification of a completely regular space T. Then each $f \in C(T)$ can be extended to a continuous mapping f^β of βT into the one-point compactification $R \cup \{\infty\}$ of R. Let $vT = \{p : p \in \beta T, f^\beta(p) \neq \infty$ for each $f \in C(T)\}$. Clearly $T \subset vT \subset \beta T$. When vT is given the topology induced on it by that of βT, it is called the *Hewitt realcompactification* of T. If $T = vT$, then T is said to be *realcompact* (Hewitt used the term Q-space). A detailed study of realcompact spaces may be found in [8].

PROPOSITION 4.9. *Let $f \in \Omega(L_k)$. If $f \in L_b^\perp$, then f vanishes at each $\mu \in L_k$ which has realcompact support.*

Proof. Let $\mu \in L_k$ and suppose that the support F of μ is realcompact. For each $\phi \in C(F)$, ϕ^t is a multiplication operator on $L_k(F)$ [cf. §2]. Thus the relation $f^t\mu(\phi) = f(\phi^t\mu)$ defines $f^t\mu$ as a bounded linear functional on $C(F)$. Since F is realcompact, $f^t\mu$ has compact support K in F [11, Theorems 21 and 22]. Let $h \in C_k$ be so that $h(x) = 1$ on K. Then $f(\mu) = f^t\mu(\mathbf{1}) = f^t\mu(h) = f(h^t\mu)$. Hence if $f \in L_b^\perp$, then $f(\mu) = 0$.

THEOREM 4.10. *If the support of each element of L_b is realcompact, then $M_k = \Omega(\sigma L_b)$.*

Proof. That $\Omega(\sigma L_b) = M_k \oplus \Omega(\sigma L_b) \cap L_b^\perp$ was shown above. In view of Proposition 4.9, it suffices to prove that the elements of σL_b have realcompact support. Let $\mu \in \sigma L_b$. Then there exists a sequence $\{\mu_n\}$ in $(L_b)_+$ such that $|\mu| = \bigvee_n \mu_n$. Now $\bigvee_n (2^{-n}\|\mu_n\|^{-1})\mu_n$ is in L_b and has the same support as μ. Thus the measures in σL_b have realcompact support if those in L_b do.

The following is an easy consequence of Proposition 4.9.

THEOREM 4.11. *If the support of each Radon measure is realcompact, then $M_k = \Omega(L_k)$.*

REMARK. Since σ-compact spaces are realcompact and since closed subspaces of realcompact spaces are realcompact, it follows from Theorem 4.11 that $M_k = \Omega(L_k)$ for each σ-compact space (more generally, for each realcompact space).

5. **A chacterization of M and its subspaces.** In [16, (7.1)], M, M_b and M_k are characterized as subrings of a ring of continuous functions. Here we shall give a slight modification of that characterization.

THEOREM 5.1. *There exist locally compact and extremally disconnected spaces Y and Z with $Y \subset Z$ for which*

(i) $M = C(Y) = C(Z)$.

(ii) $M_b = C_b(Y) = C_b(Z)$.

(iii) $C_k(Y) \subset M_k \subset C_\infty(Y) \cap C_k(Z)$. *Moreover if* $M_k \neq C_k(Y)$, *then* $M_k \neq C_\infty(Y) \cap C_k(Z)$.

