

GENERALIZED EIGENVECTORS AND SEPARATION OF VARIABLES⁽¹⁾

BY
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1. Introduction. When we apply the method of separation of variables to solve an equation like the reduced wave equation, or Helmholtz equation, $(\nabla^2 + k^2)u(x, y) = f(x, y)$, we assume the solution expanded in a series of eigenfunctions of one of the variables. The coefficients depend upon the other variable. We substitute the expansion into the equation, thereby obtaining ordinary differential equations for the coefficients. The method relies upon the completeness of the eigenfunctions corresponding to one of the variables. For selfadjoint problems this assumption is justified by the usual expansion and completeness theorems of Sturm-Liouville theory [1]. For non-selfadjoint problems it is sometimes not true. An example of the latter type is the problem (for a rectangle)

$$\begin{aligned} \nabla^2 u + k^2 u &= \frac{1}{2k_0} x \cos k_0 x, \\ (1.1) \quad u(0, y) &= u(x, 0) = u(x, 1) = 0, \\ u_x(1, y) &= cu(1, y), \end{aligned}$$

where $k_0 \neq 0$ is a solution of $k_0 = \sin k_0 \cos k_0$ and $c = \cos^2 k_0$. Here the method breaks down because of the incompleteness of the eigenfunctions of the x problem [6]. An attempt is made in this paper to extend the applicability of the method of separation of variables by including generalized eigenfunctions for the purpose of completion. This is modelled after the finite-dimensional case. Here a basis of eigenvectors and generalized eigenvectors can be found for non-selfadjoint operators, leading to the Jordan canonical form. Some work has been done on developing expansion theorems in terms of generalized eigenfunctions. It has not been applied, however, to separation of variables. George Birkhoff proved [2] that a large class of functions could be expanded in a series of functions associated with a contour integral. They arise as residues of an integral in the complex plane of the Green's function for a linear homogeneous ordinary differential equation with "regular" boundary conditions. He did not investigate the properties of these residues in the non-selfadjoint case. J. Tamarkin [3] went further and showed that

Received by the editors April 19, 1963 and, in revised form, September 11, 1963.

⁽¹⁾ The research in this paper was supported by the United States Air Force Office of Scientific Research, under Grant No. AFOSR 62-108. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

these residues were principal functions (generalized eigenfunctions) of the associated differential operator. He did not investigate their properties, nor those of the coefficient functions in the expansion, any further. Naimark [4], showed that for a special class of second-order equations these principal functions were generalized eigenfunctions and satisfied certain orthogonality relations with the coefficient functions. Friedman [6] pointed out that spectral representations of both ordinary and partial differential operators are modified by the presence of multiple poles in the Green's functions. He compared the generalized eigenvectors of an operator with those of its adjoint. J. Schwartz has treated more general operators and has shown that a function may be expanded in a series of eigenvectors, with possibly a finite number of generalized eigenvectors, of a spectral operator. H. P. Kramer [11] specifically studied even order differential operators and showed that they were spectral if their coefficients were bounded operators.

2. Properties of generalized eigenvectors. Let L be a linear operator with domain a linear manifold D in a Hilbert space H . We call x a generalized eigenvector of rank r for the operator L and eigenvalue λ when $x, (L - \lambda)x, \dots, (L - \lambda)^{r-1}x$ all belong to D , and $(L - \lambda)^r x = 0$ but $(L - \lambda)^{r-1}x \neq 0$. Thus let L be the ordinary differential operator

$$(2.1) \quad Lx = p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x, \quad p_0(t) \neq 0 \text{ on } [a, b],$$

and D the linear manifold of those functions $x(t)$ with continuous derivatives up to order n on $[a, b]$ such that

$$(2.2) \quad Ux = 0.$$

Here $p_j(t)$ are complex-valued functions in $C^{n-j}[a, b]$, and the boundary form Ux is the vector

$$U_i x = \sum_{j=1}^n (a_{ij} x^{(j-1)}(a) + b_{ij} x^{(j-1)}(b)), \quad i = 1, \dots, n$$

(here (a_{ij}) and (b_{ij}) are complex matrices). Then a generalized eigenfunction of rank r for L and λ is any solution of

$$(L - \lambda)^r x = 0, \quad (L - \lambda)^{r-1} x \neq 0, \\ Ux = 0, \quad U(L - \lambda)x = 0, \dots, U(L - \lambda)^{r-1}x = 0.$$

The notation is taken from Coddington and Levinson [5]. We see that for x to lie in the domain of L it must satisfy n linear homogeneous boundary conditions at both end points on its first $n - 1$ derivatives.

Some easily shown properties of generalized eigenvectors (not necessarily of ordinary differential operators) follow. If x is of rank r for L and λ then $x, (L - \lambda)x, \dots, (L - \lambda)^{r-1}x$ form a chain of linearly independent generalized eigenvectors of decreasing rank. Of course $(L - \lambda)^{r-1}x$ is an ordinary eigenvector (rank 1). Since two chains may merge at some lower ranked com-

ponent, we call a collection of chains "independent" when their rank one components form a linearly independent set of vectors. All the generalized eigenvectors in an independent set of chains constitute a linearly independent set of vectors. Adding a lower rank to a generalized eigenvector does not change rank, nor will the chains generated by such vectors be independent. Suppose a generalized eigenvector of rank k is linearly independent from all the generalized eigenvectors of rank k or less in an independent set of chains for L and λ . Then it is linearly independent from all the vectors in the set. If for each eigenvalue of a finite set for L we have an independent set of chains, then the totality of all the vectors is a linearly independent set. Most of these properties are known [6]. They are generalizations of properties of ordinary eigenvectors, and are proven by applying powers of $L - \lambda$ to combinations of generalized eigenvectors, and equating coefficients. Also, since the set of generalized eigenvectors of rank m or less for L and λ is an invariant subspace of H (for $Lx = (L - \lambda)x + \lambda x$), then the usual theorem on the Jordan canonical form may be phrased as follows: If the space of generalized eigenvectors of rank m or less for L and λ is finite-dimensional, then there exists a basis for this space consisting of independent chains. Furthermore, the number and lengths of these chains are unique.

