

DERIVATIONS ON $\bar{p}$ -ADIC FIELDS⁽¹⁾

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1. Introduction. In a recent paper Heerema [1] has shown that if K^* is a p -adic field, i.e., a field which is complete and unramified with respect to a discrete rank 1 valuation, with ring of integers R^* and natural place $H^*: K^* \rightarrow \{k, \infty\}$, then for every derivation $d: k \rightarrow k$ there exists a derivation $D^*: K^* \rightarrow K^*$ such that $D(R^*) \subset R^*$ and $d(H^*(a)) = H^*(D^*(a))$ for every $a \in R^*$.

It follows from this, for example, that the inertial automorphisms of K^* are Taylor-series-like expressions in powers of p using integral derivations, i.e., derivations $D^*: K^* \rightarrow K^*$ such that $D^*(R^*) \subset R^*$. The fact that every derivation on k is induced by one on K^* also yields a simple way of constructing an example to show that if K is a ramified $\bar{p}$ -adic field and K^* is unramified having the same residue field, then K^* is uniquely embedded in K if and only if k is perfect [4].

In this paper we study ramified $\bar{p}$ -adic fields K with ring of integers R , residue field k and natural place $H: K \rightarrow \{k, \infty\}$ with the property that for every derivation $d: k \rightarrow k$ there exists a derivation $D: K \rightarrow K$ such that $D(R) \subset R$, $D(\pi) \in (\pi)$ for a prime element π of R and $a \in R$ implies $d(H(a)) = H(D(a))$. For convenience of discussion we will call this property of $\bar{p}$ -adic fields property (H).

We derive several characterizations of $\bar{p}$ -adic fields with property (H). These characterizations are essentially of two kinds. The first characterization gives a condition on the Eisenstein equation of the $\bar{p}$ -adic field with respect to a given fixed p -adic subfield with the same residue field in the restricted valuation, while the other characterizations are intrinsic and yield properties this class of $\bar{p}$ -adic fields must have.

The first characterization mentioned above makes use of the valuation topology of the fixed p -adic subfield and depends on the "distance" arbitrary elements can be moved by derivations in the metric generating the valuation topology. We have found a theoretically simple way of determining this distance which allows us to assign a numerical value to the Eisenstein polynomial with the property that the given $\bar{p}$ -adic field has property (H) if and only if this numerical value is greater than zero.

The other characterizations are in terms of the derivations on the $\bar{p}$ -adic field K and automorphisms on this field. We show that K has property (H)

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if and only if every integral derivation on K is an inducing derivation, i.e., if and only if $D(R) \subset R$ implies $D((\pi)) \subset (\pi)$.

Finally, after establishing some connections between integral derivations on an arbitrary $\bar{p}$ -adic field K and inertial automorphisms we are able to obtain some information on the structure of the pseudo-ramification groups of K . We are also able to give a third characterization of property (H), namely, if p is an odd prime and if $n \geq (e + p)/(p - 1)$, where e is the ramification of K , then K has property (H) if and only if for every automorphism $T: K \rightarrow K$ such that $T(a) - a \in (\pi)^n$ for all $a \in R$, it is true that $T(\pi) - \pi \in (\pi)^{n+1}$. This third result is a corollary to an extension of a result of MacLane [2] from the case of p -adic fields to the $\bar{p}$ -adic field situation.

2. The index of inertia. As in the introduction, we assume that K^* is a p -adic field, R^* its valuation ring and $k = R^*/(p)$ its residue field. We assume that $H^*: K^* \rightarrow \{k, \infty\}$ is the natural place. For $a \in R^*$, let $\Delta(a) = \min \{V(D^*(a))\}$, where D^* ranges over all integral derivations on K^* . The symbol $\Delta(a)$ is called the index of inertia of a . If we assume $V(p) = 1$, then $\Delta(a)$ will have value a non-negative integer. If $D^*(a) = 0$ for all integral derivations D^* , then we will assign $\Delta(a)$ the value ∞ . If k_0 is the maximum perfect subfield of k , then R^* contains a unique subring R_0^* such that $H^*(R_0^*) = k_0$ [2]. It is easily seen that if $a \in R_0^*$ then $\Delta(a) = \infty$. We will show below that if $\Delta(a) = \infty$, then $a \in R_0^*$.