Proof. Let Y be the space given by [16, (7.1)]. Then Y is locally compact and extremally disconnected. Since M is the set of continuous multiplication operators on L_k, (7.1) of [16] states that $M = C(Y)$, $M_b = C_b(Y)$ and $C_k(Y) \subset M_k \subset C_\infty(Y)$. Let A be the set of idempotents e of M such that $eM \subset M_b$. If vY denotes the Hewitt realcompactification of Y, then $M = C(Y) = C(vY)$ [8, §8.8]. If we consider A to be a subset of $C(vY)$, then each $e \in A$ has compact support in vY [8, Problem 8E]. Since $e \in C(vY)$, the support of e is open (and compact) in vY. Hence the support is an open subset of βY. Set $Z = \{p : p \in vY, e(p) \neq 0 \text{ for some } e \in A\}$; then Z is an open subspace of βY. Thus Z is locally compact. Since Y is extremally disconnected there exists for each $y \in Y$ an element $e \in A$ such that $e(y) \neq 0$. Hence $Y \subset Z \subset vY$. This implies that $C(Y) = C(Z)$, $C_b(Y) = C_b(Z)$ and that Z is extremally disconnected [8, Theorem 8.6 and Problem 6M.1]. To show that $M_k \subset C_k(Z)$ we need the following:

LEMMA 5.2. *Let T be a completely regular space. If $f \in C = C(T)$ is such that $fC \subset C_b$, then $f \in C_k(vT)$.*

Proof. For $n = 1,2,3,\cdots$, every $g \in C$ is bounded on the set $\{x \in vT: |f(x)| \geqq 1/n\}$. Thus $f \in C_\infty(vT)$ [8, Problem 8E]. Suppose $f \notin C_k(vT)$. Since $C_k(vT) = \{h \in C: (gh)^\beta(p) = 0 \text{ for all } p \in \beta T - vT \text{ and all } g \in C\}$, there exists $g \in C$ and $p \in \beta T - vT$ such that $(fg)^\beta(p) \neq 0$. Hence $f^\beta(p) \cdot (fg^2)^\beta(p) = ((fg)^\beta(p))^2 \neq 0$. But $f^\beta(p) = 0$ since $f \in C_\infty(vT)$ [8, Theorem 7.2 and Problem 7F]. This contradiction completes the proof of Lemma 5.2.

We return to the proof of Theorem 5.1. If $f \in M_k$, then $fM \subset M_k$ [Theorem 4.5]. Thus $f \in C_k(vY)$. Let e be the component of $\mathbf{1}$ in the closed ideal of M generated by f. Then e and f have the same support in vY; whence $e \in C_k(vY)$ and in turn $e \in A \subset C_k(Z)$. Therefore $M_k \subset C_k(Z)$. It remains to show that if $M_k \neq C_k(Y)$, then M_k is a proper subset of $C_\infty(Y) \cap C_k(Z)$. Let $f \in M_k, f \notin C_k(Y)$. We may suppose that $0 \leqq f \leqq \mathbf{1}$ since f can be replaced by $|f| \wedge \mathbf{1}$. Then $\mathbf{1}_f = \bigvee_n f^{1/n}$ is the component of $\mathbf{1}$ in the closed ideal generated by f. Since $f \notin C_k(Y)$, $\mathbf{1}_f \notin M_k$ [16, §7]. Let $\mu \in (L_k)_+$ be such that $\mathbf{1}_f^t\mu \notin L_b$. Now $(f^{1/n})^t\mu \uparrow \mathbf{1}_f^t\mu$ [Proposition 4.8]. By the continuity of the norm, $\lim_n \|(f^{1/n})^t\mu\| = \infty$. Choose a subsequence $\{g_k\}$ of

$\{f^{1/n}\}$ so that $\| g_k^t \mu \| \geqq k^3$. Then $g = \sum k^{-2} g_k \in C_\infty(Y) \cap C_k(Z)$ while $g^t \mu \notin L_b$. This concludes the proof of Theorem 5.1.

A topological space is *pseudocompact* if every continuous function defined on it is bounded.

COROLLARY 5.3. *The support in Y of each element of M_k is pseudocompact.*

Proof. Let F be the support in Y of $f \in M_k$. Since Y is extremally disconnected, F is open. Hence the characteristic function e of F is in $C(Y) = M$. Then $e \in C_k(Z)$ [Theorem 5.1, (iii)]. Let $\phi \in C(F)$ and define $\bar{\phi}(x) = \phi(x)$ on F and $\bar{\phi}(x) = 0$ on $Y - F$. Then $\bar{\phi} \in C(Y) = C(Z)$. Thus $e\bar{\phi} \in C_k(Z)$. Therefore ϕ is bounded.

