

NEARLY MODULAR ORTHOCOMPLEMENTED LATTICES

BY

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Introduction. Let L be a complete, orthocomplemented lattice. We say that L is a dimension lattice if L is weakly modular and there is an equivalence relation on L satisfying the axioms A, B, C, and D' of Loomis [5]. We say that L is locally finite if every element of L is the join of finite elements.

If L is a dimension lattice in which every element is finite, then L is modular. Conversely, Kaplansky [4] has shown that if L is a complete orthocomplemented modular lattice, then L is a continuous geometry. From this and the results of von Neumann [7] and Iwamura [3], it follows that L is a dimension lattice in which every element is finite. Thus we conclude that L is a finite dimension lattice if and only if L is a complete orthocomplemented modular lattice. The main purpose of this paper is to obtain a similar characterization of locally finite dimension lattices.

To obtain such a characterization, we need to weaken the assumption that L is modular. It is natural to try the assumption that L is semi-modular, but this is not enough. We need to know that in some sense enough modular pairs exist. For this reason, we define a *modular element* to be an element a such that $[0, a]$ is a modular lattice and (x, a) is a modular pair for all x in L ⁽¹⁾. An atom is always a modular element, and the finite elements in a dimension lattice are modular. We say that an orthocomplemented lattice L is *nearly modular*, if L is weakly modular and semi-modular and every element of L is the join of modular elements. Our principal result is the following theorem⁽²⁾.

THEOREM A. *Let L be a complete orthocomplemented lattice. Then L is a locally finite dimension lattice if and only if L is nearly modular.*

In order to characterize the special class of Type I dimension lattices, we define an element a to be *minimal* if for every $x \leq a$ we can write $x = za$ with z in the center of L . Then we have

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⁽¹⁾ The author is indebted to G. W. Mackey for his suggesting the study of elements a for which (x, a) is always a modular pair.

⁽²⁾ Ramsay [8] has proved independently a closely related theorem that L is a locally finite dimension lattice if and only if L contains a join-dense ideal which is itself a modular lattice. The connection between his work and ours will be discussed in §3.

THEOREM B. *Let L be a complete, weakly modular, orthocomplemented lattice. Then L is a dimension lattice of Type I if and only if L is semi-modular, and there exists in L a minimal element whose central cover is 1.*

The contents of the paper are as follows: In §1 we summarize a few known definitions and results. In §2 we develop the theory of modular elements. The key theorem is that under suitable conditions, if a is modular and b is perspective to a then b is modular. An alternate characterization of nearly modular lattices is given, and some continuity theorems involving modular elements are proved. Theorem A is proved in §3 and Theorem B in §4. Finally, in §5 we introduce a simple construction, the horizontal sum of two orthocomplemented lattices. This is used to give examples showing that neither of the conditions in the definition of nearly modular can be dropped.

1. Preliminaries about modular pairs. In this section, we give a few known definitions and results which will be used without explicit mention throughout this paper.

A pair of elements (b, c) in a lattice L is said to be a *modular pair* if for all $x \leq c$ we have $(x \cup b)c = x \cup bc$. The pair (a, b) is a *d-modular pair* if for all $x \geq a$ we have $(a \cup b)x = a \cup bx$. L is said to be *semi-modular* if the relation of being a modular pair is symmetric in L .

LEMMA 1.1. *Let L be an orthocomplemented lattice. Then (b, c) is a modular pair in L if and only if (c', b') is a d-modular pair.*

COROLLARY. *L is semi-modular if and only if the relation of being a d-modular pair is symmetric in L .*

Following Loomis [5] we say that an orthocomplemented lattice is *weakly modular* if for $a \leq b$ we have $b = a \cup a'b$, that is, if (a, a') is a d-modular pair for all a in L . Some authors call a weakly-modular orthocomplemented lattice "orthomodular". From Lemma 3 of [5] we have the following useful result.

LEMMA 1.2. *Let L be a weakly-modular orthocomplemented lattice. Then if $a \cup c \perp b$ we have $(a \cup b)c = ac$.*

2. Modular elements. Let a be an element in an orthocomplemented lattice L . We say that a is *modular* if $[0, a]$ is a modular lattice and (x, a) is a modular pair for all x in L . We say that a is *d-modular* if $[a, 1]$ is a modular lattice, and (a, x) is a d-modular pair for all x in L .

LEMMA 2.1. *Let L be a semi-modular orthocomplemented lattice. The following statements about an element a of L are equivalent.*

- (1) (x, a) is a modular pair for all x in L .
- (2) (x, a') is a modular pair for all x in L .

(3) (a, x) is a d -modular pair for all x in L .

(4) (a', x) is a d -modular pair for all x in L .

Proof. From the fact that L is orthocomplemented it follows immediately that (1) and (4) are equivalent and that (2) and (3) are equivalent. Since L is semi-modular, (1) implies that (a, x) is a modular pair for all x . But this means that (x, a) is a d -modular pair for all x ; and this is equivalent to (3), because L is semi-modular. Thus (1) implies (3), and similarly (3) implies (1).

LEMMA 2.2. *The element a is modular if and only if a' is d -modular.*

Proof. The mapping $x \rightarrow x'$ is an anti-isomorphism of $[0, a]$ onto $[a', 1]$. Thus $[0, a]$ is a modular lattice if and only if $[a', 1]$ is a modular lattice. By Lemma 2.1, (x, a) is a modular pair for all x if and only if (a', x) is a d -modular pair for all x .

LEMMA 2.3. *If a is modular and $b \leq a$, b is modular. If a is d -modular and $b \geq a$, b is d -modular.*

LEMMA 2.4. *Suppose that (a, x) is a d -modular pair for all x in L . Then in the lattice $[c, 1]$, $(a \cup c, x)$ is a d -modular pair for all x .*

Proof. Suppose that $c \leq x$ and $a \cup c \leq y$. Then $[(a \cup c) \cup x]y = (a \cup x)y = a \cup xy = (a \cup c) \cup xy$.

Let L be a complete orthocomplemented lattice, and let a be an element such that $[0, a]$ is a modular lattice. We ask if a is modular. In general the answer is no, even if L is semi-modular. (See §5 for an example.) The answer is, however, yes if there are enough modular elements in L . Because of this we introduce the following condition on an orthocomplemented lattice L .

(*) If $x \neq 0$ in L then there exists $y \neq 0$ in L such that $y \leq x$ and (z, y) is a modular pair for all z in L .

Assuming this condition we can show that if $[0, a]$ is modular then a is modular. The proof depends on Kaplansky's theorem [4] that a complete orthocomplemented modular lattice is a continuous geometry. We will use this frequently without explicit mention of it.