For notational convenience we will introduce the symbol $a[b]$ meaning a^b .

If $a \in R^*$, then one can easily show that $a \in R_0^*$ or $a = \sum_{i=0}^{\infty} p[i]a_i[p[n_i]]$ and for some i , $H(a_i) \in k \setminus k^p$. Let $\mathcal{J}$ denote the set of all such subscripts.

THEOREM 1. $\Delta(a) = \min_{i \in \mathcal{J}} (i + n_i)$ if $\mathcal{J} \neq \emptyset$, $\Delta(a) = \infty$ otherwise.

Proof. Let $a \in R^*$ and decompose

$$a = \sum_{i \in \mathcal{J}} p[i]a_i[p[n_i]] + \sum_{j \notin \mathcal{J}} p[j]a_j.$$

Let $n = \min_{i \in \mathcal{J}} (i + n_i) = N(a)$. Let $i_0 < i_1 < \dots < i_q$ be the collection of indices such that $i_0 + n_{i_0} = \dots = i_q + n_{i_q} = n$. Let $a' = \sum_{v=0}^q p[i_v]a[p[n_{i_v}]]$. We have $a = a' + a''$, where $N(a') = N(a)$ and $N(a'') \geq N(a) + 1$. Thus if $\Delta(a) = N(a)$, then $\Delta(a') = N(a')$ and conversely. Hence, it suffices to show that $\Delta(a') = N(a')$. Now $a' = p[i_0]a^*$, where a^* is a unit. Also, $H^*(a^*) = \gamma_{i_0}[p[n_{i_0}]]$, where $H^*(a_{i_0}) = \gamma_{i_0} \in k \setminus k^p$. Let $\{\gamma_{i_0}, m\}$ be a p -basis for k . Then $\{\gamma_{i_0}[p[n_{i_0}]], m\}$ is a p -basis for $k^* = [p[n_{i_0}]](m)$ and $[k: k^*] = p[n_{i_0}] = p[n_{i_0}]$ with $k = k^*(\gamma_{i_0})$, where the minimum polynomial of γ_{i_0} is $X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]]$. Let m^* be a set of representatives in R^* of m . Let K_m^* be the complete closure of $K_0^*(m^*)$, where K_0^* is the unique subfield of K^* having residue field k_0 . Let R_m^* be its ring of integers and construct $R' = R_m^*[a^*, R^*[p[n_{i_0}]]]$. Then R' is an integral domain and $H^*(R') = k^*$.

Let K' be the complete closure of the quotient field of R' , then K' is the quotient field of R' , since K^* is complete and unramified. Now let c be any representative for γ_{i_0} . Then $K^* = K'(c)$ and $[K^*:K'] = p[n_{i_0}]$, since K^* is unramified. Thus the minimum polynomial of c is of the form

$$(A) \quad X[p[n_{i_0}]] + \alpha_1 X[p[n_{i_0}] - 1] + \dots = f(X).$$

Now $f(c) = 0$ implies $H^*(f(c)) = f^{H^*}(\gamma_{i_0}) = 0$, where

$$(B) \quad X[p[n_{i_0}]] + H^*(\alpha_1)X[p[n_{i_0}] - 1] + \dots = f^{H^*}(X).$$

Thus $X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]]|f^{H^*}(X)$, but since they have the same degree, $f^{H^*}(X) = X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]]$. Hence

$$(C) \quad \begin{aligned} (i) \quad & H^*(\alpha_i) \in (p) \quad \text{for } 1 \leq i \leq p[n_{i_0}] - 1, \\ (ii) \quad & H^*(f(0)) = \gamma_{i_0}[p[n_{i_0}]]. \end{aligned}$$