REMARK. Corollary 5.3 is stated in [16]. However the proof given there rests on a lemma [16, lemma following (7.8)] which is invalid for non-normal spaces. The Tychonoff plank affords a counterexample.

EXAMPLE. Let X be the real line, μ the Lebesgue measure and T the corresponding space constructed in the proof of Theorem 3.1. Also, let $h_n \in C_k(T)$ be defined as in Example 2 at the end of §3. Then $f = \sum n^{-3} h_n$ and $g = \sum n^{-2} h_n$ are elements of $C_\infty(T)$. From the information derived in Example 2 at the end of §3, it follows that $f \in M_k$ while $\sum n^{-2} \mu(h_n) = \sum (2n+1) n^{-2}$. Therefore $g \notin M_k$. Clearly f and g have the same support in Y. This example shows that in general M_k is not equal to $C_k(Y)$ and thus it answers the question raised by Kaplan in [16, §§5, 7].

PROPOSITION 5.4. *If each measure in L_b has realcompact support, then $M_k = C_k(Y)$.*

Proof. Let $f \in M_k$ and $\mu \in \sigma L_b$. Then $f^t \mu$ has compact support (see the proof of Proposition 4.9). If $\mathbf{1}_f$ is the characteristic function of the support of f, then $\mathbf{1}_f^t \mu$ also has compact support; whence $\mathbf{1}_f^t \sigma L_b \subset L_b$. It follows from Theorem 4.5 that $\mathbf{1}_f \in M_k$. Thus $M_k = C_k(Y)$ [16, (5.2) and (7.2)].

REMARK. Proposition 5.4 is equally valid whether the support of $\mu \in L_b$ is taken in X or whether μ is considered as a measure on Y and the support of μ is taken in Y.

The following is due to Kaplan [15, (3.4)] and [16, proof of 7.1, (i)].

PROPOSITION 5.5. (i) $L = \tilde{\Omega}(M_k) = \tilde{\Omega}(C_k(Y))$.
(ii) $L_b = \tilde{\Omega}(M_b)$.

PROPOSITION 5.6. *The set X can be identified with the set of isolated points of Y.*

Proof. For $x \in X$, $R\varepsilon_x$ is a closed ideal in $L_k(X) = \tilde{\Omega}(C_k(Y))$. Thus $R\varepsilon_x$ is a one-dimensional ideal in $L_k(Y) = \Omega(C_k(Y))$ and hence the support in Y of ε_x consists of a single point y. Therefore $\varepsilon_x = \varepsilon_y$ [Proposition 2.2]. Since $\varepsilon_y \in \tilde{\Omega}(C_k(Y))$, y is an isolated point [Proposition 2.3]. Conversely, suppose y is an isolated point of Y. Then $\varepsilon_y \in \tilde{\Omega}(C_k(Y)) = L_k(X)$ and $R\varepsilon_y$ is a one-dimensional ideal. Hence the support in X of ε_y consists of a single point x. Thus $\varepsilon_y = \varepsilon_x$ [Proposition 2.2].

We shall consider X both as a topological space with its original topology and as a subset of Y. When we refer to the topology on X, we shall mean the original topology rather than the topology induced on X by that of Y. These topologies differ widely unless the original topology is discrete.

Since $L_k = (L_k)_0 \oplus (L_k)_1$, it follows that $(\bigcup L(K))_0$ and $(\bigcup L(K))_1$ are direct summands of $\bigcup L(K)$. Define $M_0 = \Omega((\bigcup L(K))_0)$ and $M_1 = M_0'$. Then $M = M_0 \oplus M_1$ and M_1 is the order dual of $(\bigcup L(K))_1$ [16, §2]. For $f \in M$, let f_0 and f_1 denote the components of f in M_0 and M_1 respectively. If A is a subset of M, denote $\{f_0 : f \in A\}$ by A_0 and $\{f_1 : f \in A\}$ by A_1. Then it follows that $(M_k)_0 = M_0 \cap M_k$, $(M_k)_1 = M_1 \cap M_k$ and $M_k = (M_k)_0 \oplus (M_k)_1$. Similar statements hold for M_b.