THEOREM 2.1. *Let L be a complete, weakly-modular, semi-modular orthocomplemented lattice for which condition (*) holds. Let a be an element of L . Then if $[a, 1]$ is a modular lattice, a is d -modular. If $[0, a]$ is a modular lattice, a is modular.*

Proof. It suffices to prove the first statement. Let $b \in L$. We will show that (c, b) is a d -modular pair for all $c \geq a$. Let S be the set of $x \leq b$ such that (c, x) is a d -modular pair for all $c \geq a$. Suppose that $\{x_\alpha\}$ is a chain in S . Let $y = \bigcup_\alpha x_\alpha$. Then for $a \leq c \leq d$,

$$(c \cup y)d = \left(c \cup \bigcup_a x_a \right) d = \left(\bigcup_a (c \cup x_a) \right) d.$$

But $[a, 1]$ is a continuous geometry, so $[\bigcup_a (c \cup x_a)]d = \bigcup_a [(c \cup x_a)d] = \bigcup_a [c \cup x_a d] = c \cup [\bigcup_a (x_a d)] \leq c \cup yd$. Thus (c, y) is a d -modular pair; y is in S . By Zorn's lemma, S contains a maximal element x_0 . Let y be an element such that $y \leq x'_0 b$ and (z, y) is a modular pair for all z in L . We will show that $(c, x_0 \cup y)$ is a d -modular pair for all $c \geq a$. Since x_0 is maximal in S , this means that $y = 0$. But it then follows from (*) that $x'_0 b = 0$, i.e., that $b = x_0$; and this will prove the theorem. To show that $(c, x_0 \cup y)$ is a d -modular pair for all $c \geq a$, let $a \leq c \leq d$. Then

$$(c \cup x_0 \cup y)d \leq [(c \cup x_0) \cup (y \cup x_0)](x_0 \cup d) = c \cup x_0 \cup [(y \cup x_0)(x_0 \cup d)],$$

because, by Lemma 2.4, $(y \cup x_0, c \cup x_0)$ is a d -modular pair in $[x_0, 1]$, and $[x_0, 1]$ is semi-modular. Now $(y \cup x_0)(x_0 \cup d) = x_0 \cup d(y \cup x_0)$, because (d, x_0) is a d -modular pair, and L is semi-modular. Thus $(c \cup x_0 \cup y)d \leq [c \cup x_0 \cup d(y \cup x_0)]d = c \cup d(y \cup x_0) \cup x_0 d$, because $(c \cup d(y \cup x_0), x_0)$ is a d -modular pair. Thus $(c \cup x_0 \cup y)d \leq c \cup d(y \cup x_0)$ for all $d \geq c$, and this proves that $(c, x_0 \cup y)$ is a d -modular pair for all $c \geq a$.

LEMMA 2.5. *Let (c, d) be a pair of elements in an orthocomplemented lattice such that (c, d) is both a modular pair and a d -modular pair and $cd = 0$. Then the mapping $\theta: x \rightarrow x \cup c$ is an isomorphism of $[0, d]$ onto $[c, c \cup d]$.*

Proof. If $\theta(x) = \theta(y)$, we have $x \cup c = y \cup c$. Therefore $(x \cup c)d = (y \cup c)d$. But $(x \cup c)d = x \cup cd = x$, and $(y \cup c)d = y$. Thus $x = y$; θ is one-to-one. If $c \leq y \leq d \cup c$, $\theta(dy) = dy \cup c = (c \cup d)y = y$. Thus θ is onto. Clearly, $\theta(x) \leq \theta(y)$ if and only if $x \leq y$.

We will say that two elements a and b are *perspective* if there exists c such that $a \cup c = b \cup c$ and $ac = bc = 0$.

THEOREM 2.2. *Let L satisfy the hypotheses of Theorem 2.1. Let c be a complement of a in L . Then a is modular if and only if c is d -modular.*

Proof. Suppose that a is modular. Then (a, c) is both a modular pair and a d -modular pair. By Lemma 2.5, $[0, a]$ and $[c, 1]$ are isomorphic. Thus $[c, 1]$ is a modular lattice. By Theorem 2.1, c is d -modular. A similar argument shows that if c is d -modular, then a is modular.

COROLLARY 1. *If a is modular, and $a \cup c = 1$, then c is d -modular. If c is d -modular, and $ac = 0$, then a is modular.*

Proof. Suppose that a is modular and $a \cup c = 1$. Let $d = (ca)'c$. Then d is a complement of a , so d is d -modular. Since $d \leq c$, c is d -modular. The second part of the corollary is proved by a dual argument.

COROLLARY 2. *If a is modular, and b is perspective to a , b is modular.*

Proof. Since a and b are perspective, there exists c such that $a \cup c = b \cup c$ and $ac = bc = 0$. Let $d = c \cup (a \cup c)'$. Then d is a common complement of a and b [5, Proof of Lemma 38]. Since a is modular, d is d -modular, and therefore b is modular.

We will denote the center of a lattice L by $Z(L)$. The center of a complete, weakly-modular, orthocomplemented lattice L is a complete Boolean sublattice of L [1, Corollaries to Theorems 6 and 7]. Therefore, for any a in L , the intersection of all central elements $z \geq a$ is itself a central element. It is called the *central cover* of a and will be denoted by $e(a)$. The following characterization of central elements will be useful. The lemma is part of Theorem 3.2 of [6].

LEMMA 2.6. *Let a be an element in an orthocomplemented lattice L such that $x = xa \cup xa'$ for all x . Then $a \in Z(L)$.*

COROLLARY. *Let a be an element in a weakly-modular orthocomplemented lattice L such that $xa = 0$ implies $x \perp a$. Then $a \in Z(L)$.*

We will say that an orthocomplemented lattice L is *nearly modular* if and only if L is semi-modular and every element in L is the join of modular elements. It is natural to ask, "Can this last condition be replaced by the condition that L contains a modular element a such that $e(a) = 1$?" We suspect that the answer to this question is yes but have not been able to prove it. Statement (2) of the following theorem is the closest we have come to answering the question.

THEOREM 2.3. *Let L be a complete, weakly-modular, orthocomplemented lattice. Then the following three statements about L are equivalent.*

- (1) L is nearly modular.
- (2) L is semi-modular, condition (*) holds for L , and there exists an element c in L such that $e(c) = 1$ and $[0, c]$ is a modular lattice.
- (3) The modular elements in L form a join-dense ideal.