Now $\{a^*, m^*\}$ is a collection of representatives of a p -basis of k^* . Thus, as Heerema has shown [1], we determine a unique integral derivation $D: K' \rightarrow K'$ by letting

$$(D) \quad \begin{aligned} (i) \quad & D(a) = 0, \quad a \in K_0^*, \quad a \in m^*, \\ (ii) \quad & D(a^*) = p[n_{i_0}]\mu, \quad u \text{ some unit in } R'. \end{aligned}$$

Now consider the extension of D to K^* . By the construction of the fields and a straightforward argument on the valuation, $R^* = R'[c]$. Hence, if we can show that $V(D(c)) \geq 0$, then $D: K^* \rightarrow K^*$ is an integral derivation. We know that $D(c)$ is uniquely determined and $D(c) = -f^\nu(c)/f'(c)$. We observe that $a \in R'$ implies $V(D(a)) \geq n_{i_0}$. Thus $V(D(\alpha_i)) \geq 1 + n_{i_0}$ for all i , $1 \leq i \leq p[n_{i_0}] - 1$ by condition (C), (i). Next observe that $f(0) = a^* + p \cdot v$ and $D(f(0)) = D(a^*) + pD(v) = p[n_{i_0}]\mu + p \cdot p[n_{i_0}]v^* = p[n_{i_0}](u + pv^*) = p[n_{i_0}]\mu^*$, where u^* is a unit. Thus $V(f^\nu(c)) = n_{i_0}$.

Observe that $f'(c)$ has degree $\leq p[n_{i_0}] - 1$. Thus from the fact that $R^* = R'[c]$, $V(f'(c))$ is equal to the minimum of the values of the coefficients. Now the coefficient of $c[p[n_{i_0}] - 1]$ in $f'(c)$ is $p[n_{i_0}]$. Thus the minimum value of the coefficients is $\leq n_{i_0}$. Thus $V(f'(c)) \leq n_{i_0}$. Hence $V(D(c)) \geq 0$. Thus $D: K^* \rightarrow K^*$ is an integral derivation. Now $D(a') = D(p[i_0]a^*) = p[i_0]D(a^*) = p[i_0 + n_{i_0}]\mu$ and $V(D(a')) = i_0 + n_{i_0} = n$. Since $V(D^*(a')) \geq n$ for all D^* , it follows that $\Delta(a') = N(a') = n$ and thus $\Delta(a) = N(a) = n$ and the theorem follows.

3. $\bar{p}$ -adic fields and property (H). In this section we shall be concerned with $\bar{p}$ -adic fields K , with ring of integers R , maximal ideal (π) , residue field $k = R/(\pi)$ and natural place $H: K \rightarrow \{k, \infty\}$. If V is the valuation on K and if $V(\pi) = 1$, $V(p) = e$, then $K = K^*(\pi)$, where K^* is a p -adic field in the restricted valuation $V^* = V/K^*$, $[K:K^*] = e$ and π is the root of an Eisen-

stein polynomial $f(X) = X^e + p \sum_{i=0}^{e-1} f_i X^i$, $f_i \in R^* = R \cap K^*$ and $f_0 \in R^* \setminus (p)$, $(p) = R^* \cap pR$ [3].

It is also true that if $H^* = H/K^*$, then $H^*(R^*) = k$. Throughout the discussion we will assume that K is given and that K^* and π have been fixed once chosen.

Let $\Delta_{K|K^*} = \min\{(\Delta(f_i) + 1)e + i\} - V(f'(\pi))$, where $f'(X)$ is the derivative with respect to X of $f(X)$. Notice that since $0 \leq i \leq e-1$, then by the properties of V , $\min\{(\Delta(f_i) + 1)e + i\}$ is uniquely determined and equal to $(\Delta(f_{i_0}) + 1)e + i_0$ for some index i_0 , $0 \leq i_0 \leq e-1$. Hence $\Delta_{K|K^*} = (\Delta(f_{i_0}) + 1)e + i_0 - V(f'(\pi))$ for some index i_0 . Since $V(f'(\pi))$ is fixed, $\Delta_{K|K^*}$ depends only on the coefficients of $f(X)$ once K^* and π have been chosen.

