ON THE DEGREE OF CONVERGENCE OF EXTREMAL POLYNOMIALS AND OTHER EXTREMAL FUNCTIONS(1)

BY

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1. **Introduction.** This paper is motivated by the following problem [19, §11.3]. Let a given function $F(z)$ be of class L^p, $p > 1$, on an analytic Jordan curve γ in the plane of the complex variable z, and let $p_n(z)$ be the unique sequence of polynomials in z of respective degrees n of best approximation to $F(z)$ on γ in the sense of minimizing $\int_\gamma |F(z) - p_n(z)|^p |dz|$; these minimizing $p_n(z)$ may also be subjected to certain auxiliary conditions of interpolation $p_n(w_k) = u_k$, $k = 1, 2, \cdots, m$, which are independent of n and are not necessarily related to $F(z)$. The object is to study convergence and degree of convergence of the sequence $p_n(z)$ to a possible limit minimizing function and to study various properties of this limit function. We attack this problem first (Part I) by studying a general situation in L^p-space, and then (Part II) by specializing to the problem already mentioned. We treat likewise the analogous problem where best approximation is measured by a surface integral over the interior C of γ.

Related problems for special cases have been studied by Bieberbach [2], Julia [7], Keldyš and Lavrentieff [9], and Smirnoff [17]. For an analytic Jordan curve approximation in the case of arbitrary m and $p = 2$ was studied by Walsh [19, §11.5], including complete determination of the limit function in closed form and (where appropriate) maximal convergence of the extremal polynomials. The corresponding problem of approximation, including maximal convergence, with $m = 1$ and p arbitrary was considered by Spitzbart [18]. Approximation by functions analytic and bounded in a region D containing $\overline{C}$ was treated by Walsh and Russell [23], especially in the case $p = 2$. The minimizing functions have been studied by Kakeya [8], Doob [4], Penez [11], Macintyre and Rogosinski [10], and Rogosinski and Shapiro [13]. F. Riesz and M. Riesz [12] studied the boundary values of an analytic function in H_p. A paper by Rosenbloom and Warschawski [14] contains in outline some material related to work in the present paper, without detailed results; the present research was commenced and completed without use of that material.

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For $p \neq 2$ we rely on suitable inequalities and make large use of the orthogonality condition (Theorem 4) which characterizes the extremal function f^*. Continuity properties of the extremal function enable us to estimate $\|p_n\| - \|f^*\|$, and inequalities depending on the uniform convexity of L^p to estimate $\|f_n^* - p_n^*\|$. Conclusions are obtained involving degree of convergence, and overconvergence, i.e., convergence in a larger region. (Compare with the simpler case, $p = 2$ [19, §11.5, Theorem 9].)

In Part I we are concerned with a closed convex subset L_* of L^p with closure by the L^p-norm topology. If $f(z)$ is analytic interior to the unit circle and if the integrals $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$, $0 < r < 1$, are bounded, $f(z)$ is said to be of class H_p and it is well known that for radial approach or approach "in angle" has boundary values almost everywhere on the unit circle and that for these boundary values the integral $\int_0^{2\pi} |f(e^{i\theta})|^p d\theta$ exists. The subclass H_p of L^p on $|z| = 1$ so determined is known to be a closed subset of L^p. For the case that γ is an analytic Jordan curve, $H_p(\gamma)$ is defined in the obvious way by a conformal mapping onto the interior of the unit circle. The closure on γ of H_p in L^p is evident.

If L^p refers to the surface integral taken with respect to area over C, a limited region of the complex plane, H_p' is defined as the subset of functions of L^p analytic on C. If $f_m, f_n \in H_p'$ and $\lim_{m,n\to\infty} \iint_C |f_n - f_m|^p dS = 0$, there exists $f \in L^p$ such that $\lim_{n\to\infty} \iint_C |f - f_n|^p dS = 0$. Then $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on any closed set interior to C [19, p. 109]. Thus H_p' is closed.

A subset L_* of L^p is called a *convex subset* of L^p provided that $g, h \in L_*$ implies $[rg + (1 - r)h] \in L^*$ for all r such that $0 \leqq r \leqq 1$. We say L_n is a *nested sequence of subsets* of L^p if $L_n \subset L_{n+1} \subset L^p$, $n = 1, 2, \cdots$. In the applications considered in this paper (Part II) L_* is the closure of $\bigcup_1^\infty L_n$.

It is assumed throughout this paper that C is a limited region of the complex plane and that integration is over E, namely, the boundary γ of C or the point set C itself. In the former case we suppose γ is an analytic Jordan curve and integrate with respect to arc length; in the latter case we integrate with respect to area. If $\int_E |f_1 - f_2|^p d\mu = 0$, f_1 and f_2 are regarded as the same function on E.

A polynomial $p_n(z)$ of degree n is a function which can be expressed in the form $a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, where a_0 may be zero. We give examples for later reference of closed convex subsets L_* of L^p and nested sequences L_n of such sets.

EXAMPLE A. For $L_* = H_p$ (or H_p') and L_n, the set of polynomials of degree n, L_n is a nested sequence of closed convex subsets of the closed convex subset L_*.

EXAMPLE B. Given points $w_1, w_2, \cdots, w_m$ interior to C, let interpolation conditions $f(w_j) = u_j$, $j = 1, 2, \cdots, m$, be assigned. If the w_j are not all distinct, the appropriate u_j are to be interpreted in the usual way as values assigned to derivatives. Define $L_* = \{f \mid (1)\ f \in H_p$ (or H'_p); (2) $f(w_j) = u_j$, $j = 1, \cdots, m\}$ and $L_n = \{f \mid (1)\ f \in L_*;\ (2)\ f(z) = p_n(z)\}$. It is known that the function f_p^* of L_* which minimizes $\int_{|z|=1} |f(z)|^p\, |dz|$ is of the form

$$A \prod_{i \in S'} (z - \alpha_i)/(1 - \bar{\alpha}_i z) \prod_{i \in S} (1 - \bar{\alpha}_i z)^{2/p} \prod_{j=1}^{m} (1 - \bar{w}_j z)^{-2/p},$$

where $|\alpha_i| < 1$, S is the set of integers $1, 2, \cdots, m-1$, and S' is a certain subset of S [10, pp. 277-278].