Proof. Suppose that (1) holds. Let $\{a_\alpha\}$ be a maximal family of nonzero modular elements such that $e(a_\alpha) \perp e(a_\beta)$ for $\alpha \neq \beta$. Let $b = \bigcup_\alpha a_\alpha$. Then it is clear that $e(b) = 1$, and b is modular. Thus (2) holds. Now suppose that (2) holds, that a and b are modular, and that $x \leq a \cup b$. Let $y = (xa)'x$, $d = (ba)'b$. Then $ya = da = 0$, and $y \cup a = (y \cup a)(a \cup d) = a \cup d(y \cup a)$. Thus y and $d(y \cup a)$ are perspective. But $d \leq b$, so $d(y \cup a)$ is modular. Thus y is modular. Also xa is modular. Now if $z \geq x$ and w is any element, we have $(x \cup w)z = (xa \cup y \cup w)z = xa \cup (y \cup w)z = xa \cup y \cup wz = x \cup wz$. Thus for every $x \leq a \cup b$ and every w , (x, w) is a d -modular pair. It follows that for every $x \leq a \cup b$, x is modular. Thus if (2) holds, the modular elements form an ideal. Let a be the join of all modular elements in L . By Theorem 2.2 the element c , such that $[0, c]$ is a modular lattice and $e(c) = 1$, is a modular element. Therefore $e(a) = 1$. Suppose that x is an

element such that $xb = 0$ for all modular elements b . Let b be any modular element and let $d = b'(x \cup b)$, $x_1 = (xd)'x$. Let $y = (x_1 \cup d)b$. Then $x_1d = yd = 0$, and $y \cup d = (x_1 \cup d)b \cup d = (d \cup b)(x_1 \cup d) = (x_1 \cup d)$. Thus x_1 and y are perspective. But y is modular, and $x_1 \leq x$. Hence $x_1 = 0$; $x = xd \leq b'$ for all modular b . This means that $x \leq a'$. It follows that for all x in L , we have $x = xa \cup xa'$. Therefore a is in the center of L , and hence $a = 1$. It now follows that if $xb = 0$ for all modular b , then $x = 0$. Thus every element x is the join of modular elements. This completes the proof that (2) implies (3). Now Ramsay [8, Theorem 5.4] has shown that if a complete weakly-modular orthocomplemented lattice L contains a join-dense ideal which is modular, then L is semi-modular. It follows immediately that (3) implies (1).

We conclude this section with some continuity theorems involving modular elements. The first is a trivial consequence of Kaplansky's theorem. The other two depend on arguments using the property of being nearly modular.

THEOREM 2.4. *Let L be a complete, weakly-modular, orthocomplemented lattice, and let c be any element of L . Then if a is a modular element, and $\{b_\alpha\}$ is a chain such that $b_\alpha \leq a$ for all α , we have $c(\bigcup_\alpha b_\alpha) = \bigcup_\alpha (cb_\alpha)$. If a is a d -modular element, and $\{b_\alpha\}$ is a chain such that $b_\alpha \geq a$ for all α , then $c \cup (\bigcap_\alpha b_\alpha) = \bigcap_\alpha (c \cup b_\alpha)$.*

Proof. It suffices to prove the first statement. We have $c(\bigcup_\alpha b_\alpha) = (ca)(\bigcup_\alpha b_\alpha)$. But $[0, a]$ is a continuous geometry, so $(ca)(\bigcup_\alpha b_\alpha) = \bigcup_\alpha (cab_\alpha) = \bigcup_\alpha (cb_\alpha)$.

THEOREM 2.5. *Let L be a complete, nearly-modular, orthocomplemented lattice, and let c be any element of L . Then if a is modular, and $\{b_\alpha\}$ is a chain such that $b_\alpha \leq a$ for all α , we have $c \cup (\bigcap_\alpha b_\alpha) = \bigcap_\alpha (c \cup b_\alpha)$. If a is d -modular, and $\{b_\alpha\}$ is a chain such that $b_\alpha \geq a$ for all α , we have $c(\bigcup_\alpha b_\alpha) = \bigcup_\alpha (cb_\alpha)$.*

Proof. It suffices to prove the first statement. We need only show that $\bigcap_\alpha (b_\alpha \cup c) \leq c \cup (\bigcap_\alpha b_\alpha)$. Suppose that y is modular and $y \leq \bigcap_\alpha (b_\alpha \cup c)$. Then for all α , $y \leq (y \cup a)(b_\alpha \cup c) = b_\alpha \cup c(y \cup a)$, because b_α is modular. Thus $y \leq \bigcap_\alpha (b_\alpha \cup c(y \cup a))$. But the lattice $[0, y \cup a]$ is a continuous geometry, so $\bigcap_\alpha (b_\alpha \cup c(y \cup a)) = c(y \cup a) \cup (\bigcap_\alpha b_\alpha) \leq c \cup (\bigcap_\alpha b_\alpha)$. Thus for all modular $y \leq \bigcap_\alpha (b_\alpha \cup c)$, we have $y \leq c \cup (\bigcap_\alpha b_\alpha)$. Since the modular elements are join-dense in L , this proves that $\bigcap_\alpha (b_\alpha \cup c) \leq c \cup (\bigcap_\alpha b_\alpha)$.

THEOREM 2.6. *Let L be a complete, nearly-modular, orthocomplemented lattice, and let $\{b_\alpha\}$ be a chain of elements in L . Then if a is modular, $a \cup (\bigcap_\alpha b_\alpha) = \bigcap_\alpha (a \cup b_\alpha)$. If a is d -modular, $a(\bigcup_\alpha b_\alpha) = \bigcup_\alpha (ab_\alpha)$.*

Proof. Again it suffices to prove the first statement, and we need only

show that $\bigcap_{\alpha}(a \cup b_{\alpha}) \leq a \cup (\bigcap_{\alpha} b_{\alpha})$. Suppose that x is a modular element, and that $x \leq \bigcap_{\alpha}(a \cup b_{\alpha})$. Then

$$x \leq \bigcap_{\alpha} [(a \cup x)(a \cup b_{\alpha})] = \bigcap_{\alpha} [a \cup b_{\alpha}(a \cup x)].$$

Now $\{b_{\alpha}(a \cup x)\}$ is a chain in the continuous geometry $[0, a \cup x]$, so $\bigcap_{\alpha} [a \cup b_{\alpha}(a \cup x)] = a \cup (\bigcap_{\alpha} [b_{\alpha}(a \cup x)]) \leq a \cup (\bigcap_{\alpha} b_{\alpha})$. Thus for all modular elements x such that $x \leq \bigcap_{\alpha}(a \cup b_{\alpha})$, we have $x \leq a \cup (\bigcap_{\alpha} b_{\alpha})$. This proves that $\bigcap_{\alpha}(a \cup b_{\alpha}) \leq a \cup (\bigcap_{\alpha} b_{\alpha})$.