Let an integral derivation D on K be an integral_n derivation if $D(\pi) \in (\pi)^n$.

THEOREM 2. *K has the property that every integral derivation is an integral_n derivation, $n \geq 1$, if and only if $\Delta_{K|K^*} \geq n$. If $n \geq 1$, then it is also true that for any such D , $D((\pi)) \subset (\pi)$.*

Proof. Suppose $\Delta_{K|K^*} = n \geq 0$. Let D^* be an integral derivation on K^* . Then D^* has a unique extension $D: K \rightarrow K$ which is completely determined by $D(\pi)$. Now $D(\pi) = -f''(\pi)/f'(\pi)$. Hence $V(D(\pi)) = V(f''(\pi)) - V(f'(\pi)) \geq \Delta_{K|K^*} = n \geq 0$. Thus $D(\pi) \in (\pi)^n \subset R$.

Since $a \in R$ implies $a = g_a(\pi)$, where $g_a(X) \in R^*[X]$, it follows that $D(a) = g_a^{D'}(\pi) + g'_a(\pi)D(\pi) \in R$. Hence $D(R) \subset R$.

Notice that $n \geq 1$ implies $D(\pi d) = \pi D(a) + aD(\pi) \in (\pi)$ if $a \in R$, hence $D((\pi)) \subset (\pi)$.

Now suppose that D is an integral derivation on K , then $a = g_a(\pi)$ implies $D(a) = g_a^{D'}(\pi) + g'_a(\pi)D(\pi)$. In particular, if $a \in R^*$ then $D(a) = g_a^{D'}(\pi) = g_{a,D}(\pi)$. Thus $D = D|K^*$ is given by $D'(a) = g_{a,D'}(\pi)$.

Consider $g_{a,D}(X)$. Since $[K:K^*] = e$, we may choose $g_{a,D}(X)$ of degree at most $e-1$. Hence, if we do this, then

$$g_{a,D}(X) + g_{b,D'}(X) = g_{a+b,D'}(X)$$

and

$$ag_{b,D}(X) + bg_{a,D}(X) = g_{ab,D}(X)$$

for all $a, b \in R^*$.

Thus we may write $D' = \sum_{i=0}^{e-1} \pi^i D_i^*$, where $D_i^*(a)$ is the coefficient of X^i in $g_{a,D'}(X)$, and so $D_i^*: K^* \rightarrow K^*$ is an integral derivation on K^* . The fact that the polynomials $g_{a,D'}(X)$ are uniquely determined implies that the representation $D' = \sum_{i=0}^{e-1} \pi^i D_i^*$ is unique.

Since $\Delta_{K|K^*} = n \geq 0$, each derivation D_i^* has a unique extension D_i to K such that $D_i(R) \subset R$, $D_i(\pi) \in (\pi)^n$ and $D_i((\pi)) \subset (\pi)$ if $n \geq 1$. Hence since

D' has unique extension D to K , it follows that $D = \sum_{i=0}^{e-1} \pi^i D_i$. Thus $D(\pi) \in (\pi)^n$ and $D((\pi)) \subset (\pi)$ if $n \geq 1$.

Conversely, suppose that every integral derivation D on K is an integral $_n$ derivation, $n \geq 1$. Suppose also that $\Delta_{K|K^*} = m < n$. Suppose $\Delta_{K|K^*} = (\Delta(f_{i_0}) + 1)e + i_0 - V(f'(\pi))$ and suppose that D^* is an integral derivation on K^* such that $V^*(D^*(f_{i_0})) = \Delta(f_{i_0})e$. Then if we extend D^* to a derivation D on K , $V(D(\pi)) = m < n$.