I. Results for a general L^p-space

2. **Existence of extremal function.** It is well known that if L_* is a closed convex subset of L^p, $1 < p < \infty$, there is a unique element of L_* for which $\|g\|_p = \{\int_E |g|^p\, d\mu\}^{1/p}$ is a minimum [1, p. 103], [25, p. 129]. We assume that the measure μ of E is finite. In this paper *extremal function* of L_* denotes a function g^* which minimizes $\|g\|_p$, $g \in L_*$.

Theorem 1. *If $F \in L^p$ and L_* is a closed convex subset of L^p, then*

$$L'_* = \{g \mid g = F - f,\ f \in L_*\}$$

is also a closed convex subset of L^p. There exists a unique element f^* of L_* for which $\|F - f\|_p$ is a minimum, namely: $f^* \equiv F - g^*$, where g^* is the extremal function of L'_*.

In Theorem 2 the weight function w is understood to be positive, integrable, and bounded from zero on E.

Theorem 2. *If L_* is a closed convex subset of L^p_w (the set of functions f such that $\int_E w|f|^p d\mu$ exists) when $[\int_E w|f|^p d\mu]^{1/p}$ is taken as $\|f\|_p$, then*

$$L'_* = \{g \mid g = w^{1/p} f,\ f \in L_*\}$$

is a closed convex subset of L^p. Hence, there exists a unique element $f^* \in L_*$ for which $\|f\|_p$ is a minimum, namely: $f^* = g^*/(w)^{1/p}$, where g^* is the extremal function of L'_*.

By combining the results of Theorems 1 and 2 we obtain the following: If w is a weight function as described above and if F is an arbitrary element of L^p_w then for f in L_*, a closed convex subset of L^p_w, there is a unique f^* of L_* such that $\int w|F - f|^p d\mu$ is a minimum and $f^* = F - g^*/(w)^{1/p}$.

3. **Inequalities for norms in an arbitrary L^p-space.** As is customary,

$$\left\{ \int_E |g|^p d\mu \right\}^{1/p}$$

is abbreviated by $\|g\|_p$ or just by $\|g\|$.

LEMMA 3.1. *Suppose $p > 0$ and choose r, $0 < r < 1$. Then, for $|Z| < r < 1$, $|1+Z|^p = 1 + p \operatorname{Re} Z + R(Z)$ with $|R(Z)| < |Z|^2 A(p,r)$ with $A(p,r)$ dependent on p and r but independent of Z.*

Proof. We have

$$(1+Z)^{p/2} = 1 + (p/2)Z + M(Z) \quad \text{with } |M(Z)| < K(p,r)|Z|^2.$$

Then

$$|1+Z|^p = [1 + (p/2)Z + M(Z)][1 + (p/2)\bar{Z} + \bar{M}(Z)]$$
$$= 1 + p \operatorname{Re} Z + R(Z), \quad \text{as required.}$$

The method of proof used for Theorem 3 is similar to one used by Ahlfors [1, pp. 121-122].

THEOREM 3. *Choose r, $0 < r < 1$. Suppose $p > 1$ and $f, g \in L^p$ with integration over E. Let E_1 denote the subset of E on which $|g/f| < r$ and E_2 the subset of E on which $|g/f| \geqq r$. Then*

$$\text{(a)} \quad \int_{E_1} |f+g|^p d\mu = \int_{E_1} |f|^p d\mu + p \operatorname{Re} \int_{E_1} |f|^p (g/f)\, d\mu + \int_{E_1} R_1 d\mu,$$
$$|R_1(X)| < |f(X)|^p |g(X)/f(X)|^2 A(p,r);$$

$$\text{(b)} \quad \int_{E_2} |f+g|^p d\mu = \int_{E_2} |f|^p d\mu + p \operatorname{Re} \int_{E_2} |f|^p (g/f)\, d\mu + \int_{E_2} R_2 d\mu,$$
$$|R_2(X)| \leqq |g(X)|^p B(p,r).$$

Proof. On E_1 Lemma 3.1 implies

$$|f|^p |1 + (g/f)|^p = |f|^p [1 + p \operatorname{Re}(g/f) + R(g/f)].$$

The result (a) is immediate.

Since on E_2 we have $|g/f| \geqq r$,

$$\big||f+g|^p - |f|^p - p \operatorname{Re} |f|^p (g/f)\big| \leqq |g|^p [(1 + 1/r)^p + (1/r)^p + p(1/r)^{p-1}].$$

Inequality (b) follows directly.

A *subset* L_* of L^p with an extremal element f will be called *admissible* if for some b, which may depend on h, $0 < b \leqq \infty$, the relation $h \in L_*$ implies $f + \lambda(f-h) \in L_*$ for λ complex and $|\lambda| \leqq b$.

THEOREM 4. *Suppose f is extremal for an admissible subset L_* of L^p and let h be any element of L_*. Then*

$$\int_E |f|^p (f-h)/f d\mu = 0. \tag{4.1}$$

Proof. In Theorem 3 set $g = \lambda\phi$ with $\phi \equiv f - h$ and denote E_1 and E_2, which now depend on λ, by E_{1_λ} and E_{2_λ}. Addition of (a) and (b) yields:

$$\int_E |f + \lambda\phi|^p d\mu = \int_E |f|^p d\mu + p \operatorname{Re} \int_E |f|^p(\lambda\phi/f)\, d\mu$$

$$+ \int_{E_{1_\lambda}} R_1 d\mu + \int_{E_{2_\lambda}} R_2 d\mu, \tag{4.2}$$

$$|R_1(X)| < |\lambda|^2 |f(X)|^{p-2} |\phi(X)|^2 A(p,r), \quad |R_2(X)| \leqq |\lambda|^p |\phi(X)|^p B(p,r).$$

For $p \geqq 2$, it follows from the Hölder inequality and the fact the integral is not decreased by substituting E for E_1 that the third term on the right is bounded by $A(p,r)\|f\|_p^{p-2}|\lambda|^2\|\phi\|_p^2$. Clearly the fourth term on the right side of (4.2) is bounded by $B(p,r)|\lambda|^p\|\phi\|_p^p$. For $1 < p < 2$, the third term on the right side of (4.2) is bounded by $A(p,r)|\lambda|^2 \int_E |f|^p |\phi/f|^p d\mu$, that is, by $A(p,r)|\lambda|^2\|\phi\|_p^p$, and the fourth term by $B(p,r)|\lambda|^p\|\phi\|_p^p$.