Since we are going to consider expansion theorems in terms of generalized eigenfunctions of ordinary differential operators such as (2.1), the concept of the adjoint operator, introduced in general situations by Birkhoff [2], is essential. This is true also in the case of ordinary eigenvectors. The salient points are [5] few. First, the operator M , with domain the set of functions $y(t) \in C^n[a, b]$ satisfying $Vy = 0$, is said to be adjoint to L if $\langle Lx, y \rangle = \langle x, My \rangle$ for all $x, y \in C^n[a, b]$ such that $Ux = 0, Vy = 0$. Here V is a boundary form, and the scalar product is the standard formula $\langle x, y \rangle = \int_a^b x(t)\overline{y}(t) dt$. L and U determine M , along with its domain, uniquely. The formal adjoint M is called L^+ , and is the differential operator

$$(2.3) \quad L^+y = (-1)^n(\overline{p}_0y)^{(n)} + (-1)^{n-1}(\overline{p}_1y)^{(n-1)} + \cdots + \overline{p}_ny,$$

$$p_0(t) \neq 0 \text{ on } [a, b].$$

V is called U^+ , so that the domain of L^+ is the set of functions $y(t) \in C^n[a, b]$ satisfying

$$(2.4) \quad U^+y = 0.$$

The problem $L^+y = 0, U^+y = 0$ is said to be adjoint to the problem $Lx = 0, Ux = 0$. The basic (alternative) theorem about adjoints is that both of these problems have the same number of linearly independent solutions; $Lx = f, Ux = 0$ has a solution if and only if f is orthogonal to the null space of L^+ [5]. The generalized eigenfunctions of rank m or less for L and λ are the solutions of

$$(2.5) \quad (L - \lambda)^m x = 0, \quad Ux = 0, \quad U(L - \lambda)x = 0, \dots, U(L - \lambda)^{m-1}x = 0,$$

whereas for L^+ and $\bar{\lambda}$ they are the solutions of

$$(2.6) \quad \begin{aligned} (L^+ - \bar{\lambda})^m y &= 0, \quad U^+ y = 0, \\ U^+(L^+ - \bar{\lambda})y &= 0, \dots, U^+(L^+ - \bar{\lambda})^{m-1}y = 0. \end{aligned}$$

Note that the solution spaces of (2.5) and (2.6) are finite-dimensional. A problem is said to be selfadjoint if $L = L^+$ and both operators have the same domain. For such problems all generalized eigenfunctions are of rank one. The fact that they form a complete set is given by Sturm-Liouville theory. One of the orthogonality properties from this theory is now generalized to higher ranks for the non-selfadjoint case.

THEOREM 2.1. *If x is a generalized eigenfunction for L and λ , and y is one for L^+ and $\bar{\mu}$, $\mu \neq \lambda$, then $(x, y) = 0$.*

THEOREM 2.2. *The basis of independent chains for the space of generalized eigenfunctions of rank m or less for L^+ and $\bar{\lambda}$ matches that for L and λ in number and lengths of chains.*

Theorem 2.1 is established by induction on the ranks of x and y . Theorem 2.2 is established by showing that the vectors of rank j or less, $j = 1, \dots, m$ in each basis are the solutions of adjoint problems and thus agree in number. For details see [10]. These theorems would also follow from J. Schwartz's results [9] if it were established that L is a regular differential operator.

3. The use of generalized eigenfunctions in expansion theorems. Generalized eigenfunctions are brought into expansion theorems by use of Birkhoff's result on the contour integral of the Green's function. It will be recalled [5] that if the problem

$$(3.1) \quad (L - \lambda)x = 0, \quad Ux = 0 \quad (\text{where } L \text{ is given by (2.1)})$$

has only the trivial solution for at least one complex value of λ , then the Green's function $G(t, t', \lambda)$ for this problem exists. G satisfies

$$(L - \lambda)G = \delta(t - t'), \quad UG = 0 \quad (\delta(t - t') \text{ is the Dirac delta function})$$

for λ not an eigenvalue, and is a meromorphic function of λ with poles only at eigenvalues. $\int_a^b G(t, t', \lambda) f(t') dt'$ is the unique solution of

$$(L - \lambda)x = f, \quad Ux = 0,$$

for continuous f , and λ not an eigenvalue. Furthermore, the only possible limit point of the eigenvalues is ∞ . For selfadjoint problems the eigenvalues are real and the poles of G are simple. Birkhoff's result [2] is that if U satisfies certain mild conditions which make the problem "regular," then for functions f made up of a finite number of pieces in $[a, b]$, each real, continuous, and with a continuous derivative,

$$(3.2) \quad \frac{1}{2} [f(t+0) + f(t-0)] = \lim_{n \rightarrow \infty} \left[-\frac{1}{2\pi i} \oint_{C_n} \int_a^b G(t, t', \lambda) f(t') dt' d\lambda \right] \text{ on } (a, b).$$

At $t = a$ and $t = b$ the sequence converges to linear combinations of $f(a + 0)$ and $f(b - 0)$. Here C_n is a circle about the origin of radius n in the λ -plane. For selfadjoint problems the residue of G at the eigenvalue $\lambda = \lambda_p$ [5] is

$$- \sum_{j=m_p}^{n_p} x_j(t) \bar{\psi}_j(t'),$$

where the x_j and ψ_j are the eigenfunctions for L and L^+ , respectively. Thus (3.2) for this type of problem takes the form

$$\frac{1}{2} [f(t + 0) + f(t - 0)] = \sum_{j=1}^{\infty} x_j(t) \langle f, \psi_j \rangle.$$

This is the usual expansion theorem.

Thus we expect for the non-selfadjoint case that the residue of G changes only by the addition of generalized eigenfunctions.

THEOREM 3.1. *Let $G(t, t', \lambda)$ be the Green's function for the problem (3.1). Then*

$$\begin{aligned} \frac{1}{2} [f(t + 0) + f(t - 0)] &= \lim_{n \rightarrow \infty} \left[-\frac{1}{2\pi i} \oint_{C_n} \int_a^b G(t, t', \lambda) f(t') dt' d\lambda \right] \quad \text{on } (a, b), \\ (3.3) \qquad \qquad \qquad &= -\frac{1}{2\pi i} \lim_{n \rightarrow \infty} \oint_{C_n} \int_a^b G(t, t', \lambda) f(t') dt' d\lambda \\ &= \sum_{j=1}^{\infty} x_j(t) \langle f, \psi_j \rangle \quad \text{on } (a, b). \end{aligned}$$

Here the $x_j(t)$ and $\psi_j(t)$ are generalized eigenfunctions for L and L^+ , respectively. Also

$$(3.4) \qquad \qquad \qquad \langle x_j, \psi_k \rangle = \delta_{jk}.$$

More specifically, if l is a pole of order m of G , then the residue at l is

$$\begin{aligned} \text{Res}_l &= - [x_{1l_1}(t) \bar{\psi}_{1l_1}(t') + \cdots + x_{1n_1}(t) \bar{\psi}_{1l_1}(t')] \\ (3.5) \qquad &- [x_{2l_2}(t) \bar{\psi}_{2l_2}(t') + \cdots + x_{2n_2}(t) \bar{\psi}_{2l_2}(t')] - \cdots \\ &- [x_{pl_1}(t) \bar{\psi}_{pl_1}(t') + \cdots + x_{pn_p}(t) \bar{\psi}_{pl_1}(t')], \quad m = n_1 \geq \cdots \geq n_p. \end{aligned}$$

The $x_{ij}(t)$ constitute a chain basis for the space of generalized eigenfunctions of rank m or less. The first subscript denotes chain number and the second denotes rank. The $\psi_{ij}(t)$ constitute a corresponding chain basis to the adjoint problem (2.6) (with λ replaced by l), the subscripts having the same meaning. The orthogonality relation

$$(3.6) \qquad \qquad \qquad \langle x_{mn}, \psi_{m'n'} \rangle = \delta_{mn} \delta_{n, n_m + 1 - n'}$$

holds.