If $m \geq 0$, then $D(R) \subset R$ and hence $V(D(\pi)) = n > m$, a contradiction. If $m < 0$, then $\pi^{-m}D(R) \subset R$ and hence $V(\pi^{-m}D(\pi)) = 0 \geq n \geq 1$, a contradiction. Thus $m \geq n$ and the theorem follows.

COROLLARY. Every integral derivation D^* on K^* has an integral extension D to K if and only if $\Delta_{K|K^*} \geq 0$.

THEOREM 3. K has property (H) if and only if $\Delta_{K|K^*} \geq 1$.

Proof. Suppose $\Delta_{K|K^*} \geq 1$. Let $d: k \rightarrow k$ be any derivation.

Then there is an integral derivation D^* on K^* such that $a \in R^*$ implies $H^*(D^*(a)) = d(H^*(a)) = d(H(a))$. Since $\Delta_{K|K^*} \geq 1$, then D^* has a unique extension D to K which is at least an integral $_1$ derivation. Hence D induces a derivation on k . Since $a \in R^*$ implies $D(a) = D^*(a)$, it follows that D induces $d: k \rightarrow k$. Hence K has property (H).

Conversely, suppose that $\Delta_{K|K^*} = m \leq 0$. Then there is an integral derivation D^* on K^* such that if D is the unique extension of D^* to K , then $V(D(\pi)) = m \leq 0$. If D_0^* is any derivation such that D^* and D_0^* induce the same derivation on k , then $D_0^* = D^* + pD^*$ and thus if D_0 is the unique extension of D_0^* to K , then $V(D_0(\pi)) = V(D(\pi)) = m$. Now suppose $d: k \rightarrow k$ is induced by $D^*: K^* \rightarrow K^*$ and suppose that $D: K \rightarrow K$ also induces $d: k \rightarrow k$. Then if $D' = D|K^*$, $D' = \sum_{i=0}^{e-1} \pi^i D_i^*$, where $D_i^*(R^*) \subset R^*$. For $a \in R^*$, $H^*(D^*(a)) = H(D(a)) = H(D'(a)) = H^*(D_0^*(a))$ and thus D_0^* induces $d: k \rightarrow k$.

Now suppose D_i is the unique extension of D_i^* to K , then $V(\pi^i D_i(\pi)) \geq m + i > V(D_0(\pi))$ and thus $V(\sum_{i=0}^{e-1} \pi^i D_i(\pi)) = V(D_0(\pi)) = m \leq 0$. But

$$D = \sum_{i=0}^{e-1} \pi^i D_i$$

and thus $V(D(\pi)) = V(D_0(\pi)) = m \leq 0$. However, D is an inducing derivation and so $V(D(\pi)) \geq 1$. Hence $d: k \rightarrow k$ is not induced by any $D: K \rightarrow K$ and K does not have property (H).

COROLLARY 1. K has property (H) if and only if every integral derivation D on K is an inducing derivation.

COROLLARY 2. If K is a $\bar{p}$ -adic field of ramification e and if $(e, p) = 1$, then K has property (H).

4. Automorphisms on $\overline{p}$ -adic fields. In this section we shall be concerned with establishing a connection between derivations and inertial automorphisms on $\overline{p}$ -adic fields K . Let $G = \{T \mid T \text{ is an automorphism on } K\}$; for $n \geq 1$, let $G_n = \{T \in G \mid T(a) - a \in (\pi)^n \text{ for all } a \in R\}$;

$$\overline{G}_n = \{T \in G_n \mid T(\pi) - \pi \in (\pi)^{n+1}\}.$$

As in §2, $a[b]$ means a^b .

LEMMA 1. *Let $n \geq 1$ and let $T \in G_n$. Then if $Z = T - I$, $Z[q]: R \rightarrow (\pi)[q(n-1)+1]$.*

Proof. $Z(ab) = T(ab) - ab = aZ(b) + bZ(a) + Z(a)Z(b)$. Thus, in particular, $Z(\pi[m]) = Z(\pi \cdot \pi[m-1]) = \pi Z(\pi[m-1]) + \pi[m-1]Z(\pi) + Z(\pi[m-1])Z(\pi)$. Hence $Z(\pi[2]) \in (\pi)[n+1]$ and by induction, $Z(\pi[m]) \in (\pi)[m+n-1]$.

