A DUALITY IN FUNCTION SPACES(¹)

BY

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0. **Introduction.** James Eells, in his paper [2], has shown a fresh approach to the computation of cohomology groups of functions spaces. Roughly speaking, he proves that, if X is an infinite-dimensional manifold and if Y is a submanifold of codimension n, then there is an isomorphism relating $H^{p+n}(X, X \sim Y)$ to $H^p(Y)$ (for details, see Example 1.4 [B] below). Eells then proceeds to show that many important function spaces are indeed infinite-dimensional manifolds, and, in many instances, there are useful submanifolds of finite codimension. By this method, Eells is able to obtain many new results on function spaces as well as to give new insights to known facts. A drawback of his approach is noted by Eells himself: "It should be remarked that although Alexander-Pontrjagin duality is a theory of topological character, our applications of it require the differentiable structure of our function spaces (e.g. to establish that certain subspaces are in fact finite-codimensional submanifolds)." It is the purpose of the present paper to give a purely topological foundation to the 'duality theorem' in function spaces. It will be seen, for instance, that in the applications discussed in [2] the differentiable structures are completely irrelevant. We believe that the new setting for the 'duality theorem' makes it much easier to apply even to a situation where a differentiability is readily available. The hybrid nature of our approach should be pointed out: We study local properties of a function space by means of fiber maps and global properties by sheaf theory.

In the present paper, we shall deal exclusively with the additive cohomology. The multiplicative structure of the cohomology of function spaces will be considered in a subsequent paper.

Finally, I should like to express my gratitude to J. Eells for many lively discussions we had on the subject of this paper.

1. Since we shall be using a number of cohomology theories in the following paragraph, we shall clarify our notation. Our sheaf-theoretic terminology is that of Godement [5].

If $\mathscr{S}$ is a locally simple sheaf on a topological space X, one can define the group of *singular p-cochains of X with coefficient $\mathscr{S}$* (denoted by $C^p(X; \mathscr{S})$) as follows; A member c ot $C^p(X; \mathscr{S})$ is a function such that,

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for each singular p-simplex $s: \Delta^p \to X$, $c(s)$ is a member of $\Gamma s^* \mathscr{S}$. The boundary operator $C^p(X; \mathscr{S}) \to^\delta C^{p+1}(X; \mathscr{S})$ is defined in the usual way, taking into account the fact that $s^* \mathscr{S}$ is a simple sheaf. The *support* $\sigma(c)$ of a member c of $C^p(X; \mathscr{S})$ is a closed subset of X such that $x \notin \sigma(c)$ if and only if there is a neighborhood U of x with the property that $c(s) = 0$ whenever s is a singular p-simplex in U. A family Φ of closed subsets of X is called a *family of supports* if it satisfies the following three conditions:

(i) $A \in \Phi$ and $B = B^- \subset A$ imply $B \in \Phi$,

(ii) $A \in \Phi$ and $B \in \Phi$ imply $A \cup B \in \Phi$, and

(iii) $A \in \Phi$ implies that, for some B in Φ, $A \subset B^0$.

Let $C_\Phi^p(X; \mathscr{S})$ be the subgroup of $C^p(X; \mathscr{S})$ consisting of those p-cochains c such that $\sigma(c) \in \Phi$. Then $C_\Phi^*(X; \mathscr{S})$ is a subcomplex of $C^*(X; \mathscr{S})$, and the derived groups of $C_\Phi^{\cdot}(X; \mathscr{L})$ will be denoted by $H_\Phi^p(X; \mathscr{L})$ $(p = 0, 1, 2, \cdots)$. If Φ is the family of all closed subsets of X, it obviously satisfies (i), (ii) and (iii). In this case $H_\Phi^p(X; \mathscr{S})$ is denoted simply by $H^p(X; \mathscr{S})$. Also if $\mathscr{S}$ is the simple sheaf of an Abelian group G on X (i.e., $\mathscr{S} = X \times G$ with G having the discrete topology), then $H_\Phi^p(X; \mathscr{S})$ shall be denoted by $H_\Phi^p(X; G)$. It is clear, then, that $H^p(X; G)$ denotes the usual pth singular cohomology of X with coefficient G. Similarly, if (X, Y) is a pair, then we can define $C_\Phi^p(X, Y; \mathscr{S})$ to be the subgroup of $C_\Phi^p(X; \mathscr{S})$ consisting of all members c of $C_\Phi^p(X; \mathscr{S})$ such that $c(s) = 0$ whenever s lies in Y, and the derived groups of the complex $C_\Phi^*(X, Y; \mathscr{S})$ are defined to be $H_\Phi^p(X, Y; \mathscr{S})$ $(p = 0, 1, 2, \cdots)$. As before, the group $H_\Phi^p(X, Y; \mathscr{S})$ will be denoted by $H^p(X, Y; \mathscr{S})$ and $H_\Phi^p(X, Y; G)$ when Φ is the family of all closed subsets of X and when $\mathscr{S}$ is the simple sheaf of a group G, respectively.