We may prove this by writing out the Laurent expansion of G , applying powers of $L - \lambda$, and matching coefficients. See [10] for details. Alternatively this would follow from Theorem 3, Corollary 2 of J. Schwartz's article if it were established that L satisfies the hypotheses therein.

4. Application to separation of variables for a rectangle. We attempt now to solve the Helmholtz equation for a rectangle by separation of variables. We will find that in some cases we must use the preceding theorems to obtain the necessary expansions. The problem is to solve

$$(4.1) \quad \begin{aligned} (\nabla^2 + k^2)u(x, y) &= f(x, y), & x_1 < x < x_2, & y_1 < y < y_2, \\ u_x(x_1, y) &= z_1 u(x_1, y), & u_x(x_2, y) &= z_2 u(x_2, y), \\ u_y(x, y_1) &= I_1 u(x, y_1), & u_y(x, y_2) &= I_2 u(x, y_2). \end{aligned}$$

The impedance constants, as well as k , may be complex. By the separation procedure we will obtain systems of ordinary differential equations. We will have to investigate their solvability and the uniqueness of solutions.

The separated problems arising from (4.1) are

$$(4.2) \quad \begin{aligned} X'' + \alpha^2 X &= 0, & X'(x_1) &= z_1 X(x_1), & X'(x_2) &= z_2 X(x_2), \\ Y'' + \beta^2 Y &= 0, & Y'(y_1) &= I_1 Y(y_1), & Y'(y_2) &= I_2 Y(y_2), \end{aligned}$$

where $\alpha^2 + \beta^2 = k^2$. The nonzero eigenvalues, α_n^2 , are the squares of the solutions of

$$(4.3) \quad \omega(\alpha_n) \equiv (z_1 - i\alpha_n)(z_2 + i\alpha_n)e^{-i\alpha_n(x_2 - x_1)} - (z_1 + i\alpha_n)(z_2 - i\alpha_n)e^{i\alpha_n(x_2 - x_1)} = 0.$$

Zero is an eigenvalue if and only if $z_2(z_1 x_1 - 1) = z_1(z_2 x_2 - 1)$, and then the corresponding eigenfunction is

$$X_0 = z_1(x - x_1) + 1.$$

An application of Rouché's theorem [7] shows that the α_n asymptotically tend to $n\pi/(x_2 - x_1)$ as n approaches infinity. The adjoint problems of (3.2) are

$$(4.4) \quad \begin{aligned} X^{+''} + \bar{\alpha}^2 X^+ &= 0, & X^{+'}(x_1) &= \bar{z}_1 X^+(x_1), & X^{+'}(x_2) &= \bar{z}_2 X^+(x_2), \\ Y^{+''} + \bar{\beta}^2 Y^+ &= 0, & Y^{+'}(y_1) &= \bar{I}_1 Y^+(y_1), & Y^{+'}(y_2) &= \bar{I}_2 Y^+(y_2). \end{aligned}$$

We denote by X_{nm} that generalized eigenfunction corresponding to α_n^2 which is of rank m . Thus for each n we have a chain basis

$$X_{n1}, X_{n2}, \dots, X_{nm_n}$$

of (4.2), where m_n is the order of α_n as a pole of $G(x, x', \alpha)$, and correspondingly

$$X_{n1}^+, X_{n2}^+, \dots, X_{nm_n}^+$$

of (4.4). These bases satisfy

$$(4.5) \quad \langle X_{nm}, X_{pq}^+ \rangle = \delta_{np} \delta_{m, m_n + 1 - q}.$$

Let $\beta_n^2 = k^2 - \alpha_n^2$.

Having determined the bases of the x problem and its adjoint by the methods of §3, we proceed according to standard technique. Thus let $u(x, y)$ be a solution of (4.1). Then, expanding u and f in series of the x functions (we note that the x problem has impedance boundary conditions, and is thus regular in the sense of Birkhoff) we get

$$(4.6) \quad u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} a_{nm}(y) X_{nm}(x),$$

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} c_{nm}(y) X_{nm}(x).$$

By (3.5)

$$c_{nm}(y) = \langle f, X_{n, m_n+1-m}^+ \rangle.$$

THEOREM 3.1. *Suppose (4.1) has a solution. Substitution of (4.6) into (4.1) leads to systems of ordinary differential equations for the $a_{nm}(y)$. If k^2 is not an eigenvalue of (4.1) these systems are uniquely solvable. If k^2 is an eigenvalue of (4.1) these systems are solvable if and only if f is orthogonal to all the eigenfunctions of the adjoint of (4.1). The arbitrariness in the solutions of these systems induces an arbitrariness in the solution of (4.1) to within at most an eigenfunction. In either case, once the $a_{nm}(y)$ are found, the solution to (4.1) is given by (4.6).*

Proof. Substitution of (4.6) into (4.1) and use of (4.5) leads to the following systems of ordinary differential equations for the a_{nm} 's.

$$(4.7) \quad \begin{aligned} a''_{nm_n} + \beta_n^2 a_{nm_n} &= c_{nm_n}, \\ a''_{n, m_n-1} + \beta_n^2 a_{n, m_n-1} &= c_{n, m_n-1} + a_{nm_n}, \\ &\vdots \\ a''_{n2} + \beta_n^2 a_{n2} &= c_{n2} + a_{n3}, \quad n = 1, 2, \dots, \\ a''_{n1} + \beta_n^2 a_{n1} &= c_{n1} + a_{n2}, \\ a'_{nm}(y_1) &= I_1 a_{nm}(y_1), \\ a'_{nm}(y_2) &= I_2 a_{nm}(y_2), \quad m = 1, 2, \dots, m_n; \quad n = 1, 2, \dots \end{aligned}$$

We observe that the expression (4.6) for u already satisfies two of the boundary conditions in (4.1). Each equation in (4.7) is of the form

$$a'' + \beta_n^2 a = g, \quad a'(y_1) = I_1 a(y_1), \quad a'(y_2) = I_2 a(y_2),$$

where g differs from equation to equation. Thus the system (3.7) are uniquely solvable if and only if

$$(4.8) \quad a'' + \beta_n^2 a = 0, \quad a'(y_1) = I_1 a(y_1), \quad a'(y_2) = I_2 a(y_2)$$

has only the trivial solution. In solving (4.7) we consider separately the cases where k^2 is and is not an eigenvalue of (4.1).

