

GEOMETRY OF THE ZEROS OF THE SUMS OF LINEAR FRACTIONS

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The object of the present paper is to study the zeros of functions of the form

$$(1) \quad f(z) \equiv \sum_{k=1}^m \frac{a_k}{z - \alpha_k} - \sum_{k=1}^n \frac{b_k}{z - \beta_k}, \quad a_k > 0, \quad b_k > 0,$$

where the α_k and β_k have various geometric configurations as their loci. We investigate also functions of this form where the a_k and b_k are nonreal.

The appropriateness of this study arises from the facts that (i) Lagrange's interpolation formula for a polynomial with prescribed real values in real points α_k and β_k has a factor of precisely form (1), and a similar remark holds for nonreal values and nonreal points; (ii) Riemann sums for a Cauchy integral are of these same forms, in the respective real and nonreal cases; (iii) the logarithmic derivative of a rational function is of form (1), which enables us to study the location of the critical points. Our main theorems (Theorems 1 and 2) refer respectively to the real and nonreal cases just mentioned, where the locus of the α_k and β_k is a line segment with the a_k and b_k real, or a circular disk with the a_k and b_k not necessarily real.

Theorem 2 is a special case of a much more general theorem due to Marden [3], but is proved in detail here particularly because of the applications (i) and (ii), not mentioned by Marden. Namely, the present methods apply also to the case (Theorems 3 and 4) where the locus of the α_k, β_k , etc., is a circumference rather than a disk, a case not included in Marden's treatment yet important precisely for the study of a Cauchy or Cauchy-Stieltjes integral.

As is frequently done [2] in the study of zeros of such functions as (1), we interpret the conjugate of $f(z)$ as the force at z due to repelling particles at the α_k and attracting particles at the β_k , where each particle repels with a force equal to its mass a_k or $-b_k$ times the inverse distance; the original problem of finding the zeros of $f(z)$ is equivalent to the problem of finding the positions of equilibrium in this field of force.

THEOREM 1. *Let the conditions of (1) be satisfied, with $A = \sum a_k > B = \sum b_k$, and let all α_k and β_k lie on the interval $-1 \leq z \leq +1$. Then all nonreal zeros*

Presented to the Society, September 10, 1962; received by the editors March 22, 1963 and, in revised form, July 3, 1963.

⁽¹⁾ This research was sponsored (in part) by the Air Force Office of Scientific Research.

of $f(z)$ lie in the closed interior of the ellipse

$$(2) \quad Bx^2 + Ay^2 = \frac{AB}{A-B}.$$

All real zeros of $f(z)$ lie in the interval

$$(3) \quad |x| \leq \frac{A+B}{A-B}.$$

Indeed the sets mentioned constitute the locus of the zeros of $f(z)$ for all $f(z)$ satisfying the hypothesis.

In the field of force already introduced, if a point z_0 is considered, it is frequently convenient (cf. [2]) to replace n positive (or negative) particles α_k (or β_k) by a single *equivalent* particle whose mass is the sum of the masses of the original particles and which exerts the same force at z_0 . If the particle α_k is inverted in the unit circle whose center is z_0 , the corresponding force at z_0 is represented by the vector from the inverse of α_k to the point z_0 multiplied by the mass of the particle; the total force at z_0 due to all the particles α_k is represented by the vector (weighted by the total mass) from the center of gravity of the weighted inverses to z_0 ; the equivalent particle of the α_k is located at the inverse of this center of gravity. We often have occasion to use the fact that if a number of initial points of vectors with common terminal point and various weights are given, their center of gravity lies in their convex hull; this center of gravity is the initial point of the vector resultant, weighted by the sum of weights of the given vectors.

With the hypothesis of Theorem 1 we first choose $\text{Im}(z_0) > 0$. For fixed z_0 the particle α_0 equivalent to the given α_k lies in the circular segment $S(z_0)$ bounded by the interval $-1 \leq x \leq 1$ and by an arc of the circle through -1 , $+1$, and z_0 whose endpoints are $z = +1$ and -1 ; the arc lies in the closed half-plane $\text{Im}(z) \leq 0$. This remark follows from the fact that the inverse in the unit circle whose center is z_0 of the interval $-1 \leq x \leq +1$ is an arc of a circle through z_0 ; the convex hull of this arc is a certain segment of a circle whose inverse is $S(z_0)$. Moreover, $S(z_0)$ is the actual *locus* of the equivalent particle when all possible choices of the α_k are considered, not restricted in total number or in respective (positive) masses.