We might remark that the theorem "If a is modular, $a(\bigcup_{\alpha} b_{\alpha}) = \bigcup_{\alpha}(ab_{\alpha})$," is not true. A counterexample is found by taking the element a to be an atom in the lattice of closed subspaces of a Hilbert space.

3. A nearly-modular lattice is a locally finite dimension lattice. By a dimension lattice we will mean a complete, weakly-modular, orthocomplemented lattice on which there exists an equivalence relation satisfying the four axioms A, B, C, and D' of Loomis [3]. These axioms are:

(A) If $a \sim 0$, then $a = 0$,

(B) If $a_1 \perp a_2$ and $b \sim a_1 \cup a_2$, then there exists an orthogonal decomposition of b , $b = b_1 \cup b_2$, such that $b_1 \sim a_1$ and $b_2 \sim a_2$,

(C) If $\{a_{\alpha}\}$ is a family of pairwise orthogonal elements, and if $\{b_{\alpha}\}$ is a second orthogonal family with the same indices such that $a_{\alpha} \sim b_{\alpha}$ for all α , then $\bigcup_{\alpha} a_{\alpha} \sim \bigcup_{\alpha} b_{\alpha}$,

(D') If a and b have a common complement, then $a \sim b$.

An element a in a dimension lattice is *finite* if $b \sim a$ and $b \leq a$ implies $b = a$. A dimension lattice is *locally finite* if every element is the join of finite elements.

The purpose of this section is to prove Theorem A, i.e., to prove that a complete orthocomplemented lattice L is a locally finite dimension lattice if and only if it is nearly modular. The "only if" part of the theorem is just a restatement of results of Ramsay [8, Theorem 5.4]. Before going on with the proof of the "if" part of the theorem, it seems appropriate to point out the relationship between our work and that of Ramsay [8]. Ramsay has proved that a complete, weakly-modular, orthocomplemented lattice L is a locally finite dimension lattice if it contains a join-dense modular ideal I . Thus we could combine his theorem with Theorem 2.3 to obtain our result. On the other hand he obtains as intermediate results the facts that the elements of I are modular and that L is semi-modular. This shows that L is nearly modular. The arguments of this section could, therefore, be applied to prove that L is a locally finite dimension lattice. Ramsay's actual method of proof is different from ours. He works with a dimension function on L ; we work directly with the equivalence relation.

We now continue with the proof of Theorem A. Throughout the remainder of this section we will suppose that L is a complete, nearly-modular, orthocomplemented lattice. We will write $a \approx b$ to mean that a and b are modular elements which are perspective. If a is modular and $b \approx c \leq a$, we will write $b \leq a$. For two arbitrary elements a and b in L , we will write $a \sim b$ if and only if $e(a) = e(b)$ and for all z in $Z(L)$ such that za or zb is modular, we have $za \approx zb$. If $a \sim b \leq c$, we will write $a < c$. The remainder of this section will be devoted to proving that the relation \sim is an equivalence relation and that it satisfies axioms A, B, C, and D'. This will complete the proof of Theorem A. Note that axiom A follows immediately from the definition of \sim , for if $a \sim 0$, then $e(a) = 0$, and hence $a = 0$.

LEMMA 3.1. *If a and b are perspective in an orthocomplemented lattice, then $e(a) = e(b)$.*

Proof. This is Lemma 2.2 of [8].

COROLLARY 1. *The relation \sim satisfies axiom D'.*

Proof. Suppose that a and b have a common complement c . By the lemma, $e(a) = e(b)$. Since central elements are distributive, we have for any $z \in Z(L)$ that zc is an axis of perspectivity for za and zb .

COROLLARY 2. *If $a \approx b$, then $a \sim b$.*

LEMMA 3.2. *Let a and b be modular elements in L such that $a \approx b$. Then there exists d such that $a \cup b = a \cup d = b \cup d$ and $ad = bd = 0$. Thus a and b are perspective in $a \cup b$.*

Proof. Let c be an axis of perspectivity for a and b , and let $d = c(a \cup b)$. Then $(a \cup b) = (a \cup b)(a \cup c) = a \cup c(a \cup b)$, because a is modular. Similarly, $a \cup b = b \cup d$. We have $ad = bd = 0$, because $ac = bc = 0$.

LEMMA 3.3. *The relation \approx is an equivalence relation on the set of modular elements in L . If a is a modular element in L , then when it is restricted to the lattice $[0, a]$, the relation \approx satisfies axioms A, B, C, and D'.*

Proof. The lemma is contained in Theorem 4.5 of [8]. We also note that it follows immediately from Kaplansky's theorem [4] and results of von Neumann [7] and Iwamura [3] on continuous geometries.

COROLLARY. *The relation \sim is an equivalence relation on L .*

Proof. Suppose that $a \sim b$ and $b \sim c$. We must show that $a \sim c$. Clearly $e(a) = e(c)$, and it suffices to prove that if $z \in Z(L)$ and za is modular, then $za \approx zc$. Since $a \sim b$, zb is modular and $zc \approx zb$. By the lemma, \approx is transitive, so $zc \approx za$.

LEMMA 3.4. *The relation \sim is finitely additive. That is if a_1, a_2, b_1, b_2*

are elements in L such that $a_1 \sim b_1, a_2 \sim b_2, a_1 \perp a_2$, and $b_1 \perp b_2$, then $a_1 \cup a_2 \sim b_1 \cup b_2$.

Proof. Let $a = a_1 \cup a_2$ and $b = b_1 \cup b_2$. From elementary properties of central elements, it follows that $e(a) = e(b)$. Thus we need only show that if za or zb is modular, then $za \approx zb$; and we can assume that za is modular. Then za_1 and za_2 are both modular, and hence zb_1 and zb_2 are modular. Let $c = za \cup zb = za_1 \cup za_2 \cup zb_1 \cup zb_2$. Then c is modular; and working in the dimension lattice $[0, c]$, we see that $za \approx zb$.

In the next six lemmas we will prove that the relation \sim is additive on chains, and axiom C will be obtained as a corollary. The essential step is the proof of Lemma 3.9 that if a is modular and $\{b_\alpha\}$ is a chain such that $b_\alpha \leq a$ for all α then $\bigcup_\alpha b_\alpha \leq a$. This lemma will also be useful in the proof of axiom B. The key to proving Lemma 3.9 is to use the perspectivity theorem (2.2) and one of the continuity theorems (2.6) to reduce to the special case where $\bigcup_\alpha b_\alpha \perp a$. This special case is handled in Lemma 3.8. Lemmas 3.5 and 3.6 are simple statements about dimension lattices. The first follows from the additivity of the dimension function and the second from axiom C by a routine application of Zorn's lemma.