Case A. We assume k^2 is not an eigenvalue of (4.1). Then (4.8) has only the trivial solution for each n . For if $a_0(y)$ were a nontrivial solution corresponding to n_0 , then $u_0(x, y) = X_{n_0}(x)a_0(y)$ would be an eigenfunction of (4.1), contradicting our assumption. We may then solve (4.7) by using the Green's function $G(y, y', \beta_n)$, which is the unique solution of

$$G'' + \beta_n^2 G = \delta(y - y'), \quad G_y(y_1, y', \beta_n) = I_1 G(y_1, y', \beta_n), \\ G_y(y_2, y', \beta_n) = I_2 G(y_2, y', \beta_n).$$

By the properties of G the solution of (4.7) is

$$(4.9) \quad a_{nm}(y) = \int_{y_1}^{y_2} G(y, y', \beta_n) [c_{nm}(y') + a_{n,m+1}(y')] dy', \quad \begin{matrix} m = 1, 2, \dots, m_n, \\ n = 1, 2, \dots \end{matrix}$$

Here $a_{n,m_n+1} = 0$. The Green's function is

$$(4.10) \quad G(y, y', \beta_n) = \frac{[I_2 y_{>} + (1 - I_2 y_2)][I_1 y_{<} + (1 - I_1 y_1)]}{I_2(1 - I_1 y_1) - I_1(1 - I_2 y_2)} \quad \text{for } \beta_n = 0, \\ = \frac{-i}{2\beta_n \Delta} [(I_2 + i\beta_n)e^{i\beta_n(y_{>} - y_2)} - (I_2 - i\beta_n)e^{-i\beta_n(y_{>} - y_2)}] \\ \times [(I_1 + i\beta_n)e^{i\beta_n(y_{<} - y_1)} - (I_1 - i\beta_n)e^{-i\beta_n(y_{<} - y_1)}] \quad \text{for } \beta_n \neq 0.$$

Here $y_{>} = \max(y, y')$, $y_{<} = \min(y, y')$, and

$$\Delta = (I_1 + i\beta_n)(I_2 - i\beta_n)e^{i\beta_n(y_2 - y_1)} - (I_1 - i\beta_n)(I_2 + i\beta_n)e^{-i\beta_n(y_2 - y_1)}$$

From (4.6), (4.9), (4.10) and the form of the X_{nm}^+ , the $a_{nm}(y)$ turn out to be linear combinations of functions of the form

$$e^{\pm i\beta_n y} \int_{y_1}^y \int_{x_1}^{x_2} p_j(x) f(x, y') e^{\pm i[a_n(x - x_1) \pm \beta_n y']} dx dy'.$$

Also from (4.6) and (4.9) the solution for u is

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} \int_{y_1}^{y_2} G(y, y', \beta_n) [c_{nm}(y') + a_{n,m+1}(y')] dy' X_{nm}(x),$$

concluding Case A.

Case B. Now let k^2 be an eigenvalue. We show here that (4.7) is solvable if and only if f is orthogonal to all the solutions of the homogeneous adjoint problem of (4.1). Furthermore any arbitrariness in the solution of (4.7) introduces an arbitrariness in u to within at most an eigenfunction of (4.1). Before establishing this we must investigate the nature of the eigenfunctions of (4.1) and its adjoint problem. This adjoint problem is

$$(4.11) \quad (\nabla^2 + \bar{k}^2)v = 0 \\ v_x(x_1, y) = \bar{z}_1 v(x_1, y), \quad v_x(x_2, y) = \bar{z}_2 v(x_2, y), \\ v_y(x, y_1) = \bar{I}_1 v(x, y_1), \quad v_y(x, y_2) = \bar{I}_2 v(x, y_2).$$

We show this by using Green's theorem. Let u and v be any functions satisfying the boundary conditions (though not necessarily the differential equations) of (4.1) and (4.11), respectively. Then

$$\begin{aligned}\langle \nabla^2 u, v \rangle &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} \bar{v} \nabla^2 u \, dx dy \\ &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} u \nabla^2 \bar{v} \, dx dy + \oint \left(\bar{v} \frac{\partial u}{\partial n} - u \frac{\partial \bar{v}}{\partial n} \right) ds = \langle u, \nabla^2 v \rangle.\end{aligned}$$

The boundary integral vanishes by virtue of the boundary conditions.

We now investigate what the eigenfunctions of (4.1) and (4.11) are in terms of the solutions of the separated problems. The answer is given by the following.

LEMMA 4.1. *If k^2 is an eigenvalue of (4.1) then $k^2 = \alpha^2 + \beta^2$ for some pair of α and β (there may be more than one pair, that is, k^2 may be degenerate), where α^2 and β^2 are eigenvalues of (4.2). We denote the chains corresponding to them by*

$$(4.12) \quad X_1, X_2, \dots, X_m; Y_1, Y_2, \dots, Y_n,$$

where m, n are the indices of α^2 and β^2 , respectively. Then the eigenfunctions of (4.1) are the linear combinations of

$$(4.13) \quad \begin{aligned} &X_1 Y_1, X_1 Y_2 - X_2 Y_1, \dots, X_1 Y_{j-1} - X_2 Y_{j-2} + \dots + (-1)^j X_{j-1} Y_1, \\ &X_1 Y_j - X_2 Y_{j-1} + \dots + (-1)^{j+1} X_j Y_1, \quad j = \min(m, n). \end{aligned}$$

We refer to these as chain products. (In the case of degeneracy we have such chain products for each pair of eigenvalues α_n^2, β_n^2 such that $k^2 = \alpha_n^2 + \beta_n^2$.)