The locus of the particle β_0 equivalent to the β_k is also $S(z_0)$, and z_0 is a position of equilibrium if and only if α_0 and β_0 (in their respective loci) are collinear, with $|z_0 - \alpha_0|/|z_0 - \beta_0| = A/B$. If α'_0 and β'_0 are any two positions of α_0 and β_0 collinear with z_0 and in their proper loci, say with β'_0 on the interval $-1 \leq x \leq 1$ and α'_0 on the circular arc partially bounding $S(z_0)$, then the ratio $|z_0 - \alpha_0|/|z_0 - \beta_0|$ can be increased by rotating the line $z_0\alpha_0\beta_0$ about z_0 , and by sliding β_0 from β'_0 along the interval and sliding α_0 from α'_0 along the circular arc in the sense so as algebraically to decrease the

ordinate of α_0 ; this increase of the ratio is always possible as long as the abscissa of α_0 is not zero. The maximum of the ratio occurs when β_0 is on the interval, say $\beta_0 = \beta_0''$, and when the abscissa of α_0 is zero, say $\alpha_0 = \alpha_0''$. However, the ratio can take on all values and only values between unity and this maximum inclusive, for suitable choices of α_0 and β_0 in their proper loci and collinear with z_0 , $|z_0 - \alpha_0| \geq |z_0 - \beta_0|$. Thus z_0 can be a position of equilibrium if and only if we have

$$(4) \quad \frac{|z_0 - \alpha_0''|}{|z_0 - \beta_0''|} \geq \frac{A}{B}.$$

If we set $z_0 = x_0 + iy_0$, and note that the center of the circle an arc of which bounds $S(z_0)$ in part has the ordinate $b = (x_0^2 + y_0^2 - 1)/(2y_0)$, inequality (4) can be rewritten as

$$\frac{y_0 + (1 + b^2)^{1/2} - b}{y_0} \geq \frac{A}{B},$$

which is equivalent to

$$Bx_0^2 + Ay_0^2 \leq \frac{AB}{A - B};$$

this inequality is valid for both $y_0 > 0$ and $y_0 < 0$, so the proof of the first part of Theorem 1 is complete. It may be noted that the foci of the ellipse (2) are $z = +1$ and -1 , and its eccentricity is $[(A - B)/A]^{1/2}$; thus the ellipse corresponding to an arbitrary interval as assigned locus of the α_k and β_k is found at once.

If $z_0 = x_0 + iy_0$ is real and given, say $z_0 > 1$, the maximum of the first member of (4) is $(z_0 + 1)/(z_0 - 1)$, and z_0 can be a zero of $f(z)$ if and only if we have

$$\frac{z_0 + 1}{z_0 - 1} \geq \frac{A}{B},$$

so z_0 is a zero of some $f(z)$ if and only if we have

$$z_0 \leq \frac{A + B}{A - B}.$$

A similar discussion applies if we have $z_0 < -1$.

On the other hand, an arbitrary point of $-1 \leq x \leq 1$ belongs to the locus; for instance $z = 0$ is a zero of the particular function

$$f(z) \equiv \frac{A}{z - \alpha_0} - \frac{B}{z - \beta_0}$$

provided we have merely $A\beta_0 = B\alpha_0$, so α_0 and β_0 can be chosen positive and as small as desired. Theorem 1 is established.

Under the hypothesis of Theorem 1 except that now we take $A = B$, the locus of zeros of the totality of the functions $f(z)$ consists nontrivially of the entire plane. Indeed, let z_0 be a given nonreal point of the plane and let z_1 be an interior point of the circular segment $S(z_0)$ already defined. We set

$$(5) \quad f(z) \equiv \frac{1}{z - (z_1 + \delta)} + \frac{1}{z - (z_1 - \delta)} - \frac{2}{z - (z_1 + \epsilon)},$$

where $|\delta|$ and $|\epsilon|$ (> 0) are chosen so small that $z_1 \pm \delta$ and $z_1 + \epsilon$ lie within a circle interior to $S(z_0)$. The function $f(z)$ vanishes when $z = z_1 + \delta^2/\epsilon$, and ϵ and δ can be so chosen that this number is z_0 . This discussion does not apply if z_0 is real, but in that case a slight modification of the discussion of (5) does apply, and shows that z_0 belongs to the locus of zeros of all $f(z)$.