LEMMA 3.5. *Let a and b be finite elements in a dimension lattice such that $a \leq b$. Let $a_1 \leq a$ and $b_1 \leq b$ be such that $a_1 \sim b_1$. Then $a'_1 a \leq b'_1 b$. If $a \sim b$, then $a'_1 a \sim b'_1 b$.*

LEMMA 3.6. *Let $\{a_\alpha\}$ and $\{b_\alpha\}$ be families in a dimension lattice, both having the same linearly ordered indexing set I , and such that $\alpha \leq \beta$ implies $a_\alpha \leq a_\beta$ and $b_\alpha \leq b_\beta$. Assume further that $a_\alpha \sim b_\alpha$ for all α in I . Then $\bigcup_\alpha a_\alpha \sim \bigcup_\alpha b_\alpha$.*

LEMMA 3.7. *Let a be a modular element in L and let $\{b_\alpha\}$ be a family of elements with linearly ordered indexing set I such that $\alpha \leq \beta$ implies $b_\alpha \leq b_\beta$ and such that $b_\alpha \leq a$ for all α . Then there exists a family $\{a_\alpha\}$, also with indexing set I , such that $\alpha \leq \beta$ implies $a_\alpha \leq a_\beta$, $a_\alpha \approx b_\alpha$ for all α , and $a_\alpha \leq a$ for all α .*

Proof. Let S consist of all pairs $\{J, \{x_\alpha\}\}$, where J is a subset of I , and $\{x_\alpha\}$ is a family indexed by J such that $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta, x_\alpha \leq a$ for all α in J , and $x_\alpha \sim b_\alpha$ for all α in J . Partially order S by $\{J_1, \{x_\alpha\}\} \leq \{J_2, \{y_\alpha\}\}$ if and only if $J_1 \subseteq J_2$ and $x_\alpha = y_\alpha$ for all α in J_1 . Then clearly every chain in S has an upper bound. Hence, by Zorn's lemma, S contains a maximal element say $\{J, \{a_\alpha\}\}$. We need only prove that $J = I$. Suppose that this is not so. Then there exists β in I but not in J . Let

$$x_1 = \bigcup_{\alpha: \alpha < \beta, \alpha \in J} a_\alpha, \quad y_1 = \bigcup_{\alpha: \alpha < \beta, \alpha \in J} b_\alpha.$$

Working in the dimension lattice $[0, a_\beta \cup b_\beta]$, we see that $x_1 \approx y_1 \leq b_\beta$. If the set of α in J such that $\alpha > \beta$ is nonvoid, let $x_2 = \bigcap_{\alpha: \alpha > \beta} a_\alpha$

and $\alpha \in J$). Otherwise, let $x_2 = a$. Since $[0, a \cup b_\beta]$ is a dimension lattice, it follows from Lemma 3.6 that $b_\beta \lesssim x_2$. Now from Lemma 3.5 we conclude that there exists x_3 such that $x_1 \leq x_3 \leq x_2$ and $b_\beta \approx x_3$. Therefore setting $a_\beta = x_3$ we can adjoin β to J . This contradicts the maximality of $\{J, \{a_\alpha\}\}$ in S , thus proving that $J = I$.

LEMMA 3.8. *Let a be a modular element in L . Let $\{b_\alpha\}$ be a family of elements in L with the linearly ordered index set I such that $\alpha \leq \beta$ implies $b_\alpha \leq b_\beta$, and such that $b_\alpha \lesssim a$ and $b_\alpha \perp a$ for all α . Then $b = \bigcup_\alpha b_\alpha$ is a modular element, and $b \lesssim a$.*

Proof. Let $\{a_\alpha\}$, $\alpha \in I$, be a family such that $\alpha \leq \beta$ implies $a_\alpha \leq a_\beta$, $a_\alpha \leq a$ for all α , and $a_\alpha \approx b_\alpha$ for all α . We can assume that $a = \bigcup_\alpha a_\alpha$, and we then need only prove that b is perspective to a . We can suppose that I is infinite for otherwise the lemma is trivial. Let $c_\alpha = [\bigcup_{\beta: \beta < \alpha} a_\beta]'a_\alpha$ and $d_\alpha = [\bigcup_{\beta: \beta < \alpha} b_\beta]'b_\alpha$. Clearly, $a = \bigcup_\alpha c_\alpha$ and $b = \bigcup_\alpha d_\alpha$. Moreover, from the fact that for each α , $[0, a_\alpha \cup b_\alpha]$ is a dimension lattice, we conclude that $c_\alpha \approx d_\alpha$ for all α . Thus we have the following situation. There are two orthogonal families $\{c_\alpha\}$ and $\{d_\alpha\}$ such that $c_\alpha \approx d_\alpha$ for all α . Moreover, $\bigcup_\alpha c_\alpha \perp \bigcup_\alpha d_\alpha$. We want to prove that $\bigcup_\alpha c_\alpha$ is perspective to $\bigcup_\alpha d_\alpha$. Since $\bigcup_\alpha c_\alpha$ and all the d_α are modular, the situation is very similar to that in the theory of continuous geometry where one wants to prove the corresponding theorem for independent sets of elements. We could obtain a straightforward proof along the lines of the proof of Theorem 3.6, part 1 of [7]. However, it happens that Holland [2] has proved a theorem which we can quote. He defines two elements x and y to be *strongly perspective* if there exists z such that $x \cup y = x \cup z = y \cup z$ and $xz = yz = 0$. His theorem [2, Theorem 1] states that if L is a complete, weakly modular, orthocomplemented lattice and $\{x_\alpha\}$ and $\{y_\alpha\}$ are two families such that for $\alpha \neq \beta$, $x_\alpha \cup y_\alpha \perp x_\beta \cup y_\beta$ and such that for all α , x_α and y_α are strongly perspective, then $\bigcup_\alpha x_\alpha$ and $\bigcup_\alpha y_\alpha$ are strongly perspective. In our case the families $\{c_\alpha\}$ and $\{d_\alpha\}$ clearly satisfy the orthogonality requirements of Holland's theorem, and, by Lemma 3.2, c_α is strongly perspective to d_α for each α . Therefore $\bigcup_\alpha c_\alpha$ is strongly perspective to $\bigcup_\alpha d_\alpha$, i.e., a and b are strongly perspective.