Proof. First let

$$(4.14) \quad w(x, y) = X_1 Y_{k-1} - X_2 Y_{k-2} + \dots + (-1)^k X_{k-1} Y_1, \quad 2 \leq k \leq j+1.$$

We will now show that w is an eigenfunction of (4.1). We define the operators L and M by $Lw(x, y) = -w_{xx}(x, y)$, $Mw(x, y) = -w_{yy}(x, y)$. Thus (calling $X_0 = Y_0 = 0$) to preserve notation)

$$\begin{aligned}(\nabla^2 + k^2)w(x, y) &= -(L - \alpha^2 + M - \beta^2) \sum_{p=1}^{k-1} (-1)^{p+1} X_p(x) Y_{k-p}(y) \\ &= - \sum_{p=1}^{k-1} (-1)^{p+1} [X_{p-1}(x) Y_{k-p}(y) + X_p(x) Y_{k-1-p}(y)] \\ &= \sum_{p=2}^{k-1} (-1)^p X_{p-1}(x) Y_{k-p}(y) + \sum_{p=1}^{k-2} (-1)^p X_p(x) Y_{k-1-p}(y) \\ &= \sum_{p=2}^{k-1} (-1)^p X_{p-1}(x) Y_{k-p}(y) - \sum_{p'=2}^{k-1} (-1)^{p'} X_{p'-1}(x) Y_{k-p'}(y) \\ &= 0.\end{aligned}$$

Also w satisfies the boundary conditions by virtue of the conditions on the X_p and Y_p , and is thus an eigenfunction. Next we assume that w is an eigenfunction of (4.1). We will find its form. From previous arguments

$$w(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} a_{nm}(y) X_{nm}(x),$$

where the a_{nm} are to be found. Substitution into (4.1) with $f = 0$ leads to (4.7), with $c_{n_1} = c_{n_2} = \dots = c_{nm_n} = 0$.

For those n for which β_n^2 is not an eigenvalue, $a_{n_1} = a_{n_2} = \dots = a_{nm_n} = 0$. For some $n = n_0$, β_n^2 is an eigenvalue. Otherwise $a_{nm} = 0$ for all n, m and u would not be an eigenfunction. Let $\beta^2 = \beta_{n_0}^2$, $\alpha^2 = \alpha_{n_0}^2$. Dropping the chain subscript from the a_{n_0m} and from the generalized eigenfunctions, the n_0 th system of (4.7) becomes

$$\begin{aligned} a_m'' + \beta^2 a_m &= 0, \\ a_{m-1}'' + \beta^2 a_{m-1} &= a_m \\ (4.15) \quad &\cdot \quad \cdot \quad \cdot \quad a_p'(y_1) = I_1 a_p(y_1), \quad a_p'(y_2) = I_2 a_p(y_2), \\ & a_2'' + \beta^2 a_2 = a_3, \quad p = 1, 2, \dots, m. \\ & a_1'' + \beta^2 a_1 = a_2, \end{aligned}$$

We solve this separately for the cases $m \leq n$ and $m > n$.

Case a. $m \leq n$. The general solution of the first equation is $a_m = cY_1$. The second equation is then

$$a_{m-1}'' + \beta^2 a_{m-1} = cY_1,$$

whose general solution is $a_{m-1} = -cY_2 + dY_1$, since $-cY_2$ is a particular solution. Similarly the general solution of the third equation is $a_{m-2} = cY_3 - dY_2 + eY_1$. Continuing this way we arrive at the last equation, whose general solution is

$$a_1 = (-1)^{m-1}cY_m + (-1)^{m-2}dY_{m-1} + \dots + (-1)^1eY_2 + (-1)^0jY_1.$$

Thus (assuming $\beta_{n_0}^2$ is the only β_n^2 that is an eigenvalue) we find

$$\begin{aligned} u(x, y) = \sum_{j=1}^m a_j(y) X_j(x) &= (-1)^{m-1}c(X_1Y_m - X_2Y_{m-1} + \dots + (-1)^{m-1}X_mY_1) \\ &+ (-1)^{m-2}d(X_1Y_{m-1} - X_2Y_{m-2} + \dots + (-1)^{m-2}X_{m-1}Y_1) + \dots \\ &+ (-1)^0jX_1Y_1, \end{aligned}$$

which is of the asserted form.

Case b. $m > n$. Again the solution of the first equation is $a_m = c_1Y_1$. The second has solution $a_{m-1} = -c_1Y_2 + c_2Y_1$, and so on, until we come to the $(n+1)$ st equation. This equation is

$$a_{m-n}'' + \beta^2 a_{m-n} = a_{m-n+1} = (-1)^{n-1}c_1Y_n + (-1)^{n-2}c_2Y_{n-1} + \dots + (-1)^0c_nY_1.$$

Because this equation has a solution, the alternative theorem implies

$$0 = \langle a_{m-n+1}, Y_1^+ \rangle = \langle (-1)^{n-1} c_1 Y_n, Y_1^+ \rangle = (-1)^{n-1} c_1.$$

Thus its general solution is

$$a_{m-n} = (-1)^{n-1} c_2 Y_n + (-1)^{n-2} c_3 Y_{n-1} + \cdots + (-1)^1 c_n Y_2 + (-1)^0 c_{n+1} Y_1.$$

If $m - n = 1$ we are finished. If not, the next equation is

$$a''_{m-n-1} + \beta^2 a_{m-n-1} = a_{m-n}.$$

Again, since this has a solution we get $c_2 = 0$. We continue in this manner, using the alternative theorem. When we get to the m th equation we have $c_1 = c_2 = \cdots = c_{m-n-1} = 0$. The m th equation is

$$a''_1 + \beta^2 a_1 = a_2 = (-1)^{n-1} c_{m-n} Y_n + (-1)^{n-2} c_{m-n+1} Y_{n-1} + \cdots + (-1)^0 c_{m-1} Y_1.$$

Thus $c_{m-n} = 0$. Its solution is

$$a_1 = (-1)^{n-1} c_{m-n+1} Y_n + (-1)^{n-2} c_{m-n+2} Y_{n-1} + \cdots + (-1)^0 c_m Y_1.$$

Thus since $a_{n+1} = a_{n+2} = \cdots = a_m = 0$ we finally get

$$\begin{aligned} w(x, y) = & (-1)^0 c_m (X_1 Y_1) + (-1)^1 c_{m-1} (X_1 Y_2 - X_2 Y_1) + \cdots \\ & + (-1)^{n-1} c_{m-n+1} (X_1 Y_n + \cdots + (-1)^{n-1} X_n Y_1). \end{aligned}$$

This is of the asserted form, proving the Lemma.