Since f is extremal for L_* and $(f + \lambda\phi) \in L_*$ for $|\lambda| \leqq b$, we have

$$\|f + \lambda\phi\|^p \geqq \|f\|^p,$$

whence

$$0 \leqq p \operatorname{Re} \int \lambda |f|^{p-2} \bar{f} \phi\, d\mu + O(|\lambda|^S), \qquad \lambda \to 0, \tag{4.3}$$

with $S = 2$ for $p \geqq 2$, $S = p$ for $1 < p < 2$. If (4.1) is not satisfied, $\arg\lambda$ can be assigned so that the argument of the first term on the right side of (4.3) is π. For $|\lambda|$ sufficiently small the modulus of the first term exceeds that of the second; this gives a contradiction and completes the proof. For this proof, compare [5a].

THEOREM 5. *Suppose f is extremal for an admissible subset L_* of L^p and that π_n is a sequence in L_* such that $\|f - \pi_n\|_p < 1$. We conclude the following*:

(a) $\|\pi_n\|_p^p - \|f\|_p^p < M_1\|f - \pi_n\|_p^2$ *for* $p \geqq 2$.

(b) *If f is bounded on E and if $|f(X) - \pi_n(X)| < 1$ almost everywhere on E, then*

$$\|\pi_n\|_p^p - \|f\|_p^p < M_2\|f - \pi_n\|_2^2 \quad \text{for } p \geqq 2;$$

if $|f|^{p-2}$ is bounded on E and if $|(f(X) - \pi_n(X))/f(X)| < r < 1$ almost everywhere on E, then

$$\|\pi_n\|_p^p - \|f\|_p^p < M_3\|f - \pi_n\|_2^2 \quad \text{for } 1 < p < 2.$$

(c) *If $|(f(X) - \pi_n(X))/f(X)|$ is bounded on E, then*

$$\|\pi_n\|_p^p - \|f\|_p^p \leqq M_4\|f - \pi_n\|_p^p \quad \text{for } 1 < p < 2.$$

(d) *If* $|(f(X) - \pi_n(X))/f(X)| < r < 1$ *almost everywhere on* E, *then*

$$\|\pi_n\|_p^p - \|f\|_p^p \leqq M_5 \int_E |f|^{p-2}|f - \pi_n|^2 d\mu \quad \textit{for } p > 1.$$

Proof. Identify the present f and $(\pi_n - f)$ with f and g of Theorem 3. The sets E_1 and E_2, now denoted by E_{1n} and E_{2n}, depend on n. By combining (a) and (b) of Theorem 3 and applying Theorem 4, we have

$$\|\pi_n\|_p^p - \|f\|_p^p \leqq A(p,r) \int_{E_{1n}} |f|^{p-2}|f - \pi_n|^2 d\mu + B(p,r) \int_{E_{2n}} |f - \pi_n|^p d\mu. \tag{5.1}$$

If $p \geqq 2$, by applying Hölder's inequality to the integral in the first term of the second member of (5.1) and noting that

$$\int_{E_{2n}} |f - \pi_n|^p d\mu \leqq \left[\int_{E_{2n}} |f - \pi_n|^p d\mu \right]^{2/p},$$

we deduce that the right side of (5.1) is less than or equal to

$$\max \left[A(p,r) \|f\|_p^{p-2}; B(p,r) \right] \cdot \|f - \pi_n\|_p^2,$$

which completes the proof of (a).

In (5.1) we factor $\sup|f|^{p-2}$ from the first integral and, in case $|f - \pi_n| < 1$ and $p \geqq 2$, substitute $|f - \pi_n|^2$ for $|f - \pi_n|^p$ in the second integral on the right without changing the sense of the inequality or the constant factors, thus obtaining (b) for $p \geqq 2$. If $1 < p < 2$ and $|(f - \pi)/f| < r < 1$ on E, since E_{2n} is the null set, the result (b) is obtained from (5.1) by factoring $\sup|f|^{p-2}$ from the first integral on the right.

Noting $|f|^{p-2}|f - \pi_n|^2 = |(f - \pi_n)/f|^{2-p}|f - \pi_n|^p$ and substituting into (5.1) in the case $1 < p < 2$, we obtain (c). In case (d), E_{2n} is the null set.

COROLLARY 5.1. *In the inequalities of Theorem* 5, *if* $\|f\| \neq 0$, $\|\pi_n\| - \|f\|$ *may be substituted for* $\|\pi_n\|^p - \|f\|^p$.