LEMMA 3.9. *Let a be a modular element in L . Let $\{b_\alpha\}$ be a family with linearly ordered indexing set I such that $\alpha \leq \beta$ implies $b_\alpha \leq b_\beta$ and such that $b_\alpha \lesssim a$ for all α . Then $b = \bigcup_\alpha b_\alpha$ is a modular element, and $b \lesssim a$.*

Proof. Let $c = a'b$, and let $d = (a'b)'b$. Then $b = c \cup d$. We have $c = a'(\bigcup_\alpha b_\alpha)$. But a' is d -modular, and $\{b_\alpha\}$ is a chain, so, by Theorem 2.6, $c = \bigcup_\alpha (a'b_\alpha)$. Clearly $a'b_\alpha \lesssim a$ and $a'b_\alpha \perp a$ for all α . Therefore, by the preceding lemma, c is a modular element. Now a' is d -modular and $da' = 0$. Therefore, d is modular (Theorem 2.2). Thus b is the join of two

modular elements; b is modular. Because b is modular, $[0, a \cup b]$ is a dimension lattice, and we can apply Lemmas 3.7 and 3.6 to conclude that $b \leq a$.

LEMMA 3.10. *Let $\{a_\alpha\}$ and $\{b_\alpha\}$ be families in L , both with the same linearly ordered indexing set I , such that $\alpha \leq \beta$ implies $a_\alpha \leq a_\beta$ and $b_\alpha \leq b_\beta$. Assume further that $a_\alpha \sim b_\alpha$ for all α . Then $\bigcup_\alpha a_\alpha \sim \bigcup_\alpha b_\alpha$.*

Proof. Let $a = \bigcup_\alpha a_\alpha$ and $b = \bigcup_\alpha b_\alpha$. Clearly, $e(a) = e(b)$. Let z be a central element such that za or zb is modular. It is only necessary to prove that $za \approx zb$, and we can assume that za is modular. Then za_α is modular for all α . Since $a_\alpha \sim b_\alpha$ for all α , $za_\alpha \approx zb_\alpha$ for all α . Thus for all α , $zb_\alpha \leq za$. By the preceding lemma, $\bigcup_\alpha (zb_\alpha)$ is modular. Working in the dimension lattice $[0, za \cup \bigcup_\alpha (zb_\alpha)]$ we conclude that $\bigcup_\alpha (za_\alpha) \approx \bigcup_\alpha (zb_\alpha)$. But $za = \bigcup_\alpha (za_\alpha)$ and $zb = \bigcup_\alpha (zb_\alpha)$, so the lemma is proved.

COROLLARY. *The relation \sim on L satisfies axiom C.*

Proof. Let $\{c_\alpha\}$ and $\{d_\alpha\}$ be two orthogonal families with the same indexing set I and such that $c_\alpha \sim d_\alpha$ for all α . We must prove that $\bigcup_\alpha c_\alpha \sim \bigcup_\alpha d_\alpha$. Well order the set I , and let $a_\alpha = \bigcup_{\beta \leq \alpha} c_\beta$ and $b_\alpha = \bigcup_{\beta \leq \alpha} d_\beta$. We claim that $a_\alpha \sim b_\alpha$ for all α . Suppose that this is not so. Then there exists a smallest β such that " $a_\beta \sim b_\beta$ " does not hold. Let $x = \bigcup_{\alpha: \alpha < \beta} a_\alpha$ and $y = \bigcup_{\alpha: \alpha < \beta} b_\alpha$. By the lemma, $x \sim y$. Moreover, $a_\beta = c_\beta \cup x$, $b_\beta = d_\beta \cup y$, $c_\beta \perp x$, $d_\beta \perp y$, and $c_\beta \sim d_\beta$. Therefore by Lemma 3.4, $a_\beta \sim b_\beta$, which is a contradiction. Now clearly $\bigcup_\alpha c_\alpha = \bigcup_\alpha a_\alpha$ and $\bigcup_\alpha d_\alpha = \bigcup_\alpha b_\alpha$. By the lemma, $\bigcup_\alpha a_\alpha \sim \bigcup_\alpha b_\alpha$, so the corollary is proved.

To complete the proof of Theorem A, we must show that axiom B holds. Suppose for the moment that $Z(L)$ is trivial. Then any two nonmodular elements are equivalent. Let $a \sim b$, and $a = a_1 \cup a_2$, where $a_1 \perp a_2$. We seek an orthogonal decomposition $b = b_1 \cup b_2$ such that $b_1 \sim a_1$ and $b_2 \sim a_2$. If a is modular, such a decomposition obviously exists. If a_1 is modular but a_2 is not, we find $b_1 \leq b$ such that $b_1 \sim a_1$. Then $b'_1 b$ is not modular, and so $b'_1 b \sim a_2$. If neither a_1 nor a_2 is modular, we need only split b into two nonmodular elements.

The proof in the general case, where $Z(L)$ is nontrivial, proceeds along the same lines. However, to take care of nonmodular elements which are not equivalent, the notion of a purely infinite element is introduced. We will say that the element $a \in L$ is *purely infinite* if and only if whenever $z \in Z(L)$ and za is modular then $za = 0$. From the definition of a purely infinite element and the equivalence relation we get immediately the following lemma.

LEMMA 3.11. *If a and b are both purely infinite and $e(a) = e(b)$, then $a \sim b$. If a is purely infinite and $b \sim a$, then b is purely infinite.*

We will say that two elements a and b of L are *related* if there exist modular elements $a_1 \leq a$ and $b_1 \leq b$ such that $a_1 \sim b_1 \neq 0$. It follows from the definition of the relation \sim that if $e(a) \perp e(b)$ then a and b are not related. We need the converse of this, and it comes as a corollary to the following theorem, which is of some interest in itself.

THEOREM 3.1. *Let a be an element in the complete, nearly-modular, ortho-complemented lattice L . Then the center of $[0, a]$ is just the set $\{za : z \in Z(L)\}$.*

Proof. Since the modular elements of L form an ideal, this theorem is contained in Theorem 4.3 of [8]. The author independently obtained a proof using results of von Neumann [7].