As an application of Theorem 1 we formulate

COROLLARY 1. *Let $r(z)$ be a rational function of z whose finite zeros and poles lie on the segment $-1 \leq z \leq +1$, of respective total orders A and B or B and A , $A > B$. Then all finite nonreal critical points of $r(z)$ lie in the closed interior of the ellipse (2), and all finite real critical points lie in the closed interval (3).*

The logarithmic derivative of $r(z)$ is of form (1), where the α_k are the zeros of $r(z)$ and the β_k are the poles, each enumerated a number of times according to its multiplicity, and where all a_k and b_k are unity. The corollary follows from Theorem 1.

As a second application we have

COROLLARY 2. *Let $f(z)$ be defined by the Stieltjes integral*

$$f(z) = \int_{-1}^1 \frac{d\sigma(t)}{t - z}, \quad -1 \leq t \leq 1,$$

where the total positive variation of $\sigma(t)$ on $-1 \leq t \leq 1$ is A and the total negative variation is $-B$, $A > B$. Then all finite nonreal zeros of $f(z)$ lie in the closed interior of the ellipse (2), and all real zeros lie in the closed interval (3).

The proof of Corollary 2 follows by considering the partial sums approximating the Stieltjes integral, and by Theorem 1. If the total negative variation of $\sigma(t)$ is greater than the total positive variation it suffices to consider the zeros of $-f(z)$.

In the proof of theorems such as Theorem 1 on the geometry of zeros of functions, two methods of proof are frequently used: (i) study of the loci of particles equivalent to various categories of particles; (ii) study of the total forces due to various categories of particles. We have just employed method (i), and now proceed to use method (ii) in a different problem.

THEOREM 2 (MARDEN). *Let the function $f(z)$ be of the form*

$$f(z) \equiv \sum_{k=1}^m \frac{a_k}{z - \alpha_k} - \sum_{k=1}^n \frac{b_k}{z - \beta_k} + \sum_{k=1}^p \frac{ic_k}{z - \gamma_k} - \sum_{k=1}^q \frac{id_k}{z - \delta_k},$$

where all the a_k, b_k, c_k , and d_k are non-negative. We set $A = \sum a_k, B = \sum b_k, C = \sum c_k, D = \sum d_k$, and suppose $(A - B)^2 + (C - D)^2 \neq 0$. If $\Gamma: |z| \leq 1$ is the simultaneous locus of the points $\alpha_k, \beta_k, \gamma_k$, and δ_k for all a_k, b_k, c_k, d_k satisfying the conditions given, then the locus of the zeros of $f(z)$ is the disk

$$(6) \quad |z| \leq \frac{A + B + C + D}{[(A - B)^2 + (C - D)^2]^{1/2}}.$$

We continue to interpret the conjugate of $f(z)$ as defining a field of force in the z -plane. If z_0 is a zero of $f(z)$, then ωz_0 with $|\omega| = 1$ is a zero of $f(z)$ with the original $\alpha_k, \beta_k, \gamma_k, \delta_k$ replaced by $\omega\alpha_k, \omega\beta_k, \omega\gamma_k, \omega\delta_k$, so it is sufficient for us to study $z_0 = a$, real; we take $a > 1$, and then we make a translation of the plane so that Γ becomes $\Gamma_1: |z + a| \leq 1$ and z_0 becomes $z_1 = 0$. The inverse of Γ_1 in the unit circle whose center is z_1 is

$$\left| z + \frac{a}{a^2 - 1} \right| \leq \frac{1}{a^2 - 1},$$

and the force exerted at z_1 due to all the particles α_k is represented by a vector with initial point z_1 and terminal point in the disk

$$C_1: \left| z - \frac{aA}{a^2 - 1} \right| \leq \frac{A}{a^2 - 1};$$

in fact C_1 is the locus of the terminal points of such vectors for all possible choices of the a_k and α_k , with A fixed; compare [2, p. 13]. The "disk" C_1 is represented by the formula given even if $A = 0$.

Likewise, the locus of the terminal points of the vectors with initial points in z_1 and representing the force at z_1 for all possible choices of the b_k and β_k with B fixed is the disk

$$C_2: \left| z + \frac{aB}{a^2 - 1} \right| \leq \frac{B}{a^2 - 1}.$$

The locus of the terminal points of the vectors with initial points in z_1 representing the total force at z_1 due to the particles at the α_k and β_k is the disk which is the "sum" of C_1 and C_2 :

$$(7) \quad \left| z - \frac{a(A - B)}{a^2 - 1} \right| \leq \frac{A + B}{a^2 - 1},$$

in the sense that if C_1 and C_2 are the loci of z_1 and z_2 then (7) is the locus of $z_1 + z_2$.