Proof. If $\|\pi_n\| = \|f\|$, the result is trivial. Otherwise, the law of the mean applied to the function X^p implies there exists ξ_n, $\|f\| < \xi_n < \|\pi_n\|$, so that

$$\|\pi_n\|^p - \|f\|^p = [\|\pi_n\| - \|f\|]p\xi_n^{p-1} > [\|\pi_n\| - \|f\|]p\|f\|^{p-1}.$$

THEOREM 6. *Suppose* f *is extremal for a convex subset* L_* *of* L^p *and suppose* π_n *is a sequence in* L_* *such that* $\lim_{n\to\infty}\|\pi_n\| = \|f\| \neq 0$. *Then for any* ψ_n $(\in L_*)$ *such that* $\|\psi_n\| \leqq \|\pi_n\|$,

$$\|f - \psi_n\| = O[(\|\pi_n\| - \|f\|)^{1/r}], \qquad n \to \infty, \tag{6.1}$$

with $r = p$ *for* $p \geqq 2$ *and* $r = 2$ *for* $1 < p < 2$. *For* $p \geqq 2$, *if* L_* *is admissible and if* $\|f - \pi_n\| = O(\gamma_n)$ *with* $\lim_{n\to\infty}\gamma_n = 0$, *then*

$$\|f - \psi_n\| = O(\gamma_n^{2/p}), \qquad n \to \infty. \tag{6.2}$$

Proof. We apply an inequality on uniformly convex spaces proved by Hanner [6] for $p > 1$. In the case $p \geqq 2$ a result of Clarkson [3] could be used. Since $\|\psi_n\| - \|f\| \leqq \|\pi_n\| - \|f\|$, it is sufficient to show

$$\|f - \psi_n\| = O[(\|\psi_n\| - \|f\|)^{1/r}].$$

Let $d = \|f\|$ and $\eta_n = \|\psi_n\| - \|f\|$. Then $\|f/d\| = 1$ and $\|\psi_n/(d+\eta_n)\| = 1$. Now

$$\begin{aligned}\|f/2d + \psi_n/2(d+\eta_n)\| &\geqq \|(f+\psi_n)/2d\| - \|\psi_n\|[(1/2d) - 1/2(d+\eta_n)]\\ &\geqq 1 - \eta_n/2d.\end{aligned}$$

By the theorem of Hanner $\|\psi_n/(d+\eta_n) - f/d\| \leqq \epsilon_n$, with $\epsilon_n = O(\eta_n^{1/r})$, $n \to \infty$, with $r = 2$ for $1 < p < 2$ and $r = p$ for $p \geqq 2$. Now

$$\begin{aligned}\|\psi_n - f\| &\leqq d[\|\psi_n/(d+\eta_n) - f/d\| + \|\psi_n(1/d - 1/(d+\eta_n))\|]\\ &\leqq d[\epsilon_n + \eta_n/d] = O(\eta_n^{1/r}).\end{aligned}$$

Combining Corollary 5.1, as applied to (a) of Theorem 5 and the result of the present theorem, we obtain (6.2).

For $2 \leqq p < \infty$, if f is extremal for L_*, a convex subset of L^p, and h is an arbitrary element of L_*, it can be proved easily from an inequality of Clarkson [3] that $\|h - f\|^p \leqq 2^{p-1}[\|h\|^p - \|f\|^p]$. For by Clarkson's inequality $\|h+f\|^p + \|h - f\|^p \leqq 2^{p-1}[\|h\|^p + \|f\|^p]$. Since $\|(h+f)/2\| \geqq \|f\|$, we have $-\|h+f\|^p \leqq -2^p\|f\|^p$. Thus, $\|h - f\|^p \leqq 2^{p-1}[\|h\|^p - \|f\|^p]$. For $1 < p < 2$ there does not exist M such that $\|h - f\|^p \leqq M[\|h\|^p - \|f\|^p]$. For if this inequality held it could be combined with the conclusion in Theorem 5, $\|\pi_n\|^p - \|f\|^p < M_3\|f - \pi_n\|_2^2$, to obtain a degree of convergence stronger than maximal convergence in the case f is analytic and bounded from zero on γ.

When $\{\epsilon_n\}$ is used henceforth in this paper, it is to be understood that for n sufficiently large $0 < \epsilon_{n+1} \leqq \epsilon_n < 1$ and $\lim_{n\to\infty} \epsilon_n = 0$.

THEOREM 7. *Suppose f, such that $\|f\| \neq 0$, is extremal for an admissible convex subset L_* of L^p, $p > 1$. Suppose, for given $\{\epsilon_n\}$, there exists π_n $(\in L_*)$ such that $\|f - \pi_n\|_p < 1$ and such that for some M*

$$|[f(X) - \pi_n(X)]/f(X)| < M\epsilon_n \quad \textit{when } X \in (E - E_0), \tag{7.1}$$

where E_0 is a set of measure zero and is independent of n. If $\psi_n \in L_*$ and $\|\psi_n\| \leqq \|\pi_n\|$, then $\|f - \psi_n\| = O(\epsilon_n^{2/r})$, $n \to \infty$, with $r = p$ for $p \geqq 2$ but $r = 2$ for $1 < p < 2$.

Proof. Corollary 5.1, applied to (d) of Theorem 5, followed by (7.1) and Theorem 6, yields the result.

THEOREM 7′. *Suppose f is extremal for an admissible convex subset L_* of L_p, $p > 1$, and that for given $\{\epsilon_n\}$ there exists π_n $(\in L_*)$ such that*

$$\|f - \pi_n\|_p < 1, \tag{7.1'}$$

$$\|(f - \pi_n)/f\|_2 = O(\epsilon_n), \tag{7.2'}$$

$$|(f - \pi_n)/f| < r < 1 \quad \text{on } E. \tag{7.3'}$$

Then, if $\|f\|_p \neq 0$ and if f is bounded on E, the conclusion of Theorem 7 holds.

THEOREM 8. *Suppose f with $\|f\| \neq 0$ is extremal for an admissible convex subset L_* of L^p, $p \geqq 2$, and that, for a given $\{\epsilon_n\}$, there exists f_n $(\in L_*)$ such that $\|f - f_n\|_p < 1$, and that either*

$$\|f - f_n\|_p = O(\epsilon_n), \qquad n \to \infty, \tag{1}$$

or

(2) *f is bounded on E, $|f_n(X) - f(X)| < 1$ almost everywhere on E, and*

$$\|f - f_n\|_2 = O(\epsilon_n), \qquad n \to \infty.$$

If ψ_n $(\in L_*)$ is defined so that $\|\psi_n\|_p \leqq \|f_n\|_p$, then

$$\|f - \psi_n\|_p = O(\epsilon_n^{2/p}), \qquad n \to \infty.$$

Proof. Under the hypotheses (1) or (2), Corollary 5.1 as applied to (a) or (b) of Theorem 5 implies $\|f_n\| - \|f\| = O(\epsilon_n^2)$. Theorem 6 now gives the required result.