Applying Lemma 4.1 to obtain the eigenfunction of (4.11) we see that they are linear combinations of

$$(4.16) \quad \begin{aligned} & X_1^+ Y_1^+, X_1^+ Y_2^+ - X_2^+ Y_1^+, \dots, X_1^+ Y_{j-1}^+ - X_2^+ Y_{j-2}^+ + \cdots + (-1)^j X_{j-1}^+ Y_1^+, \\ & X_1^+ Y_j^+ X_2^+ Y_{j-1}^+ + \cdots + (-1)^{j+1} X_j^+ Y_1^+, \quad j = \min(m, n). \end{aligned}$$

The X_p^+ and Y_p^+ are generalized eigenfunctions of (4.4). The orthogonality relations among these functions are

$$(4.17) \quad \langle X_i, X_j^+ \rangle = \delta_{i, m+1-j}, \quad \langle Y_i, Y_j^+ \rangle = \delta_{i, n+1-j}.$$

We are now ready to establish our initial assertion on the necessary and sufficient conditions for the solvability of (4.7) when k^2 is an eigenvalue. Dropping the chain subscript the n_0 th system of (4.7) becomes

$$\begin{aligned} & a''_m + \beta^2 a_m = c_m, \\ & a''_{m-1} + \beta^2 a_{m-1} = c_{m-1} + a_m, \\ (4.18) \quad & \cdot \quad \cdot \quad \cdot \quad a'_p(y_1) = I_1 a_p(y_1), \quad a'_p(y_2) = I_2 a_p(y_2), \\ & a''_2 + \beta^2 a_2 = c_2 + a_3, \quad p = 1, 2, \dots, m. \\ & a''_1 + \beta^2 a_1 = c_1 + a_2, \end{aligned}$$

First assume that (4.7) has a solution for the a_{nm} . Define

$$\tilde{u}(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} a_{nm}(y) X_{nm}(x).$$

Then

$$(\nabla^2 + k^2)\tilde{u}(x, y) = f(x, y).$$

Let $u^+(x, y)$ be any solution of (4.11). Then

$$\begin{aligned} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{u}^+(x, y) dx dy &= \langle f, u^+ \rangle = \langle (\nabla^2 + k^2)\tilde{u}, u^+ \rangle \\ &= \langle \tilde{u}, (\nabla^2 + \bar{k}^2)u^+ \rangle = 0, \end{aligned}$$

establishing the necessity. Now assume that f is orthogonal to all solutions of (4.11). Those systems of (4.7) for which β_n^2 is not an eigenvalue are uniquely solvable, as before, by the method of Green's function. For the remaining systems, a typical one being (4.18), we have to show that in each equation the inhomogeneous term satisfies the orthogonality criterion of the alternative theorem. By our assumption

$$\begin{aligned} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_1^+(x) \bar{Y}_1^+(y) dx dy &= 0, \\ \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_1^+(x) \bar{Y}_2^+(y) dx dy - \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_2^+(x) \bar{Y}_1^+(y) dx dy &= 0, \\ (4.19) \quad &\cdot \quad \cdot \quad \cdot, \\ \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_1^+(x) \bar{Y}_{j-1}^+(y) dx dy - \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_2^+(x) \bar{Y}_{j-2}^+(y) dx dy + \dots \\ &+ (-1)^j \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_{j-1}^+(x) \bar{Y}_1^+(y) dx dy = 0, \\ \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_1^+(x) \bar{Y}_j^+(y) dx dy - \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_2^+(x) \bar{Y}_{j-1}^+(y) dx dy + \dots \\ &+ (-1)^{j+1} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_j^+(x) \bar{Y}_1^+(y) dx dy = 0. \end{aligned}$$

We want to show

$$(4.20) \quad \langle c_m, Y_1^+ \rangle = \langle c_{m-1} + a_m, Y_1^+ \rangle = \dots = \langle c_1 + a_2, Y_1^+ \rangle = 0.$$

Starting with the first equation,

$$\langle c_m, Y_1^+ \rangle = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_1^+(x) \bar{Y}_1^+(y) dx dy = 0.$$

Thus this equation is solvable, its general solution being

$$(4.21) \quad a_m = a_{mp} + d_1 Y_1,$$

where a_{mp} is a particular solution and d_1 is arbitrary. Then

$$\begin{aligned} \langle c_{m-1} + a_m, Y_1^+ \rangle &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_2^+(x) \bar{Y}_1^+(y) dx dy + \langle a_m, Y_1^+ \rangle \\ &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \bar{X}_1^+(x) \bar{Y}_2^+(y) dx dy + \langle a_m, Y_1^+ \rangle \\ &= \int_{y_1}^{y_2} c_m(y) \bar{Y}_2^+(y) dy + \langle a_m, Y_1^+ \rangle \\ &= \int_{y_1}^{y_2} [a_m''(y) + \beta^2 a_m(y)] \bar{Y}_2^+(y) dy + \langle a_m, Y_1^+ \rangle \\ &= [a_m' \bar{Y}_2^+ - a_m \bar{Y}_2^{+'}]_{y_1}^{y_2} - \langle a_m, Y_1^+ \rangle + \langle a_m, Y_1^+ \rangle = 0 \end{aligned}$$

by (4.18). The general solution of the second equation is then

$$(4.22) \quad a_{m-1} = a_{(m-1)p} - d_1 Y_2 + d_2 Y_1,$$

where $a_{(m-1)p}$ is any particular solution of $a'' + \beta^2 a = c_{m-1} + a_{mp}$. Note that this equation is solvable, just set $d_1 = 0$ in (4.20). We continue proving (4.20) in the same manner. We replace each c_p by $\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) \cdot \bar{X}_{m+1-p}^+(x) dx dy$. Then we use (4.19) to replace these expressions by ones involving lower ranked x adjoint functions. We now replace these latter integrals by higher indexed c_q 's; use (4.18) to replace these by derivatives of the higher a_q 's. Then we integrate by parts, to get $(L^+ - \beta^2)$ operating on the adjoint y functions. Finally we use the boundary conditions to show that the resulting expression is zero. After establishing

$$(4.23) \quad \langle c_m, Y_1^+ \rangle = \langle c_{m-1} + a_m, Y_1^+ \rangle = \cdots = \langle c_{m-(j-1)} + a_{m-(j-2)}, Y_1^+ \rangle = 0$$

we have for the general solution of the system of the first j equations of (4.18),

$$\begin{aligned} a_m &= a_{mp} + d_1 Y_1, \\ a_{m-1} &= a_{(m-1)p} - d_1 Y_2 + d_2 Y_1, \\ (4.24) \quad &\cdot \quad \cdot \quad \cdot \\ a_{m-(j-2)} &= a_{[m-(j-2)]p} + (-1)^{j-2} d_1 Y_{j-1} + (-1)^{j-3} d_2 Y_{j-2} + \cdots \\ &\quad + (-1)^1 d_{j-2} Y_2 + d_{j-1} Y_1, \\ a_{m-(j-1)} &= a_{[m-(j-1)]p} + (-1)^{j-1} d_1 Y_j + (-1)^{j-2} d_2 Y_{j-1} + \cdots \\ &\quad + (-1)^1 d_{j-1} Y_2 + d_j Y_1. \end{aligned}$$