A real-valued function on a topological space is said to be *locally bounded* if it is bounded on some neighborhood of each point of the space.

PROPOSITION 5.7. (i) *M_0 is lattice and ring isomorphic with the lattice-ordered ring of all locally bounded functions on X. (The lattice and ring operations are defined pointwise.)*

(ii) *$(M_b)_0$ is isometric and lattice and ring isomorphic with the Banach lattice-ordered ring of all bounded real-valued functions on X. (The norm is given by: $\|f\| = \sup\{|f(x)| : x \in X\}$.)*

Proof. We shall show that the isomorphism is given by $f \to f|X$. Suppose $f \in M_0$ is not locally bounded on X. Then there exists $p \in X$ such that f is unbounded on each neighborhood of p. Let K be a compact neighborhood of p and select a sequence $\{x_n\}$ from K so that $|f(x_n)| \geqq n^3$. Let $\mu \in (L_k)_0$ be such that $\mu(x_n) = n^{-2}$ and $\mu(x) = 0$ for $x \neq x_n$ $n = 1,2,3,\cdots$. Then $\mu \in L(K)$. On the other hand $|f|(\mu) \geqq n^{-2}|f(x_n)| \geqq n$. This proves that each $f \in M_0$ is locally bounded on X. If ϕ is a locally bounded function on X, then ϕ is bounded on each compact subset of X. Hence $f(\mu) = \sum_{x \in X} \phi(x)\mu(x)$, $\mu \in (\bigcup L(K))_0$ defines f as an element of M_0 such that $f|X = \phi$. Since $M = \tilde{\Omega}(\bigcup L(K))$ and since $(L_k)_0$ is the closure of the linear space generated by $\{\varepsilon_x : x \in X\}$, it follows that $f \in M_0$ is > 0 if and only if $f|X > 0$. Thus $f \to f|X$ is a lattice isomorphism. Clearly it is also a ring isomorphism. Part (ii) follows from (i) and the order unit property of $\mathbf{1}$.

Let Y_0 be the closure with respect to the topology on Y of the set X. Then define $Y_1 = Y - Y_0$. Since X is a discrete and hence open subset of the extremally disconnected space Y, the sets Y_0 and Y_1 are open. Thus we have:

PROPOSITION 5.8. (i) $M_0 = C(Y_0)$, $M_1 = C(Y_1)$,
(ii) $(M_b)_0 = C_b(Y_0)$, $(M_b)_1 = C_b(Y_1)$,
(iii) $C_k(Y_0) \subset (M_k)_0 \subset C_\infty(Y_0)$, $C_k(Y_1) \subset (M_k)_1 \subset C_\infty(Y_1)$.

In general it is not possible to characterize $(M_k)_0$ by means of topological terms alone. However, we can obtain the following:

PROPOSITION 5.9. (i) *Let $f \in M_0$. If f has compact support in X, then $f \in M_k$.*
(ii) *The support in X of each $f \in (M_k)_0$ is pseudocompact.*

Proof. Part (i) follows from the characterization of $(L_k)_0$ given in Proposition 2.6. To prove (ii), let $f \in M_0$ and suppose that the support F in X of f is not pseudocompact. Then there exists an unbounded $\phi \in C(F)$. Define $\bar{\phi}(x) = \phi(x)$ on F and $\bar{\phi}(x) = 0$ on $X - F$; then $\bar{\phi}$ is locally bounded on X and hence $\bar{\phi} \in M_0$. Also, $\bar{\phi}$ is unbounded on the set $\{x : x \in X, f(x) \neq 0\}$. Therefore $\bar{\phi}$ is unbounded on the support in Y of f. Hence $f \notin M_k$ [Corollary 5.3].