THEOREM 9. *Suppose f is extremal for an admissible subset L_* of L^p, $p \geqq 2$. For given $\{\epsilon_n\}$ suppose $\pi_n (\in L_*)$ defined so that $\|f - \pi_n\|_p = O(\epsilon_n)$, $n \to \infty$. Let ψ_n be any sequence in L_* such that $\|\psi_n\|_p \leqq \|\pi_n\|_p$. Then if $[|f|^{p-2}]^{-\beta}$ is integrable for some β (> 0), we have*

$$\int_E |f - \psi_n|^{2(1-a)} \, d\mu = O(\epsilon_n^{2(1-a)}) \quad \text{for } a = 1/(1+\beta).$$

Proof. We first derive (9.2), which holds for $p > 1$ if f is extremal for an admissible subset L_* of L^p, and (9.3). Finally (9.5), (9.3), and (9.4) are combined to complete the proof.

We begin with the identity

$$\int |f|^{p-2}|f - g|^2 d\mu = \int |f|^{p-2}\bar{f}(f - g)\, d\mu - \int |f|^{p-2} f\bar{g}\, d\mu + \int |f|^{p-2}|g|^2 d\mu, \tag{9.1}$$

which holds if $g \in L_*$. Theorem 4 implies $\int |f|^{p-2}\bar{f}(f - g)\, d\mu = 0$, whence since $\int |f|^p d\mu$ is real,

$$\int |f|^{p-2}f\bar{g}\,d\mu = \int |f|^{p-2}\bar{f}g\,d\mu = \int |f|^p d\mu.$$

Substitution of these results into (9.1) yields

$$\int |f|^{p-2}|f-g|^2 d\mu = \int |f|^{p-2}[|g|^2 - |f|^2]d\mu. \tag{9.2}$$

Hölder's inequality implies

$$\int |f|^{p-2}|g|^2 d\mu \leqq \left\{\int |f|^p d\mu\right\}^{(p-2)/p}\left\{\int |g|^p d\mu\right\}^{2/p}.$$

Since f is extremal, the right member is not greater than $\int |g|^p d\mu$. This result combined with (9.2) implies that

$$\int |f|^{p-2}|f-g|^2 d\mu \leqq \int |g|^p d\mu - \int |f|^p d\mu. \tag{9.3}$$

We note now that if $p > 0$ and $r > 0$, if $|f|^{p-2}|f-g|^r$ is integrable and if, for some $\beta\,(>0)$, also $[|f|^{p-2}]^{-\beta}$ is integrable, then

$$\int |f-g|^{r(1-a)} d\mu \leqq \left[\int d\mu/\{|f|^{p-2}\}^{(1-a)/a}\right]^a \cdot \left[\int |f|^{p-2}|f-g|^r d\mu\right]^{1-a}, \tag{9.4}$$

where $a = 1/(1+\beta)$, a result obtained by using $|f|^{p-2}$ as the weight function in an inequality previously proved [19, p. 105]. (The method of proof in which Hölder's inequality is applied holds for an arbitrary L^p-space.)

Now by hypothesis, $\|\pi_n\| \geqq \|\psi_n\| \geqq \|f\|$. Then

$$\|\psi_n\|_p^p - \|f\|_p^p = O(\epsilon_n^2) \tag{9.5}$$

is an immediate consequence of (a) of Theorem 5.

Combining (9.5) with (9.3) and (9.4) when g is replaced by ψ_n, we obtain the required result.

II. Applications to analytic functions

4. Convergence of sequences of functions extremal for line and surface integrals in the complex plane. The remainder of this paper is concerned with convergence, in particular, degree of convergence of the extremal polynomials to the corresponding extremal function in H_p on certain subsets of the complex plane. We frequently use

HYPOTHESIS H. *Let D be a closed limited Jordan region, with interior C, bounded by a Jordan curve γ which, for the line integral case, is supposed analytic. Assume integration over C for the surface integral, over γ for the line integral. Define L_* and L_n as in Example* B *of the introduction. Suppose $F \in L^p$*

defined on E, the set over which the integral is taken, and let f^* *and* f_n^* *minimize* $\|F - f\|_p$, $f \in L_*$, *and* $\|F - f_n\|_p$, $f_n \in L_n$, *respectively.*

Before proceeding to consider degree of convergence under assumption of special behavior of the extremal function we consider convergence in the general case.

Theorem 10. *Assume Hypothesis* H. *Then* $f_n^*(z)$ *converges to* $f^*(z)$ *everywhere interior to* γ, *uniformly on any closed set interior to* γ.

Proof. For the line integral case there exist [23, p. 431] polynomials P_n such that $\lim_{n\to\infty}\|f^* - P_n\| = 0$, since $f^* \in H_p$. We define $p_n(z)$ as $G_n(z) + P_n(z)$ with $G_n(z)$ the polynomial [19, §11.1] of degree $m - 1$ such that $G_n(w_k) = u_k - P_n(w_k)$, $k = 1, 2, \cdots, m$, and such that on γ we have $|G_n(z)| \leqq M \max_k |u_k - P_n(w_k)|$ if the points w_k are distinct; a suitably modified inequality holds even if some w_k are multiple. Then $p_n \in L_n$ and

$$\|f^* - p_n\| \leqq \|f^* - P_n\| + \|P_n - p_n\| = \|f^* - P_n\| + \|G_n\|$$
$$\leqq \|f^* - P_n\| + M' \max |f^*(w_k) - P_n(w_k)|.$$

Now $P_n(z)$ converges to $f^*(z)$ uniformly on any closed set interior to γ [19, §5.8] and, since the w_k are interior to γ,

$$\lim_{n\to\infty} \max |f^*(w_k) - P_n(w_k)| = 0.$$

Thus,

$$\lim_{n\to\infty} \|f^* - p_n\| = 0.$$

Minkowski's inequality, applied again, yields

$$\|F - p_n\| - \|F - f^*\| \leqq \|f^* - p_n\|. \tag{10.1}$$

By Theorem 6, when $L_*' = \{g \mid g = F - f,\ f \in L_*\}$ is taken as the L_*, $\pi_n = F - p_n$, as the π_n, and $F - f_n^*$ as the ψ_n of that theorem with $F - f^*$ extremal for L_*', we have $\|f^* - f_n^*\| = O[(\|F - p_n\| - \|F - f^*\|)^{1/r}]$. Combining this result with (10.1), we obtain $\lim \|f^* - f_n^*\| = 0$. It follows [19, §5.8] that $\lim f_n^*(z) = f^*(z)$ everywhere interior to γ, uniformly on any closed set interior to γ.