Here the d 's are arbitrary and $a_{(m-q)p}$, $q = 0, 1, \dots, j-1$, is any particular solution of

$$(4.25) \quad a'' + \beta^2 a = c_{m-q} + a_{[m-(q-1)]p},$$

where $a_{(m+1)p} = 0$. By (4.23) this equation is solvable. (Just set the d 's = 0). If $j = m$ we have covered all the equations of (4.18) and have established our assertion. The general solution is (4.24). Thus the contribution of this system to u is

$$\begin{aligned}
 \sum_{p=1}^m a_p(y) X_p(x) &= d_m(X_1 Y_1) + (-1)^1 d_{m-1}(X_1 Y_2 - X_2 Y_1) \\
 (4.26) \quad &+ (-1)^2 d_{m-2}(X_1 Y_3 - X_2 Y_2 + X_3 Y_1) + \cdots \\
 &+ (-1)^{m-1} d_1(X_1 Y_m - X_2 Y_{m-1} + \cdots + (-1)^m X_{m-1} Y_2 + (-1)^{m+1} X_m Y_1) \\
 &+ X_1 a_{1p} + X_2 a_{2p} + \cdots + X_m a_{mp}.
 \end{aligned}$$

This is arbitrary to within an eigenfunction of (4.1). If instead $j = n < m$ we have to go further and show that

$$\begin{aligned}
 (4.27) \quad \langle c_{m-n} + a_{m-(n-1)}, Y_1^+ \rangle &= \langle c_{m-(n+1)} + a_{m-n}, Y_1^+ \rangle = \cdots \\
 &= \langle c_2 + a_3, Y_1^+ \rangle = \langle c_1 + a_2, Y_1^+ \rangle = 0.
 \end{aligned}$$

To do this we will have to fix some of the d 's, thus restricting the generality of our solution.

Case a. $j = n < m = n - r \leq 2n$, therefore $r \leq n$.

In this case the general solution of (4.18) is

$$\begin{aligned}
 a_m &= a_{mp} + d_{10} Y_1, \\
 a_{m-1} &= a_{(m-1)p} + (-1)^1 d_{10} Y_2 + d_{20} Y_1, \\
 a_{m-2} &= a_{(m-2)p} + (-1)^2 d_{10} Y_3 + (-1)^1 d_{20} Y_2 + d_{30} Y_1, \\
 &\quad \cdot \quad \cdot \quad \cdot \\
 a_{m-(r-1)} &= a_{|m-(r-1)|p} + (-1)^{r-1} d_{10} Y_r + (-1)^{r-2} d_{20} Y_{r-1} + \cdots \\
 &\quad + (-1)^1 d_{(r-1)0} Y_2 + d_{r0} Y_1, \\
 a_{m-r} &= a_{(m-r)p} + (-1)^r d_{10} Y_{r+1} + (-1)^{r-1} d_{20} Y_r + \cdots \\
 &\quad + (-1)^1 d_{r0} Y_2 + d_{r+1} Y_1 \quad (\text{delete if } r = n), \\
 &\quad \cdot \quad \cdot \quad \cdot \\
 a_{m-(n-1)} &= a_{|m-(n-1)|p} + (-1)^{n-1} d_{10} Y_n + (-1)^{n-2} d_{20} Y_{n-1} + \cdots \\
 &\quad + (-1)^{n-r} d_{r0} Y_{n-(r-1)} + (-1)^{n-(r+1)} d_{r+1} Y_{n-r} + \cdots + (-1)^1 d_{n-1} Y_2 + d_n Y_1, \\
 a_{m-n} &= a_{(m-n)p} + (-1)^{n-1} d_{20} Y_n + \cdots + (-1)^{n-(r-1)} d_{r0} Y_{n-(r-2)} \\
 &\quad + (-1)^{n-r} d_{r+1} Y_{n-(r-1)} + \cdots + (-1)^1 d_n Y_2 + d_{n+1} Y_1, \\
 &\quad \cdot \quad \cdot \quad \cdot \\
 a_2 &= a_{2p} + (-1)^{n-1} d_{r0} Y_n + (-1)^{n-2} d_{r+1} Y_{n-1} + \cdots \\
 &\quad + (-1)^1 d_{m-2} Y_2 + d_{m-1} Y_1, \\
 a_1 &= a_{1p} + (-1)^{n-1} d_{r+1} Y_n + \cdots + (-1)^1 d_{m-1} Y_2 + d_m Y_1.
 \end{aligned}$$

Here $d_{r+1}, d_{r+2}, \dots, d_m$ are arbitrary, and $a_{1p}, \dots, a_{mp}, d_{10}, \dots, d_{r0}$ are defined as follows: $a_{(m-q)p}$ for $q = 0, 1, \dots, n-1$ is any particular solution of

$$(4.29) \quad a'' + \beta^2 a = c_{m-q} + a_{|m-(q-1)p},$$

where we set $a_{(m+1)p} = 0$. This is solvable by (4.23). We define in order $d_{10}, a_{(m-n)p}, d_{20}, a_{|m-(n+1)p}, \dots, d_{r0}, a_{1p}$ by

$$(4.30) \quad d_{q0} = -(-1)^{n-1}[\langle cm - (n+q-1), Y_1^+ \rangle + \langle a_{|m-(n+q-2)p}, Y_1^+ \rangle],$$

$$q = 1, 2, \dots, r,$$

$$a_{(m-q)p}'' + \beta^2 a_{(m-q)p} = c_{m-q} + a_{|m-(q-1)p} + (-1)^{n-1} d_{(q+1-n)0} Y_n,$$

$$q = n, n+1, \dots, m-2, m-1.$$

Note that the choice of $d_{10}, \dots, d_{r0}$ makes this latter equation solvable. One can now check that (4.27) holds and that (4.28) is indeed the general solution.

Case b.

$$j = n < m = n + r > 2n, \quad \text{therefore } r > n.$$

Again we define $a_{1p}, \dots, a_{mp}, d_{10}, \dots, d_{r0}$ by (4.29) and (4.30). The general solution of (4.18) is then

$$\begin{aligned} a_m &= a_{mp} + d_{10} Y_1, \\ a_{m-1} &= a_{(m-1)p} + (-1)^1 d_{10} Y_2 + d_{20} Y_1, \\ &\quad \cdot \quad \cdot \quad \cdot \\ a_{m-(n-1)} &= a_{|m-(n-1)p} + (-1)^{n-1} d_{10} Y_n + (-1)^{n-2} d_{20} Y_{n-1} + \dots \\ &\quad + (-1)^1 d_{(n-1)0} Y_2 + d_{n0} Y_1, \\ a_{m-n} &= a_{(m-n)p} + (-1)^{n-1} d_{20} Y_n + \dots + (-1)^1 d_{n0} Y_2 + d_{(n+1)0} Y_1, \\ (4.31) \quad &\quad \cdot \quad \cdot \quad \cdot \\ a_{m-(r-1)} &= a_{|m-(r-1)p} + (-1)^{n-1} d_{(r-n+1)0} Y_n + \dots + (-1)^1 d_{(r-1)0} Y_2 + d_{r0} Y_1, \\ a_{m-r} &= a_{|m-rp} + (-1)^{n-1} d_{(r-n+2)0} Y_n + \dots + (-1)^1 d_{r0} Y_2 + d_{r+1} Y_1, \\ &\quad \cdot \quad \cdot \quad \cdot \\ a_2 &= a_{2p} + (-1)^{n-1} d_{r0} Y_n + (-1)^{n-2} d_{r+1} Y_{n-1} + \dots \\ &\quad + (-1)^1 d_{m-2} Y_2 + d_{m-1} Y_1, \\ a_1 &= a_{1p} + (-1)^{n-1} d_{r+1} Y_n + (-1)^{n-2} d_{r+2} Y_{n-1} + \dots \\ &\quad + (-1)^1 d_{m-1} Y_2 + d_m Y_1. \end{aligned}$$