By a similar method, and with the note that the conjugate of $f(z)$ defines

the forces, it follows that the locus of the terminal points of the vectors with initial points in z_1 representing the total force at z_1 due to the particles at the γ_k and δ_k is the disk

$$(8) \quad \left| z + \frac{ia(C-D)}{a^2-1} \right| \leq \frac{C+D}{a^2-1}.$$

For the two sets of forces we have vectors with initial points in z_1 ($= 0$) and terminal points whose loci are the respective disks (7) and (8). The total resultant force is represented by a vector whose initial point is z_1 and the locus of whose terminal points lies in the disk

$$(9) \quad \left| z - \frac{a(A-B) - ia(C-D)}{a^2-1} \right| \leq \frac{A+B+C+D}{a^2-1}.$$

A necessary and sufficient condition that z_1 be a possible position of equilibrium is that the total force may be zero, or that z_1 ($= 0$) should lie in the disk (9), namely

$$\left| \frac{a(A-B) - ia(C-D)}{a^2-1} \right| \leq \frac{A+B+C+D}{a^2-1}, \quad a = |z_0| > 1,$$

which is essentially (6). The second member of (6) is greater than unity unless three of the four numbers A, B, C, D are zero.

If z_0 is a zero of $f(z)$ for a particular choice of the α_k etc., and if $0 < \rho < 1$, the point ρz_0 is a zero of $f(z)$ with the a_k, b_k, c_k, d_k unchanged and the $\alpha_k, \beta_k, \gamma_k, \delta_k$ multiplied by ρ . Moreover $z_0 = 0$ is a zero of $f(z)$ with suitably chosen $\alpha_k, \beta_k, \gamma_k, \delta_k$ small in modulus, and this completes the proof of Theorem 2.

COROLLARY 1. *If $f(z)$ in Theorem 2 is of the form*

$$f(z) \equiv \sum_{k=1}^m \frac{a_k}{z - \alpha_k} + \sum_{k=1}^n \frac{ic_k}{z - \gamma_k}, \quad A + C \neq 0,$$

the locus of its zeros is the disk

$$|z| \leq \frac{A+C}{(A^2+C^2)^{1/2}}.$$

COROLLARY 2. *If $f(z)$ in Theorem 2 is of the form*

$$(10) \quad f(z) \equiv \sum_{k=1}^m \frac{a_k}{z - \alpha_k} - \sum_{k=1}^n \frac{b_k}{z - \beta_k}, \quad A > B,$$

the locus of its zeros is the disk

$$(11) \quad |z| \leq \frac{A+B}{A-B}.$$

If $r(z)$ is a rational function not identically constant, and if the exact degrees of its numerator and denominator are A and B , its logarithmic derivative is of form (10); the conclusion of Corollary 2 is essentially that all zeros of the derivative lie in the disk (11), a result [1] proved by the present author in 1918.

If $f(z)$ is multiplied by ω with $|\omega| = 1$, the zeros of the new function $f_1(z)$ ($\equiv \omega f(z)$) are unchanged, yet the second member of (6) is not unchanged by such an arbitrary transformation; indeed the denominator in the second member of (6) is precisely $|A - B + iC - iD|$, which is invariant, but the numerator is not invariant. This seeming paradox is resolved if we consider for instance the special case $B = C = D = 0$. The original theorem refers to the zeros of $\sum a_k(z - \alpha_k)^{-1}$, $a_k > 0$ whereas if we set $\omega = \cos\theta + i\sin\theta$, $0 < \theta < \pi/2$; the new function $f_1(z)$ is to be written $\sum a_k(\cos\theta + i\sin\theta) \cdot (z - \alpha_k)^{-1}$, $\sum a_k = A$, which is quite different from the function

$$\sum a'_k(z - \alpha_k)^{-1} + \sum ic'_k(z - \beta_k)^{-1}$$

for all a'_k, c'_k having prescribed sums $\sum a'_k = A \cos\theta$, $\sum c'_k = A \sin\theta$.