REMARK. Since compactness and pseudocompactness are equivalent for σ-compact spaces (more generally, for realcompact spaces), Proposition 5.9 gives a complete characterization of $(M_k)_0$ for such spaces.

EXAMPLE. Let X be the real line, μ be the Lebesgue measure and T be the corresponding space constructed in the proof of Theorem 3.1. Let f be the element of M_k defined in the example following Corollary 5.3. Then the support of f_0 in T (T plays the role of X in Proposition 5.9) is T itself. However, T is not compact since it admits a measure, namely μ, which is not finite.

We now investigate the connection between M and the subspaces C, C_b and C_k.

PROPOSITION 5.10. (i) *Let $f \in M$, $g \in M$. Then $f \leqq g$ if and only if $fh \leqq gh$ for all $h \in (C_k)_+$.*
(ii) *Let $\{f_\alpha\}$ be a net in M. Then $\{f_\alpha\}$ converges if and only if $\{f_\alpha h\}$ converges for each $h \in (C_k)_+$.*

Proof. Statement (i) and the necessity part of (ii) follow easily from Theorem 5.1 and Proposition 2.12. We shall prove the sufficiency part of (ii). If $\mu \in L_k$, $h \in (C_k)_+$, then $[\lim_\alpha f_\alpha h](\mu) = \lim_\alpha f_\alpha h(\mu)$ [15, (3.7)] $= \lim_\alpha f_\alpha(h^t\mu)$. Since $C_k L_k = \bigcup L(K)$ [Proposition 2.12], it follows that $\lim_\alpha f_\alpha(\mu)$ exists for each $\mu \in \bigcup L(K)$. Clearly, the relation $f(\mu) = \lim_\alpha f_\alpha(\mu)$ defines f as a linear functional on $\bigcup L(K)$. Now $\{|f_\alpha| h\}$ converges if $\{f_\alpha h\}$ converges; thus $\lim_\alpha |f_\alpha|(\mu)$ exists for each $\mu \in \bigcup L(K)$. If $|\mu| \leqq \lambda$, then $|f(\mu)| = |\lim_\alpha f_\alpha(\mu)| \leqq \lim_\alpha |f_\alpha|(\lambda)$. Therefore f is bounded and $f \in M$. Now $fh(\mu) = f(h^t\mu) = \lim_\alpha f_\alpha(h^t\mu) = [\lim_\alpha f_\alpha h](\mu)$; whence $fh = \lim_\alpha f_\alpha h$ for each $h \in (C_k)_+$. It remains to show that $f = \lim_\alpha f_\alpha$. Let A_α be the set of suprema of finite subsets of $\{|f - f_\beta| : \beta \geqq \alpha\}$. Then $\{g(\mu) : g \in A_\alpha\}$ is bounded for each $\mu \in \bigcup L(K)$; therefore $\bigvee A_\alpha = \bigvee_{\beta \geqq \alpha} |f - f_\beta|$ exists in M [6, Théorème 1]. Set $g_\alpha = \bigvee_{\beta \geqq \alpha} |f - f_\beta|$. For any compact set $K \subset X$ and any $g \in M$, g_K denotes the component of g in $M(K)$. Choose $\phi \in (C_k)_+$ so that $\phi(x) = 1$ on K; then $(\bigwedge_\alpha g_\alpha)_K = \bigwedge_\alpha \bigvee_{\beta \geqq \alpha} |f - f_\beta|_K$ [14, (1.4)] $\leqq \bigwedge_\alpha \bigvee_{\beta \geqq \alpha} |f\phi - f_\beta \phi|$. The latter is 0 since $\lim_\alpha f_\alpha \phi = f\phi$; whence $(\bigwedge_\alpha g_\alpha)_K = 0$. Since $\bigcup M(K)$ is dense in M, $\bigwedge_\alpha g_\alpha = 0$. This proves that $\lim_\alpha f_\alpha = f$.