In the proof for the surface integral case [5], or [19, p. 45] and [19, p. 109] are used.

Theorem 11. *Suppose* L_* *defined as in Example* B (*line integral over* $|z| = 1$). *Let* $P(z)$ *be a polynomial of degree* $m - 1$ *such that* $P(w_j) = u_j$, $j = 1, \cdots, m$, *and* $B(z) = \prod_1^m (z - w_j)/(1 - \bar{w}_j z)$. *Then, if* $F \in L^p$, f^* *minimizes* $\|F - f\|_p$, $f \in L_*$, *if and only if* $g^*(z) = (f^*(z) - P(z))/B(z)$ *minimizes* $\|F_1 - g\|_p$, $g \in H_p$ *with* $F_1(z) = (F(z) - P(z))/B(z)$.

Proof. $\|F-f\| = \|(F-P)/B-(f-P)/B\|$.

COROLLARY 11.1. *f^* minimizes $\|f\|_p$, $f \in L_*$, if and only if g^* minimizes $\|(-P/B)-g\|_p, g \in H_p$.*

The following theorem was proved in more general form by Rogosinski and Shapiro ([13] or [16]).

THEOREM A. *Suppose $F(z)$ is analytic for $t<|z|<1/t$, $0<t<1$. Then for some t', $t \leqq t' < 1$, the minimizing function f^* for $\|F-f\|_p$, $f \in H_p$, norm on $|z|=1$, satisfies the following conditions. For $p=1$, $f^*(z)$ is analytic for $|z|<1/t'$. For $1<p<\infty$, $f^*(z)$ is analytic for $|z|<1/t'$ except possibly for isolated branch points in $1<|z|<1/t'$.*

COROLLARY 11.2. *The extremal function for L_* defined as in Example* B *for the line integral over an analytic Jordan curve γ is always analytic on $C+\gamma$.*

Proof. The general case reduces to the unit circle case by mapping the interior of γ onto the interior of $|w|=1$.

5. **Maximal convergence of sequences of polynomials extremal for line and surface integrals in the complex plane.** We now consider some cases where the sequence of polynomials $p_n^*(z)$, extremal for polynomials of degree n contained in a closed convex subset L_* of L^p, can be shown to converge maximally to the extremal function of L_*.

For a more complete definition of maximal convergence the reader is referred to [19, Chapter IV]. We let D denote a closed limited set in the complex plane whose complement K in the extended plane is connected; D_R the equipotential locus in K, $G(x,y)=\log R>0$, and $\rho=\rho(f)$ the greatest number such that $f(z)$ is single-valued and analytic everywhere interior to D_ρ. A sequence of polynomials $\{p_n(z)\}$ with the property that for every $R<\rho$ there exists M (which may depend on R but is independent of n and z) such that $|f(z)-p_n(z)| \leqq M/R^n$, $z \in D$, is said to *converge maximally* to $f(z)$ on D.

The case $F \in L_*$, $f^*=F$, is already known. For the sake of simplicity in certain theorems we exclude this case.

THEOREM 12. *Assume Hypothesis* H. *If f^* is single-valued and analytic on D, we conclude the following. In case $p \geqq 2$, if $[(F-f^*)^{p-2}]^{-\beta}$ is integrable for some $\beta>0$, then the sequence $f_n^*(z)$ converges maximally to $f^*(z)$ on D. In case $1<p<2$, if $F-f^* \equiv Q_q\phi$ for some function ϕ continuous and nonvanishing on $\overline{E}$ and Q_q a polynomial, say of degree q, then the convergence is maximal on D.*

Proof. We note that *if F is analytic on $\overline{E}$* (that is on D in case of the surface integral, on γ for the line integral case) *and if f^* is analytic on D* as in the line integral case (by Corollary 11.2), the requirements in the hypothesis for $(F - f^*)$ are satisfied, whence, according to the theorem, *maximal convergence is implied*. (See Corollary 12.1.)

Define L'_* as $\{g \mid g = F - f, f \in L_*\}$ and L'_n as $\{g_n \mid g_n = F - f_n, f_n \in L_n\}$. By Theorem 1 the extremal element g^* of L'_* is just $(F - f^*)$ and g_n^*, extremal for L'_n, is just $(F - f_n^*)$. We note that g^* is not identically zero on E.

Case (i). $p \geqq 2$. Since f^* is supposed analytic on D, the Tchebycheff polynomials $\pi_n(z)$ have the property that for any R_1, $R < R_1 < \rho_{f^*}$, we have $|f^*(z) - \pi_n(z)| < M'_1/R_1^n$ when $z \in D$. Such polynomials can be chosen so that $\pi_n(w_i) = f^*(w_i) = u_i$, as assigned. Thus, $\|g^* - (F - \pi_n)\| = O(1/R_1^n)$. Since $(F - \pi_n) \in L'_n$ and g_n^* is extremal for L'_n, $\|g_n^*\| \leqq \|F - \pi_n\|$. Thus, the hypothesis for Theorem 9 is satisfied with L'_* and L'_n taken as the L_* and L_n of that theorem, g^* and g_n^* as the f and ψ_n, and $\{1/R_1^n\}$ as $\{\epsilon_n\}$. Hence, Theorem 9 yields

$$\int |(F - f^*) - (F - f_n^*)|^{2(1-a)} d\mu = O(1/R_1^{2n(1-a)}) \tag{12.1}$$

for some a, $0 < a < 1$.