As a consequence of the facts just discussed, we formulate the following

REMARK. *In the application of Theorem 2 we may replace $f(z)$ by $\omega f(z)$, where ω is a constant of modulus unity; this change may modify the second member of (6). In particular if $f(z)$ can be written so as to contain one or more terms of the form*

$$\frac{\lambda_k}{z - \zeta_k}$$

where $\arg \lambda_k$ is independent of k , then as far as those terms are concerned it is favorable to choose $\arg \omega = -\arg \lambda_k$.

Corollary 2 to Theorem 1 has an analogue here, concerning the integral

$$\phi(z) \equiv \int_{\gamma} \frac{d\alpha(t)}{t - z}.$$

If γ lies in the closed interior of the unit circle, if $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ where $\alpha_1(t)$ and $\alpha_2(t)$ are real, and if A and $-B$, and C and $-D$, are the respective total positive and negative variations of $\alpha_1(t)$ and $\alpha_2(t)$ on γ , and if $(A - B)^2 + (C - D)^2 \neq 0$, then all zeros of the approximating sums of $\phi(z)$ (which approach $\phi(z)$) lie in the closed interior of a variable disk that approaches (6), so by Hurwitz's theorem all zeros of $\phi(z)$ lie in the closed interior of (6).

The remarks just made concerning $\phi(z)$ suggest the study of the hypothesis of Theorem 2 except that now the α_k and β_k are required to lie on the unit circumference γ . If the locus of positive particles α_k is γ , and if z_0 lies exterior to γ , the locus of the equivalent particle is the closed interior of γ , as becomes obvious at once by inversion in the unit circle whose center

is z_0 . If the locus of these α_k is γ and z_0 lies interior to γ , the locus of the equivalent particle is the closed exterior of γ including the point at infinity. If z_0 lies interior to γ , we may consider the equivalent particles for each category of particles to lie at infinity, whence $f(z_0) = 0$. We have

THEOREM 3. *Let the hypothesis of Theorem 2 be modified so that all particles $\alpha_k, \beta_k, \gamma_k, \delta_k$ lie on $\gamma: |z| = 1$. As far as concerns points z not on γ , the locus of the zeros z of $f(z)$ is the disk (6).*

This proof of Theorem 3 involves essentially applying the method of proof of Theorem 2, but not applying Theorem 2 itself.

To study the points z on γ , we consider (as in the proof of Theorem 2) the actual forces at z_0 due to the various categories of particles, and the locus of the terminal points of the vectors representing these forces, when the initial points lie in z_0 . We omit the assumption $(A - B)^2 + (C - D)^2 \neq 0$. As before, let us choose z_0 positive and then translate the plane, so that γ becomes $|z + 1| = 1$ and z_0 becomes $z_1 = 0$. The inverse of γ in the unit circle whose center is z_1 is the line (better, the finite points of the line) $x = -1/2$, and the locus of the terminal points of all vectors each corresponding to a set of particles α_k is the line $x = A/2$ unless $A = 0$. The locus of the terminal points of all vectors each corresponding to a set of particles β_k is the line $x = -B/2$ unless $B = 0$, and for the composition of a pair of these vectors we have as locus the line $x = (A - B)/2$; however, it is to be noted that all vectors are null vectors if we have $A = B = 0$. The loci for the vectors corresponding to the γ_k and δ_k are respectively the lines $y = -C/2$ and $y = D/2$ unless $C = 0$ or $D = 0$; for the composition of a pair of these vectors we have as locus of the terminal points the line $y = -(C - D)/2$, except if $C = D = 0$. The locus for the negatives of these last mentioned vectors having their initial points in z_1 is $y = (C - D)/2$, which always intersects the line $x = (A - B)/2$; the total sum of all vectors is null for a suitable configuration depending on given A, B, C, D , with the exceptions noted.

THEOREM 4. *With the hypothesis of Theorem 3, the locus of the zeros of $f(z)$ contains the entire circumference γ provided we have $A + B \neq 0$ and $C + D \neq 0$. The locus contains the entire circumference also if $A = B, C + D = 0$, or if $A + B = 0, C = D$. The locus contains no point of γ if $A \neq B, C + D = 0$ or if $A + B = 0, C \neq D$.*

The case $A = B = C = D = 0$ is of course trivial.

It is of interest to indicate how Theorems 1 and 2 apply to the study of zeros of restricted infrapolynomials; for these methods, compare [4], [5]. The category of restricted infrapolynomials on a set E as used here includes the category of similarly restricted polynomials of least norm on E , where

norm is in the sense of least weighted p th powers ($p > 0$) or in the sense of (weighted) Tchebycheff.