An element $f \in C$ is completely determined by its values on X. Hence we have

PROPOSITION 5.11. *C_0 is ring and lattice isomorphic with C. Furthermore, $(C_b)_0$ is isometric with C_b.*

PROPOSITION 5.12. *Let $f \in C$ and $A \subset C$; then $f = \bigvee A$ if and only if $f_0 = \bigvee A_0$.*

Proof. Since adding to A the suprema of finite subsets of A will alter neither $\bigvee A_0$ nor $\bigvee A$, we may suppose that A is directed upward. Then $f = \bigvee A$ if and only if $fh = \bigvee hA$ [Proposition 5.10] for every $h \in (C_k)_+$. Since $hA \subset M(K)$ for some compact set K, the proposition now follows from [14, (5.5)].

6. **The duality between L_k and M.** For a subset A of L_k, let A^π denote the annihilator of A in M, i.e., $A^\pi = \{f : f \in M, f^t\mu = 0 \text{ for all } \mu \in A\}$. Similarly for $B \subset M$, define $B^\pi = \{\mu : \mu \in L_k, f^t\mu = 0 \text{ for all } f \in B\}$. We need the following:

LEMMA 6.1. *Let $f \in M, \mu \in L_k$. Then $|f^t\mu| = |f|^t|\mu|$, $(f^t\mu)^+ = (f^+)^t\mu^+ + (f^-)^t\mu^-$ and $(f^t\mu)^- = (f^+)^t\mu^- + (f^-)^t\mu^+$.*

Proof. First suppose that $\mu \geqq 0$. Then $(f^+)^t\mu$ and $(f^-)^t\mu$ are disjoint [Theorem 5.1 and 16, (8.2)]. Whence $(f^t\mu)^+ = (f^+)^t\mu$. Now let μ be an arbitrary element of L_k. Since f^t is a multiplication operator $f^t\mu^+$ and $f^t\mu^-$ are disjoint. Therefore $(f^t\mu)^+ = (f^t\mu^+ + (-f)^t\mu^-)^+ = (f^t\mu^+)^+ + ((-f)^t\mu^-)^+$. The assertion concerning $(f^t\mu)^+$ now follows from the first part of this proof. The remainder of the lemma follows from standard vector lattice properties.

PROPOSITION 6.2 (i) *For any subset A of L_k, A^π is a closed ideal in M.*
(ii) *For any subset B of M, B^π is a closed ideal in L_k.*

Proof. (i) Let $g \in M$, $f \in A^\pi$ be such that $|g| \leqq |f|$. Then $|g^t\mu| = |g|^t|\mu| \leqq |f|^t|\mu| = |f^t\mu| = 0$ for all $\mu \in A$. Thus $g \in A^\pi$. This shows that A^π is an ideal. Since the mapping $f \to f^t\mu$ is continuous A^π is closed. The proof of (ii) is similar to that of (i).

REMARK. If $A \subset \bigcup L(K)$, then A^π is the null ideal [15, §2] of A in M. Likewise if $B \subset M_k$, then B^π is the null ideal in L_k of B. More generally, if $A \subset L_k$ and I is the ideal in L_k generated by A, then $A^\pi = (I \cap \bigcup L(K))^\perp$. A similar statement holds for subsets of M.