Since the integrand in the left member of (12.1) is just $|f^* - f_n^*|^{2(1-a)}$, we may apply a theorem previously proved [19, p. 94] or [19, p. 97], to obtain $|f^*(z) - f_n^*(z)| < M/R^n$ on D for some M, as required.

Case (ii). $1 < p \leqq 2$. This discussion holds for all $p > 1$, but for $p > 2$ the degree of convergence deduced is weaker than maximal convergence although it implies convergence throughout $C_{\rho^{2/p}}$.

By hypothesis, $F - f^* = Q_q\phi$ with Q_q a polynomial of degree q and ϕ continuous and nonvanishing on $\overline{E}$. Since f^* is analytic and single-valued on D, $f^*(z)/Q_q(z) = h(z) + q_r(z)/Q_q(z)$ with h analytic on D and q_r a polynomial of degree not greater than q. Hence, the existence of polynomials $p_n(z)$, independent of R, of respective degrees n [19, p. 20] such that

$$|h(z) - p_n(z)| < M'_1/R_1^n$$

when $z \in D$ implies $|f^*(z)/Q_q(z) - [p_n(z) + q_r(z)/Q_q(z)]| < M'_1/R_1^n$ when $z \in D$ and the removable singularities are removed, that is,

$$|\{f^*(z) - [p_n(z)Q_q(z) + q_r(z)]\}/Q_q(z)| < M'_1 R_1^q/R_1^{n+q}.$$

The expression in square brackets is a polynomial π_{n+q} of degree $n + q$. If $p_n(z)$ is required [19, p. 310] to equal $h(z)$ at the points w_k, then $\pi_n(w_k) = p_n(w_k)Q_q(w_k) + q_r(w_k) = f^*(w_k)$. Thus $\pi_n(w_k) = u_k$, that is, $\pi_n \in L_{n+q}$.

Since ϕ is continuous and does not vanish on $\overline{E}$, it follows that

$$|[(F(z) - f^*(z)) - (F(z) - \pi_{n+q}(z))]/Q_q(z)\phi(z)| \leqq M''_1/R_1^{n+q} \quad \text{on } E.$$

Since $(F - \pi_{n+q}) \in L'_{n+q}$, we have

$$\|g_{n+q}^*\| \leqq \|F - \pi_{n+q}\|, \qquad g_{n+q}^* \text{ extremal for } L'_{n+q}.$$

Now Theorem 7 yields

$$\|g^* - g_{n+q}^*\| = O[(1/R_1^{n+q})^{(2/r)}], \qquad g^* \text{ extremal for } L_*,$$

that is,

$$\|f^* - f_{n+q}^*\| = O[(1/R_1^{n+q})^{(2/r)}]$$

with $r = 2$ for $1 < p < 2$, but $r = p$ for $p \geqq 2$. As in Case (i), this implies the existence of M_2' such that

$$|f^*(z) - f_{n+q}^*(z)| \leqq M_2'/R^{2(n+q)/r} \quad \text{for } z \text{ on } D.$$

COROLLARY 12.1. *For L_* and L_n defined as in Example* B *for the line integral, if F is analytic on γ but $F \notin L_*$, then $f_n^*(z)$ converges maximally to $f^*(z)$, where f^* minimizes $\|F - f\|_p$, $f \in L^*$, and f_n^* minimizes $\|F - f_n\|$, $f_n \in L_n$.*

6. **Degree of convergence if the extremal function satisfies an integrated Lipschitz condition.** In the following, classes of functions previously studied in [22] are used. If γ is an analytic Jordan curve, a function $f(z)$ belongs to class $H(k,\alpha,p)$ on γ (k a non-negative integer, $0 < \alpha < 1$, $p > 1$) provided $f^{(k)}(z)$ is of class $H_p(\gamma)$, and provided, with respect to arc-length s on γ, $f^{(k)}(z)$ satisfies a pth power integrated Lipschitz condition of order α. The Zygmund class $Z(k,p)$ is similarly defined except that $f^{(k)}(z) \equiv \phi(s)$ is continuous and satisfies a condition of form

$$\int_\gamma |\phi(s+h) + \phi(s-h) - 2\phi(s)|^p ds = O(h)$$

instead of an integrated Lipschitz condition.

By Corollary 12.1, $\{f_n^*(z)\}$ converges maximally to $f^*(z)$ if $F(z)$ is assumed to be analytic on γ. Weaker conclusions are obtained in Theorem 13 under the hypothesis $F \in L^p$.

THEOREM 13. *Assume Hypothesis* H, *line integral case, $p \geqq 2$. Suppose $F \notin L_*$. If for some $k \geqq 0$, either $f^* \in H(k,\alpha,p)$, $0 < \alpha < 1$, or $f^* \in Z(k,p)$, $\alpha = 1$, on γ we conclude the following*:

(a) $\int_\gamma |f^* - f_n^*|^p |dz| = O(1/n^{2(k+\alpha)})$ *and if* $2(k+\alpha) > 1$

$$|f^*(z) - f_n^*(z)| \leqq M/n^{[2(k+\alpha)-1]/p} \quad \text{on } C + \gamma;$$

(b) *if* $[|F - f^*|^{p-2}]^{-\beta}$ *is integrable for some* $\beta > 0$, *then*

$$\int_\gamma |f^* - f_n^*|^{2(1-a)} |dz| = O(1/n^{2(k+\alpha)(1-a)})$$

for $a = 1/(1+\beta)$. *If* $2(1-a)(k+\alpha) > 1$, *then*

$$|f^*(z) - f_n^*(z)| \leqq M'/n^{k+\alpha-1/2(1-a)} \quad \text{on } C + \gamma.$$

Proof. There exist polynomials $\pi_n(z)$ [22, Theorem 1] such that $\|f^* - \pi_n\|_p = O(1/n^{k+\alpha})$. Such polynomials may be assumed to belong to L_*. (See the proof of Theorem 10.)