COROLLARY. *If a and b are not related, $e(a) \perp e(b)$.*

LEMMA 3.12. *Let $a \neq 0$ be purely infinite. Then a has an orthogonal decomposition $a = a_1 \cup a_2$ such that $e(a_1) = e(a_2)$ and both a_1 and a_2 are purely infinite.*

Proof. Let w be a modular element such that $w \leq a$. We will show that there exists a modular element $x \leq w'a$ such that $x \sim w$. Let S be the set of all modular elements $y \leq w'a$ such that $y \lesssim w$. It follows from Lemma 3.9 that every chain in S has an upper bound. Hence S has a maximal element x_0 , and for some $x_1 \leq w$, $x_0 \sim x_1$. Suppose that $x_1 < w$. Then $(x_0 \cup w)'a$ is not related to $x_1'w$, for otherwise x_0 would not be maximal in S . It follows that $e(x_1'w)[x_0 \cup w]'a = 0$, so we have

$$e(x_1'w)a = e(x_1'w)(x_0 \cup w).$$

But $e(x_1'w)(x_0 \cup w)$ is modular and a is purely infinite. Therefore $e(x_1'w)a = 0$. This contradicts the assumption that $x_1 < w$. Thus $x_1 = w$; $x_0 \sim w$. Now we can construct the orthogonal decomposition of a . Let $c_1 \leq a$ be a modular element, such that $e(c_1) = e(a)$. Let $d_1 < c_1'a$ be a modular element such that $d_1 \sim c_1$. Let $b_1 = 0$. Construct by induction sequences of elements $\{c_n\}$, $\{d_n\}$, $\{b_n\}$ such that $b_{n+1} = c_1 \cup d_1 \cup \dots \cup c_n \cup d_n$, $c_n \leq b_n'a$, $c_n \sim b_n$, $d_n \leq (c_n \cup b_n)'a$, $d_n \sim c_n \cup b_n$. Let $a_1 = \bigcup_n c_n$, $a_2 = \bigcup_n d_n$. If $z \in Z(L)$ and $za_1 \neq 0$, we have $zc_1 \neq 0$, because $e(c_1) = e(a_1)$. Then for all n , $zc_n > zc_1 \neq 0$. It follows that za_1 is not modular. Thus a_1 is purely infinite. Similarly, a_2 is purely infinite. By their construction, $a_1 \perp a_2$, and $e(a_1) = e(a_2) = e(a)$.

LEMMA 3.13. *Let $a = a_1 \cup a_2$, where $a_1 \perp a_2$, a_1 is modular, and a_2 is purely infinite. Let b be an element such that $b \sim a$. Then there exists an orthogonal decomposition of b , $b = b_1 \cup b_2$, such that $b_1 \sim a_1$, and $b_2 \sim a_2$.*

Proof. Let S consist of all elements $x \leq b$ such that $x \lesssim a_1$. Then if $\{x_\alpha\}$ is a chain in S , $\bigcup_\alpha x_\alpha$ is in S by Lemma 3.9. Thus S contains a maximal element b_1 . Let $b_2 = b'b$. We will first show that $za_2 = 0$ if and only if

$zb_2 = 0$. Suppose that $za_2 = 0$. Then $za = za_1$, so $za_1 \sim zb_1 \cup zb_2$. Let $b_3 = b_1 \cup zb_2$. Then $zb_3 = zb_1 \cup zb_2 \sim za_1$ and $z'b_3 = z'b_1 < z'a_1$. Thus $b_3 < a_1$. Since b_1 was maximal in S this means $b_3 = b_1$, i.e., $zb_2 = 0$. Suppose on the other hand that $zb_2 \neq 0$. Then $zb = zb_1$ and $zb \approx za_1 \cup za_2$. Thus za_2 is modular; $za_2 = 0$ because a_2 is purely infinite. Now it follows immediately that $e(a_2) = e(b_2)$. Further, if za_2 is modular, $za_2 = 0$, so $zb_2 = 0$; $za_2 \approx zb_2$. If zb_2 is modular, zb is modular, which means $za_2 = 0$. This gives $zb_2 = 0$, so again $za_2 \approx zb_2$. Thus $a_2 \sim b_2$. Now suppose that $b_1 \sim c_1 < a_1$. Then $c'_1 a_1$ is not zero. But $c'_1 a_1$ is unrelated to b_2 , for otherwise b_1 would not be maximal in S . Taking $z = e(c'_1 a)$, this gives $z \perp b_2$ and $z \perp a_2$. Hence $za = za_1$, $zb = zb_1$, and $za \sim zb$. This gives $zb_1 \sim za_1 = zc_1 \cup zc'_1 a_1 = zc_1 \cup c'_1 a_1$. But $zc_1 \sim zb_1$, and $a_1 \cup b_1$ is modular, so we must have $c'_1 a_1 = 0$. This contradiction proves that $a_1 \sim b_1$.

LEMMA 3.14. *The relation \sim on L satisfies axiom B.*

Proof. Let $a = a_1 \cup a_2$ and let $b \sim a$. We must show that there exists an orthogonal decomposition of b , $b = b_1 \cup b_2$, such that $b_1 \sim a_1$ and $b_2 \sim a_2$. Let e_1 be the join of all central elements $z \leq e(a)$ such that za is modular. Note that $e_1 a$ is modular. Let e_2 be the join of all central elements $z \leq e'_1 e(a)$ and such that za_1 is modular. Then $e_2 a_1$ is modular and $e_2 a_2$ is purely infinite. Let e_3 be the join of all central elements $z \leq (e_1 \cup e_2)' e(a)$ such that za_2 is modular. Then $e_3 a_2$ is modular and $e_3 a_1$ is purely infinite. Finally, let $e_4 = (e_1 \cup e_2 \cup e_3)' e(a)$. Then $e_4 a_1$ and $e_4 a_2$ are both purely infinite. It is now clear from the preceding results that for $i = 1, 2, 3, 4$ the element $e_i b$ has an orthogonal decomposition $e_i b = x_i \cup y_i$ such that $x_i \sim e_i a_1$ and $y_i \sim e_i a_2$. Taking $b_1 = \bigcup_i x_i$ and $b_2 = \bigcup_i y_i$, we get the desired decomposition of b .

This completes the proof of Theorem A.

4. Minimal elements and Type I dimension lattices. Let L be an orthocomplemented lattice. We will say that an element $a \in L$ is *minimal* if for every $x \leq a$ there exists $z \in Z(L)$ such that $x = za$. If L is modular this definition is equivalent to von Neumann's definition of minimal elements in a continuous geometry. (See [7, p. 277, especially Lemma 3.2].) Clearly, if a is minimal, $[0, a]$ is a Boolean algebra. We remark that if L is complete and nearly modular, and $[0, a]$ is a Boolean algebra, then a is minimal. This follows from Theorem 3.1.

LEMMA 4.1. *Let a be a minimal element in a complete, weakly-modular, orthocomplemented lattice L . Then a is modular.*

Proof. Let $x \leq a$, and let y be any element of L . There exists $z \in Z(L)$ such that $za = x$. We have then $(x \cup y)a = (za \cup zy \cup z'y)(za \cup z'a) = z(a \cup y)a \cup (z'(ay)) = za \cup zay \cup z'ay = x \cup ya$. Thus (y, a) is a mod-

ular pair for all y . We know that $[0, a]$ is a Boolean algebra. Therefore a is modular.