For $\mu \in L_k$, let $(L_k)_\mu$ denote the closed ideal in L_k generated by μ. For $A \subset L_k$, let A_μ denote the set of components in $(L_k)_\mu$ of the elements of A. In particular $(L_b)_\mu = L_b \cap (L_k)_\mu$. Next define $M_\mu = \{\mu\}^{\pi\prime}$. Then $M'_\mu = \{\mu\}^\pi$ and $M = M_\mu \oplus M'_\mu$. It follows that $M_\mu = \Omega((L_k)_\mu \cap \bigcup L(K))$ and that M_μ can be identified with the set of all multiplication operators on $(L_k)_\mu$. For $f \in M$, let f_μ denote the component of f in M_μ. Given a subset B of M, set $B_\mu = \{f_\mu : f \in B\}$. Observe that $(M_b)_\mu = M_\mu \cap M_b$ and $(M_k)_\mu = M_\mu \cap M_k$. Also $\mathbf{1}_\mu$ is the ring identity and strong order unit for M_μ. It is also easy to verify that $\mathbf{1}_\mu M = M_\mu$.

Proposition 6.3. *Let* $\mu \in (L_k)_+$, $f \in M_\mu$, $g \in M_\mu$. *Then* $g \leqq f$ *if and only if* $g^t\mu \leqq f^t\mu$.

Proof. The "only if" part follows from the definition of the order in M. Suppose $g^t\mu \leqq f^t\mu$. Then $((f-g)^-)^t\mu = ((f-g)^t\mu)^-$ [Lemma 6.1] $= 0$. Thus $(f-g)^- \in M'_\mu \cap M_\mu$; whence $(f-g)^- = 0$. This proves that $g \leqq f$.

Theorem 6.4. *For each* $\mu \in L_k$, $\mu M = \{f^t\mu; f \in M\}$ *is a dense ideal in* $(L_k)_\mu$.

Proof. Since f^t is a multiplication operator on M, $f^t\mu \in (L_k)_\mu$; whence $\mu M \subset (L_k)_\mu$. It follows from [16, (9.1)] that $(L_k)_\mu$ is the closure of μM. It remains to show that μM is an ideal. Lemma 6.1 insures that μM is a subvector lattice of L_k. The theorem will now follow from

Lemma 6.5. *Let* $\mu \in (L_k)_+, f \in M_+$ *and* $\lambda \in L_k$ *be such that* $0 \leqq \lambda \leqq f^t\mu$. *Then there exists a* $g \in M_+$ *such that* $g^t\mu = \lambda$.

Proof. Set $g = \bigvee \{h : h \in M_\mu,\ h^t\mu \leqq \lambda\}$. Observe that the h's are bounded by f [Proposition 6.3]. Thus g exists in M_+. It remains to show that $g^t\mu = \lambda$. In any case $\nu = \lambda - g^t\mu \geqq 0$ [Proposition 4.8]. Assume $\nu > 0$. Now $(\nu - n^{-1}\mu)^+ \uparrow \nu$. Thus there is a positive number r such that $(\nu - r\mu)^+ > 0$. Let $e = \mathbf{1}_{(\nu - r\mu)^+}$. Then $e \in M_\mu$ and $e^t(\nu - r\mu) = (\nu - r\mu)^+$. Hence $\nu - re^t\mu \geqq e^t(\nu - r\mu) > 0$. Thus $(g + re)^t\mu = \lambda - \nu + re^t\mu < \lambda$. Since $g + re \in M_\mu$, it follows from the manner in which g was defined that $g + re \leqq g$. This is a contradiction since $r > 0$ and $e > 0$. Thus $\lambda = g^t\mu$.

Theorem 6.6. *Let* $\mu \in L_k$. *Then each* $\lambda \in (L_k)_\mu$ *is the limit of a sequence in* $\mu M_b = \{f^t\mu : f \in M_b\}$.

Proof. Since $(L_k)_\mu = (L_k)_{|\mu|}$ and since λ^+ and λ^- can be considered separately we may assume that $\mu \geqq 0$ and $\lambda \geqq 0$. Since μM is a dense ideal in $(L_k)_\mu$, there exists a net $\{f_\alpha\}$ in M such that $f_\alpha^t\mu \uparrow \lambda$. Set $g_n = \bigvee_\alpha f_\alpha \wedge n\mathbf{1}$, $n = 1,2,3\cdots$. It follows easily that $g_n^t\mu \uparrow \lambda$.