Thus in either case (4.27) holds and one can check that the contribution of (4.18) to u is arbitrary to within an eigenfunction of (4.1). Thus (4.7) is solvable for the case k^2 an eigenvalue to within the asserted degree of arbitrariness. Thus, we have proved Theorem 4.1.

It is worth noting that the actual solution of (4.1) may be carried out in two steps. The first uses the unmodified procedure of separation of variables, dealing only with ordinary eigenvectors. It involves solving

$$(\nabla^2 + k^2)u_1(x, y) = f(x, y) - \sum_{n=1}^{\infty} \sum_{m=2}^{m_n} c_{nm}(y) X_{nm}(x),$$

$$u_{1_x}(x_1, y) = z_1 u_1(x_1, y), \quad u_{1_x}(x_2, y) = z_2 u_1(x_2, y),$$

$$u_{1_y}(x, y_1) = I_1 u_1(x, y_1), \quad u_{1_y}(x, y_2) = I_2 u_1(x, y_2)$$

by substituting in

$$u_1(x, y) = \sum_{n=1}^{\infty} a_{n1}(y) X_{n1}(x),$$

$$f(x, y) - \sum_{n=1}^{\infty} \sum_{m=2}^{m_n} c_{nm}(y) X_{nm}(x) = \sum_{n=1}^{\infty} c_{n1}(y) X_{n1}(x).$$

This leads to the single equation

$$a''_{n1} + \beta_n^2 a_{n1} = c_{n1}, \quad a'_{n1}(y_1) = I_1 a_{n1}(y_1), \quad a'_{n1}(y_2) = I_2 a_{n1}(y_2).$$

Any consideration of higher ranks in this part would only give us systems of ordinary differential equations whose solutions are

$$a_{nm} = 0 \quad \text{for } m > 1.$$

The second step involves solving

$$(\nabla^2 + k^2)u_2(x, y) = \sum_{n=1}^{\infty} \sum_{m=2}^{m_n} c_{nm}(y) X_{nm}(x),$$

$$u_{2_x}(x_1, y) = z_1 u_2(x_1, y), \quad u_{2_x}(x_2, y) = z_2 u_2(x_2, y),$$

$$u_{2_y}(x, y_1) = I_1 u_2(x, y_1), \quad u_{2_y}(x, y_2) = I_2 u_2(x, y_2).$$

Here we substitute in the complete expansion

$$u_2(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} a_{nm}(y) X_{nm}(x),$$

but we need only solve those systems corresponding to chains along which f has high rank components. For we must have

$$a_{n1} = a_{n2} = \dots = a_{nm} = 0$$

for the other systems. Thus there are relatively few systems to solve. Of course, the solution to the original problem is then

$$u = u_1 + u_2.$$

5. Extensions. This extended procedure of separation of variables, using generalized eigenfunctions, may be applied to other coordinate systems as well as to a larger class of partial differential equations. Let

$$(5.1) \quad \xi = \xi(x, y), \quad \eta = \eta(x, y)$$

be any conformal transformation. The Helmholtz equation then transforms to

$$(5.2) \quad u_{\xi\xi} + u_{\eta\eta} + k^2 h^2(\xi, \eta) = 0.$$

The metrical coefficient h^2 is given by

$$h^2(\xi, \eta) = x_\xi^2 + x_\eta^2.$$

A necessary and sufficient condition for separability of (5.2) with impedance boundary conditions in this system is

$$h^2(\xi, \eta) = f(\xi) + g(\eta)$$

for some f and g [8]. Thus all separable coordinates obtained by conformal maps of the $z = x + iy$ plane are obtained by solving

$$x_\xi^2 + x_\eta^2 = f(\xi) + g(\eta).$$

Then $y(\xi, \eta)$ is obtained as the harmonic conjugate of $x(\xi, \eta)$. In order that the impedance conditions

$$(5.3) \quad \frac{\partial u}{\partial n} = z(\xi, \eta)u$$

also separate for the finite domain bounded by the coordinate curves

$$(5.4) \quad \xi = \xi_1, \quad \xi = \xi_2, \quad \eta = \eta_1, \quad \eta = \eta_2,$$

$z(\xi, \eta)$ must be restricted to certain forms along the boundaries. By expressing $\partial u / \partial n$ in terms of u_ξ and u_η , we see that z must satisfy

$$(5.5) \quad \begin{aligned} z(\xi_1, \eta) &= \frac{-C_1}{h(\xi_1, \eta)}, & z(\xi_2, \eta) &= \frac{C_2}{h(\xi_2, \eta)}, \\ z(\xi, \eta_1) &= \frac{-D_1}{h(\xi, \eta_1)}, & z(\xi, \eta_2) &= \frac{D_2}{h(\xi, \eta_2)}, \end{aligned}$$

where C_1, C_2, D_1, D_2 are complex constants. If we substitute

$$u(\xi, \eta) = F(\xi)G(\eta)$$

into (5.2) and (5.3) they separate into

$$(5.5) \quad \begin{aligned} F'' + (k^2 f - \alpha^2)F &= 0, & F'(\xi_1) &= C_1 F(\xi_1), & F'(\xi_2) &= C_2 F(\xi_2), \\ G'' + (k^2 g + \alpha^2)G &= 0, & G'(\eta_1) &= D_1 G(\eta_1), & G'(\eta_2) &= D_2 G(\eta_2), \end{aligned}$$

where α^2 is the separation constant.

Thus, for example, for elliptic coordinates [8]

$$f(\xi) = d^2 \cosh^2 \xi, \quad g(\eta) = -d^2 \cos^2 \eta,$$

where d is a real constant. Then (5.2), (5.3), and (5.5) become

$$L_1 = \sum_{n=0}^p a_n(x) \frac{\partial^n}{\partial x^n}, \quad L_2 = \sum_{n=0}^q b_n(y) \frac{\partial^n}{\partial y^n}.$$

Acknowledgement. The author wishes to express his gratitude to Professor Joseph B. Keller for his guidance in this work.

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