Thus $a \in R$ implies

$$Z(a) = \pi[n]a_1,$$

$$Z[2](a) = Z(\pi[n]a_1) = a_1Z(\pi[n]) + \pi[n]Z(a_1) + Z(\pi[n])Z(a_1) \in \pi[2n-1].$$

Hence by induction $Z[q](a) \in (\pi)[qn - (q-1)]$ and the lemma follows.

COROLLARY. *If $n \geq 1$ and $T \in \overline{G}_n$, then $Z[q]: R \rightarrow (\pi)[qn]$.*

Proof. Since $Z(\pi) \in (\pi)[n+1]$, it follows that $Z(\pi[m]) \in (\pi)[m+n]$. Hence $Z[2](a) \in \pi[2n]$ and by induction $Z[q](a) \in (\pi)[qn]$.

Now suppose that $n \geq (e+p)/(p-1)$, $q \geq 2$. Then if $q = p[s]t$, it follows that $p[s]t(n-1) + 1 - se \geq n+1$. Since $q \in (\pi)[se]$, it follows that $Z[q]/q: R \rightarrow \pi[n+1]$ for all $q \geq 2$.

LEMMA 2. *Suppose $1 \leq i \leq p[\mu]$, then*

$$i \binom{p[\mu]}{i} \in p[\mu]R.$$

Proof. Consider

$$i \binom{p[\mu]}{i} = 1/(i-1)! \{p[\mu](p[\mu]-1) \cdots (p[\mu]-(i-1))\}.$$

In the expansion we obtain terms of the form

$$p[\mu] \binom{p[\mu]}{i_1} \cdots \binom{p[\mu]}{i_l},$$

$l \geq 0$, $1 \leq i_1 < i_2 < \cdots < i_l \leq i-1$. Since $(p[\mu]/i_v) \in R$, the lemma follows.

THEOREM 4. *Let p be an odd prime and suppose $n \geq (e+1)/(p-1)$. Suppose $T \in G_n$, or, if $n = 1$, $T \in \overline{G}_1$. Suppose $T = I + \pi[n]T'$. Then there*

is a derivation $D(T)_n$ on K such that $T' - D(T)_n: R \rightarrow (\pi)$. Hence $T \in \overline{G}_n$ if and only if $D(T)_n$ is an inducing derivation.

Proof. Since $T = I + Z$, it follows that

$$(1) \quad \begin{aligned} & T[p[\mu + 1]] - T[p[\mu]] \\ &= \sum_{i=1}^{p[\mu]} \left(\binom{p[\mu + 1]}{i} - \binom{p[\mu]}{i} \right) Z[i] + \sum_{i=p[\mu]+1}^{p[\mu+1]} \binom{p[\mu + 1]}{i} Z[i]. \end{aligned}$$

If $i \geq 2$, then by Lemma 2 and the fact that $Z[i]/i: R \rightarrow \pi[n + 1]$ we get

$$(2) \quad \begin{aligned} & T[p[\mu + 1]] - T[p[\mu]] = p[\mu](p - 1)Z + p[\mu]Z^*, \text{ where} \\ & Z_\mu^*: R \rightarrow (\pi)[n + 1]. \end{aligned}$$

Let

$$\begin{aligned} T_\mu &= p[-\mu](T[p[\mu + 1]] - T[p[\mu]]) \\ &= (p - 1)Z + Z_\mu^* = -Z + pZ + Z_\mu^*. \end{aligned}$$