THEOREM 4.1. *Let L be a complete, semi-modular, weakly-modular, ortho-complemented lattice. Let a be a minimal element of L . Then if b is perspective to a , b is minimal.*

Proof. Let c be an axis of perspectivity for a and b . Let $d = c'(a \cup c)$. Then d and a are perspective with axis c . Since a is modular, (c, a) is both a modular pair and a d -modular pair. Further, d is the orthocomplement of c in the weakly-modular orthocomplemented lattice $[0, a \cup c]$. Thus (c, d) is both a modular pair and a d -modular pair in $[0, a \cup c]$. Let θ be the map from $[0, a]$ to $[0, d]: \theta(x) = (x \cup c)d$. By Lemma 2.5, θ is onto. Let ϕ be the map of $Z(L)$ into $[0, d]: \phi(z) = \theta(za)$. Since a is minimal, the map $z \rightarrow za$ is onto. Therefore ϕ is onto. Now

$$\phi(z) = (za \cup c)d = (za \cup zc \cup z'c)(zd \cup z'd) = z(a \cup c)d \cup z'cd = zd.$$

Thus the map $z \rightarrow zd$ from $Z(L)$ to $[0, d]$ is onto; d is minimal. To prove that b is minimal we carry out a similar argument using the fact that c is an axis of perspectivity for b and d . The only point to note is that c is the orthocomplement in $[0, a \cup c]$ of the modular element d . Thus in $[0, a \cup c]$, (c, b) is both a modular pair and a d -modular pair.

THEOREM 4.2. *Let L be a complete, semi-modular, weakly-modular, ortho-complemented lattice. Suppose further that L contains a minimal element a such that $e(a) = 1$. Then every element is the join of minimal elements, and hence L is nearly modular.*

Proof. Let d be the join of all minimal elements, and suppose that c is an element such that $cx = 0$ for all minimal elements x . For x minimal let $y = x'(x \cup c)$ and $c_1 = (cy)'c$. Then $c_1 \cup y = (c_1 \cup y)(x \cup y) = y \cup x(c_1 \cup y)$. Clearly, $c_1 y = [x(c_1 \cup y)]y = 0$. Thus c_1 and $x(c_1 \cup y)$ are perspective. Therefore c_1 is minimal, and this means $c_1 = 0$. This proves that if $cx = 0$ for all minimal x , $c \perp x$. Thus if $cd = 0$, $c \perp d$. Therefore, $d \in Z(L)$, and every element $y \leq d$ is the join of minimal elements. But by hypothesis there exists a minimal element a such that $e(a) = 1$. Therefore $d = 1$, and hence every element is the join of minimal elements.

Let us recall a few definitions from [5]. Two elements a and b in a dimension lattice L are said to be *related* if there exist $a_1 \leq a$ and $b_1 \leq b$ such that $a_1 \sim b_1 \neq 0$. An element a is *simple* if $x \cup y \leq a$ and $x \perp y$ imply that x is not related to y . The dimension lattice L is of Type I if 1 is the join of simple elements. Theorem B, which we restate below for the reader's convenience, gives a natural characterization of Type I dimension lattices. It is somewhat more pleasing than the theorem for arbitrary locally finite dimension lattices, because we need only assume that one

minimal element with central cover 1 exists, rather than requiring that every element be the join of minimal elements.

THEOREM B. *Let L be a complete weakly-modular orthocomplemented lattice. Then L is a dimension lattice of Type I if and only if L is semi-modular and there exists in L a minimal element whose central cover is 1.*

Proof. Suppose that L is semi-modular and contains a minimal element a such that $e(a) = 1$. Then by the preceding theorem, L is nearly modular and hence is a locally finite dimension lattice. Moreover, every element in L is the join of minimal elements. But if b is a minimal element, $[0, b]$ is a Boolean algebra. Therefore if $x \cup y \leq b$ and $xy = 0$, then x and y are not related by the perspectivity relation. Thus every minimal element is simple, and L is of Type I. Suppose now that L is a Type I dimension lattice. Then L is locally finite and hence is semi-modular. In L the element 1 is the join of simple elements. Hence there exists a simple element a such that $e(a) = 1$. We need only show that a is minimal. By Theorem 3 of [5] there exists for any $x \leq a$ an invariant element z such that $za = x$. (The element z is invariant if z and z' are unrelated.) By Theorem 2 of [5], z invariant implies $y = yz \cup yz'$ for all z . Therefore if z is invariant, $z \in Z(L)$. Thus for any $x \leq a$, there exists $z \in Z(L)$ such that $za = x$. That is, a is minimal.

5. Horizontal sums and examples. Let L_1 and L_2 be two orthocomplemented lattices which have no elements in common, except perhaps 0 and 1. Let L be the set union of L_1 and L_2 with the two zero elements identified and with the two unit elements identified. We can introduce a partial ordering in L by $x \leq y$ if and only if for some i both x and y are in L_i and $x \leq y$ in L_i . Then L is an orthocomplemented lattice. If $x \in L_i$, the orthocomplement of x in L is just the orthocomplement of x in L_i . We call L the *horizontal sum* of L_1 and L_2 and write $L = L_1 \circ L_2$. If x and y are different from 0 and 1, $x \in L_1$, and $y \in L_2$, then in $L_1 \circ L_2$ the elements x and y are not related by the partial ordering. The next theorem follows trivially from this fact.

THEOREM 5.1. *Let $L = L_1 \circ L_2$, where L_1 and L_2 are orthocomplemented lattices. Then L is complete if and only if L_1 and L_2 are both complete. L is weakly modular if and only if L_1 and L_2 are both weakly modular.*

Now let L_1 and L_2 be complete, modular, orthocomplemented lattices which contain no atoms. For example, let L_1 be the lattice of projections in a factor of Type II₁. Then $L_1 \circ L_2$ is semi-modular, but $L_1 \circ L_2$ is not nearly modular. If a is different from 0 and 1 in $L_1 \circ L_2$, then $[0, a]$ is a modular lattice, but a is not modular.

Let L_1 and L_2 be complete, weakly-modular, atomic, orthocomplemented

lattices, and suppose that L_1 contains at least three orthogonal atoms. Then every element in $L_1 \circ L_2$ is the join of atoms and hence of modular elements, but $L_1 \circ L_2$ is not semi-modular. Also it is easily verified that $L_1 \circ L_2$ is irreducible. It follows that for any atom a in $L_1 \circ L_2$, $e(a) = 1$. Thus $L_1 \circ L_2$ contains a minimal element whose central cover is 1.

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