Proof of (a). Theorem 8, when applied to $L'_* = \{g \mid g = F - f, f \in L_*\}$ with f_n of Theorem 8 taken as $F - \pi_n$ and ψ_n as $F - f_n^*$, implies the first part of (a). A theorem of Sewell [15, Theorem 4.2.1] yields the second part of (a).

Proof of (b). Theorem 9, applied to $L'_* = \{g \mid g = F - f, f \in L_*\}$ with π_n of Theorem 9 taken as $F - \pi_n$ and ψ_n as $F - f_n^*$ implies the first part of (b); [15, p. 123] implies the second part of (b).

In the next theorem the surface integral is used. A degree of convergence of the minimizing polynomials to the minimizing function f^* of H'_p is obtained provided $f^* \in H(k, \alpha, 2)$. When the norm notation is used here, the surface integral is to be understood.

THEOREM 14. *Assume Hypothesis* H (*surface integral case*) *with the additional requirement that the boundary* γ *is an analytic Jordan curve.*

Case (i). $p \geqq 2$. *Suppose* F *is bounded on* γ *but* $F \notin L_*$. *If either* $f^* \in H(k, \alpha, 2)$ *for* $0 < \alpha < 1$ *or* $f^* \in Z(k, 2)$ *for* $\alpha = 1$ *on* γ *and if* $2(k + \alpha) > 1$, *then* $\|f^* - f_n^*\|_p = O(1/n^{[2(k+\alpha)+1]/p})$ *and* $|f^*(z) - f_n^*(z)| < M/n^{[2(k+\alpha)+1]/p}$ *on* D.

Case (ii). $1 < p < \infty$. *Let* $F(z) \equiv 0$. *If* $f^* = Q_q \phi$ *for some polynomial* Q_q, *say of degree* q, *and* ϕ *bounded from zero on* D *and of class* $H(k, \alpha, 2)$, $0 < \alpha < 1$, *or of class* $Z(k, 2)$, $\alpha = 1$, *on* γ, *then if* $2(k + \alpha) > 1$,

$$\|f^* - f_n^*\|_p = O(1/n^{(k+\alpha+1/2)2/r}),$$

where $r = 2$ *for* $1 < p < 2$, $r = p$ *for* $p \geqq 2$, *and*

$$|f^*(z) - f_n^*(z)| < M'/n^{(k+\alpha+1/2)2/r-2/p}$$

on D.

Proof of Case (i). There exists a function π_n of L_n such that $\|f^* - \pi_n\|_2 = \|(F - f^*) - (F - \pi_n)\|_2 = O(1/n^{k+\alpha+1/2})$, by [21], [1a], or [24]. To verify that π_n can be supposed to belong to L_* see the proof of Theorem 10. Then [15, p. 137, Exercise 4.6.14] implies $|f^*(z) - \pi_n(z)| < M/n^{k+\alpha-1/2}$ on D. (There is a misprint in [15, p. 137, line 2 from the bottom]: $|f(z) - P_n(z)|$ should be raised to the pth power.)

When Theorem 8, (2) is applied with $F - f^*$ and $F - f_n^*$ taken respectively as f^* and f_n^*, the theorem yields the required result for $\|f^* - f_n^*\|_p$. Application of [15, Exercise 4.6.14] completes the proof for Case (i).

Proof of Case (ii). $1 < p < \infty$. We proceed to show Theorem 7′ can be applied. By [21], [1a], or [24], there exist polynomials π_n of respective degree n such that

$$\|\phi - \pi_n\|_2 = O(1/n^{k+\alpha+1/2}). \tag{14.1}$$

These polynomials can be chosen so that $\pi_n(w_k) = u_k/Q_q(w_k)$. By [15, Exercise 4.6.14], $|\phi(z) - \pi_n(z)| < M_1/n^{k+\alpha-1/2}$ on D, whence, since $f^*(z) = \phi(z)Q_q(z)$ with $\phi(z)$ bounded from zero on D, $|[f^*(z) - \pi_n(z)Q_q(z)]/f^*(z)| < M_1/n^{k+\alpha-1/2} < r < 1$ on D. Clearly $\|f^* - \pi_n Q_q\|_p < 1$ for large n. When $n > q$, $1/n^{k+\alpha+1/2} < 2^{k+\alpha}/(n+q)^{k+\alpha+1/2}$, so (14.1) implies $\|(f^* - \pi_n Q_q)/f^*\|_2 = O(1/(n+q)^{k+\alpha+1/2})$. When the $\pi_n Q_q$ are taken as the polynomials of degree $n+q$ of Theorem 7′, that theorem yields $\|f^* - f_n^*\|_p = O(1/n^{(k+\alpha+1/2)2/r})$, whence $|f^*(z) - f_n^*(z)| \leqq M'/n^{(k+\alpha+1/2)2/r-2/p}$ on D, with $r = p$ for $p \geqq 2$ but $r = 2$ for $1 < p < 2$.

Theorems 12-14 have immediate application to the specific problems mentioned in the introduction. A simple illustration is the (Bieberbach) case of a surface integral with $m = 1$, $p = 2$ and the condition $p_n(0) = 1$; the extremal function $f(z)$ maps C one-to-one onto a circular disc, and its continuity properties on γ or D_ρ yield by Theorems 12 and 14 immediate results on degree of convergence of the extremal polynomials. The application of Theorem 12, involving maximal convergence, is considered in [19, Chapter 11, Theorem 7]; less general results than this application of Theorem 14 are given in [9a]. Likewise for the (Julia) problems of surface or line integrals with $m = 2$, $p > 1$, and the conditions $p_n(0) = 0$, $p_n'(0) = 1$, the extremal function is $f(z)[f'(z)]^{2/p}$ or $f(z)[f'(z)]^{1/p}$, respectively, where $f(z)$ is the mapping function; continuity properties on γ or D_ρ of the latter yield by Theorems 12-14 immediate results on degree of convergence of the extremal polynomials. Maximal convergence has already been extablished in the case with $p = 2$ for the line integral [19, Chapter 11, Theorem 9].

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