Now consider $T_{\mu+1} - T_\mu$. This map is given by

$$(3) \quad \begin{aligned} & \sum_{i=1}^{p[\mu]} g_i p[-\mu - 1] Z[i] + \sum_{i=p[\mu]+1}^{p[\mu+1]} g'_i p[-\mu - 1] Z[i], \\ & + \sum_{i=p[\mu+1]+1}^{p[\mu+2]} \binom{p[\mu + 2]}{i} p[-\mu - 1] Z[i], \end{aligned}$$

where

$$g_i = \left(\binom{p[\mu + 2]}{i} - \binom{p[\mu + 1]}{i} (1 + p) + \binom{p[\mu]}{i} p \right), \quad 1 \leq i \leq p[\mu],$$

$$g'_i = \left(\binom{p[\mu + 2]}{i} - \binom{p[\mu + 1]}{i} (1 + p) \right), \quad p[\mu] + 1 \leq i \leq p[\mu + 1].$$

Note that $g_1 = 0$. For $i \geq 2$, we get

$$(4) \quad \begin{aligned} & ig_i = 1/(i - 1)! \{ p[\mu + 1]((p[\mu + 2] - 1) \cdots (p[\mu + 2] - (i - 1))) \\ & \quad - (p[\mu + 1] - 1) \cdots (p[\mu + 1] - (i - 1))) \\ & \quad - p[\mu + 1]((p[\mu + 1] - 1) \cdots (p[\mu + 1] - (i - 1))) \\ & \quad - (p[\mu] - 1) \cdots (p[\mu] - (i - 1))) \}. \end{aligned}$$

Now suppose we pick μ such that $p[\mu/3] \geq 2(\mu + 1)e$. Then $i \geq p[\mu/3]$ implies $Z[i]: R \rightarrow p[2(\mu + 1)]R$. Also, if $1 \leq i < p[\mu/3]$, then by Lemma 2, and the fact that on the right side of (4) the terms involving $(i - 1)!$ are cancelled, it follows that $p[-\mu - 1]g_i Z[i]: R \rightarrow p([\mu] - [\mu/3] + 1)R$, where the inner

$[\mu/3]$ denotes the greatest integer function. Hence

$$T_{\mu+1} - T_\mu: R \rightarrow p([\mu] - [[\mu/3] + 1])R$$

and

$$\lim_{\mu \rightarrow \infty} T_\mu = T_{\mu_0} + \sum_{i=\mu_0}^{\infty} (T_{i+1} - T_i)$$

is a well-defined map.

Since

$$T[p[\mu]] - I = \sum_{i=1}^{p[\mu]} \binom{p[\mu]}{i} Z[i],$$

then by Lemma 2, $T[p[\mu]] - I: R \rightarrow p[\mu]R$. Thus since

$$\begin{aligned} T_\mu(ab) - aT_\mu(b) - bT_\mu(a) &= p[-\mu] \{ (T[p[\mu+1]](a) - a)(T[p[\mu+1]](b) - b) \\ &\quad - (T[p[\mu]](a) - a)(T[p[\mu]](b) - b) \} \quad (p \neq 2) \end{aligned}$$

it follows that for μ large enough T_μ is a derivation modulo $p[\mu]$. Hence $\lim_{\mu \rightarrow \infty} T_\mu$ is a derivation.

Now $\lim_{\mu \rightarrow \infty} T_\mu = -Z + pZ + \lim_{\mu \rightarrow \infty} Z_\mu^*$. Thus if we let $D(T)_n = -\pi[-n] \cdot \lim_{\mu \rightarrow \infty} T_\mu$, then $D(T)_n = +T' - pT' - \pi[-n] \lim_{\mu \rightarrow \infty} Z_\mu^*$ and $T' - D(T)_n: R \rightarrow (\pi)$, since $Z_\mu^*: R \rightarrow (\pi)[n+1]$. Hence the theorem follows.

THEOREM 5. Suppose $n \geq (e+1)/(p-1)$ and suppose $D: K \rightarrow K$ is an integral derivation on K . Then $D_n = I + \sum_{i=0}^{\infty} \pi[ni]/i! D[i]$ is an automorphism on K , $D_n \in G_n$ and $D_n \in \overline{G}_n$ if and only if $D(\pi) \in (\pi)$.