Remark. Theorem 6.4 and Theorem 6.6 are weakened forms of the Radon-Nikodym theorem.

Since $L_k(X) = \tilde{\Omega}(C_k(Y)) \subset \Omega(C_k(Y)) = L_k(Y)$, each $\mu \in L_k(X)$ can be considered both as a measure on X and as a measure on Y. Of particular interest is the support of μ in Y. Whenever we write L_k, we shall mean $L_k(X)$ rather than $L_k(Y)$.

Proposition 6.7. *Let* $\mu \in L_k$. *Then the supports in* Y *of* μ *and* $\mathbf{1}_\mu$ *are identical*

Proof. Since $\mathbf{1}_\mu^t\mu = \mu$, the support of $\mathbf{1}_\mu$ contains that of μ [3, Proposition 10, p. 73]. For $p \in Y$ not belonging to the support of μ, let $f \in M = C(Y)$ vanish on the support of μ while $f(p) \neq 0$. Then $f^t\mu = 0$ [3, p. 72]. Hence $(f\mathbf{1}_\mu)^t\mu = 0$; thus $f\mathbf{1}_\mu \in M'_\mu \cap M_\mu$. This implies that $f\mathbf{1}_\mu = 0$; whence it follows that $\mathbf{1}_\mu$ vanishes on a neighborhood of p.

COROLLARY 6.8. *The support in Y of each element of L_k is open (and closed).*

PROPOSITION 6.9. $\tilde{\Omega}(M) = \{\mu \in L_b,\ \mu$ *has pseudocompact support in* $Y\}$.

Proof. By [16, (2.7)], $\tilde{\Omega}(M) \subset \tilde{\Omega}(M_b) = L_b$. Let $\mu \in L_b$ have pseudocompact support in Y. Then $\mathbf{1}_\mu M \subset M_b$; thus the relation $\mu(f) = \mu(f\mathbf{1}_\mu)$ extends μ to become an element of $\Omega(M)$. Clearly, then $\mu \in \tilde{\Omega}(M)$. To prove the converse inclusion it suffices to show that if $\mu \in \tilde{\Omega}(M)$, then $f\mathbf{1}_\mu \in M_b$ for all $f \in M$ [cf. Lemma 5.2 and Corollary 5.3]. Let $\mu \in \tilde{\Omega}(M)$. It follows from [11, Theorems 13 and 14] that there exists $g \in M_b$ such that $|\mu|(|f-g|) = 0$. Then $f-g$ vanishes on the support of μ [3, §3, Proposition 9, Chapitre III] and hence $f\mathbf{1}_\mu = g\mathbf{1}_\mu \in M_b$.

PROPOSITION 6.10. (i) $\bigcup L(K) \subset C_k(Y)L_k \subset M_kL_k \subset C_k(Z)L_b = \tilde{\Omega}(M)$.
(ii) *If I is any one of the ideals listed in* (i), *then* $M = \Omega(I) = \tilde{\Omega}(I)$.

Proof. Clearly the first two inclusions in (i) are valid. Now $M_kL_k \subset L_b$ and $M_kC_k(Z) = M_k$; whence $M_kL_k \subset C_k(Z)L_b$. That $\tilde{\Omega}(M) = C_k(Z)L_b$ follows from Proposition 6.9 and the manner in which the space Z was constructed [cf. Theorem 5.1]. We now consider the proof of (ii). Since $\bigcup L(K)$ is a dense ideal in L_k and $M = \tilde{\Omega}(\tilde{\Omega}(M))$, it follows from [16,(2.7)] that $M = \tilde{\Omega}(I)$ for each of the ideals mentioned in part (i). The proof that $\Omega(I) = \tilde{\Omega}(I)$ can be patterned after the proof of Proposition 4.1 (i).

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