THEOREM 5. *Let the two disjoint point sets $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_n$ consist of the distinct points indicated and lie on $-1 \leq z \leq 1$. Let the real polynomial $P(z) \equiv Nz^{m+n-1} + \dots$ have the coefficient N prescribed, and also the (real) values $P(\alpha_j)$, and be a thus restricted infrapolynomial (i.e., have no restricted underpolynomial) on $E: \{\beta_j\}$. Set $\sum_1^m [P(\alpha_j)/\omega'(\alpha_j)] = N_0$, where $\omega(z) \equiv \prod_1^m (z - \alpha_j) \cdot \prod_1^n (z - \beta_j)$. Let A and $-B$ be the sum of the positive and negative numbers respectively among $P(\alpha_j)/\omega'(\alpha_j), N - N_0$; we suppose $A > B$. Then all nonreal zeros of $P(z)$ lie in the closed interior of the ellipse (2), and all real zeros lie in the interval (3).*

The polynomial $P(z)$ can be expressed by the Lagrange formula

$$(13) \quad P(z) \equiv \omega(z) \sum_1^m \frac{P(\alpha_j)}{\omega'(\alpha_j)(z - \alpha_j)} + \omega(z) \sum_1^n \frac{P(\beta_j)}{\omega'(\beta_j)(z - \beta_j)};$$

here the coefficients $P(\alpha_j)$ are prescribed, and the coefficients $P(\beta_j)$ are not prescribed, but are subject to the condition

$$(14) \quad \sum_1^n P(\beta_j)/\omega'(\beta_j) = N - N_0.$$

It is then clear that for $P(z)$ thus restricted to be an infrapolynomial on E the condition

$$(15) \quad \text{sg}[P(\beta_j)/\omega'(\beta_j)] = \text{sg}(N - N_0), \quad j = 1, 2, \dots, n,$$

is necessary and sufficient. Indeed, if (15) is satisfied, there exists no restricted underpolynomial $Q(z)$ of $P(z)$ on E , for there exists no set of values $Q(\beta_j)$ with

$$(16) \quad \sum_1^n Q(\beta_j)/\omega'(\beta_j) = N - N_0$$

such that

$$(17) \quad |Q(\beta_j)| < |P(\beta_j)| \quad \text{if} \quad P(\beta_j) \neq 0$$

and

$$(18) \quad Q(\beta_j) = 0 \quad \text{if} \quad P(\beta_j) = 0.$$

Conversely, if $P(z)$ is a restricted infrapolynomial and if we have both (14) and $\sum_1^n |P(\beta_j)/\omega'(\beta_j)| > |N - N_0|$, then we can set

$$\frac{Q(\beta_j)}{\omega'(\beta_j)} = \frac{|P(\beta_j)/\omega'(\beta_j)| \cdot |N - N_0|}{\sum_1^n |P(\beta_j)/\omega'(\beta_j)|},$$

whence (16), (17), and (18) are valid and $Q(z)$ is an underpolynomial of $P(z)$ on E .

By virtue of (13) with (14) and (15), Theorem 5 now follows from Theorem 1. If a given polynomial $P(z)$ satisfies all the requirements of Theorem 5 except that now $A < B$, we need merely reverse the signs of the $P(\alpha_j)$ and of N to apply Theorem 5 as stated. But we draw no conclusion if $A = B$, namely if $N = 0$.

A similar application of Theorem 2, still by use of equations (13), (14), and (15), yields

THEOREM 6. *Let the two disjoint point sets $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_n$ consist of the distinct points indicated, and lie in the disk $\Gamma: |z| \leq 1$. Let the polynomial $P(z) \equiv Nz^{m+n-1} + \dots$ have the coefficient N prescribed, and also the values $P(\alpha_j)$, and be a thus restricted infrapolynomial on $E: \{\beta_j\}$. Let N_0 and $\omega(z)$ be as defined in Theorem 5. Let A, B, C, D be respectively the sum of the positive numbers among $\operatorname{Re}[S], \operatorname{Re}[-S], \operatorname{Re}[-iS], \operatorname{Re}[iS]$, where S is the set $\{P(\alpha_j)/\omega'(\alpha_j), N - N_0\}$, and where we suppose $(A - B)^2 + (C - D)^2 \neq 0$. Then all zeros of $P(z)$ lie in the disk (6).*

In connection with Theorem 6, the remark following the proof of Theorem 2 is significant.

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