Proof. Observe that $V(i!) < ie/(p-1)$. Thus $ni - V(i!) \geq ni - ie/(p-1) \geq (e+1)i/(p-1) - ie/(p-1)$ and $\lim_{i \rightarrow \infty} V(\pi[ni]/i!) = \infty$. Thus D_n is a well-defined map, $D_n(R) \subset R$. D_n is additive since $D[i]$ is additive. Also, by a straightforward computation $D_n(ab) = D_n(a) \cdot D_n(b)$. Since $D_n(1) = 1$, it follows that $D_n \in G$.

By a straightforward computation $i \geq 2$ implies $V(\pi[ni]/i!) \geq n+1$, if $n \geq (e+1)/(p-1)$. Hence $D_n = I + \pi[n](D + \pi D_n^*)$, where $D_n^*(R) \subset R$. Hence $D_n \in G_n$ and $D_n \in \overline{G}_n$ if and only if $D(\pi) \in (\pi)$.

THEOREM 6. Let $\mathcal{D}$ be the additive group of integral derivations on K and let $\overline{\mathcal{D}}$ be the additive group of derivations on the residue field of K which are induced, then if p is odd and $n \geq (e+p)/(p-1)$, G_n/G_{n+1} is isomorphic to $\mathcal{D}/\pi\mathcal{D}$ and if $n \geq (e+p)/(p-1)$, $\overline{G}_n/\overline{G}_{n+1}$ is isomorphic to $\overline{\mathcal{D}}$.

Proof. Define $H_n(TG_{n+1}) = D(T)_n + \pi\mathcal{D}$ and $H_n^*(TG_{n+1}) = d$, where d is induced by $D(T)_n$ if $T \in \overline{G}_n$. Theorem 4 implies H_n and H_n^* are well-defined since $D(T)_n: R \rightarrow (\pi)$ if and only if $T \in G_{n+1}$. Since $D(T_1 \cdot T_2)_n = T'_1 + T'_2$

$+ T'_1(\pi[n]T'_2)$, it follows that $D(T_1 \cdot T_2)_n \equiv D(T_1)_n + D(T_2)_n \pmod{\pi}$, i.e., H_n and H_n^* are homomorphisms. H_n and H_n^* are monomorphisms by Theorem 4. By Theorem 5, since $D(D_n) - D: R \rightarrow (\pi)$, they are epimorphisms. Thus they are isomorphisms and the theorem follows.

COROLLARY. K has property (H) if and only if $\bar{G}_n = G_n$ for all $n \geq (e+p)/(p-1)$.

Proof. Suppose $\bar{G}_n = G_n$. Then $T \in G_n$ implies $D(T)_n$ is inducing. Since $D \in \mathcal{D}$ implies $D = D(T)_n - \pi D'$ for some T and D' , every D is inducing. Thus K has property (H). If K has property (H), then every $D(T)_n$ is inducing, which implies $T \in \bar{G}_n$. Hence $G_n = \bar{G}_n$.

CONJECTURE. Theorem 6 holds for all $n \geq 1$.

Notice that if $T \in G_n$ ($n \geq (e+p)/(p-1)$), we can obtain $D(T)_n$. Thus by Theorem 5, $(D(T)_n)_n$ is an automorphism with derivation $D((D(T)_n)_n) - D(T)_n \in \pi\mathcal{D}$. Hence $T = T_1 \cdot ((D(T)_n)_n)$, where $T_1 \in G_{n+1}$. Proceeding in this fashion we obtain

$$T = T_j((D(T_{j-1})_{n+j-1})_{n+j-1}) \cdots ((D(T)_n)_n), \quad T_j \in G_{n+j}$$

and since $\bigcap G_n = I$, we obtain

$$T = \lim_{j \rightarrow \infty} ((D(T_j)_{n+j})_{n+j}) \cdots ((D(T)_n)_n),$$

i.e., T has a Taylor-series-like expansion in terms of derivations.

Theorem 6 is an extension of a well-known result of MacLane [2], to $\bar{p}$ -adic fields.

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