By a straightforward modification of the usual proof (see, for example, Cartan [1, Exposé 20]), one can prove that $H_\Phi^p(X; \mathscr{S})$ is isomorphic to the pth Čech cohomology group $\check{H}_\Phi^p(X; \mathscr{S})$ provided X is paracompact and satisfies a certain local condition. For our purpose, the following proposition is good enough. A topological space A is *acyclic* if A is pathwise connected and $H_q(A; Z) = 0$ for $q > 0$. A topological space X is *locally acyclic* if acyclic open subsets of X form a base for the topology.

1.1. PROPOSITION. *Let $\mathscr{S}$ be a locally simple sheaf on a locally acyclic and paracompact space X, and let Φ be a family of supports. Then there is a canonical isomorphism*

$$H_\Phi^p(X; \mathscr{S}) \overset{\cong}{\longrightarrow} \check{H}_\Phi^p(X; \mathscr{S}) \quad (p = 0, 1, 2, \cdots).$$

Let X be a topological space and Y a closed subspace of X. For an Abelian group G and an integer $p \geqq 0$, we define a presheaf on X by assigning to each open subset U of X the group $H^p(U, U \sim Y; G)$. If V is another open subset such that $V \subset U$, then the restriction homomorphism $H^p(U, U \sim Y; G) \to H^p(V, V \sim Y; G)$ is the map induced by the inclusion

$(V, V \sim Y) \rightarrow (U, U \sim Y)$. The sheaf generated by this presheaf is concentrated on Y and its restriction to Y' will be denoted by $\mathscr{T}^p_G(X, Y)$. The following theorem is basic in our subsequent investigation. The essence of a proof is in Eells [2], and we shall give only a very brief outline of it.

1.2. **Theorem.** *Let X be a paracompact space and let Y be a closed subspace. If Φ is an arbitrary family of supports, we have a spectral sequence such that $E^{p,q}_2 = \check{H}^p_{\Phi|Y}(Y; \mathscr{T}^q_G(X, Y))$ and E_∞ is the graded group associated with $H^*_\Phi(X, X \sim Y; G)$, suitably filtered, where $\Phi|Y = \{A \cap Y : A \in \Phi\}$.*

   Proof. Let $\mathscr{L}^q$ be the sheaf generated by the presheaf $U \rightarrow C^q(U, U \sim Y; G)$. The boundary homomorphism $C^q(U, U \sim Y; G) \rightarrow C^{q+1}(U, U \sim Y; G)$ induces $\delta: \mathscr{L}^q \rightarrow \mathscr{L}^{q+1}$. Since Φ is a paracompactifying family and $\mathscr{L}^q$ is fine for each q, we can apply Theorem 4.6.1 of [5, p. 178] (replacing H by $\check{H}$ because of Theorem 5.10.1); hence, we conclude that there is a spectral sequence such that $E^{p,q}_2 = \check{H}^p_\Phi(X; \mathscr{H}^q(\mathscr{L}^*))$ and E_∞ is the associated graded group of the filtered cohomology of $\Gamma_\Phi(\mathscr{L}^*)$. Now $\mathscr{H}^q(\mathscr{L}^*)$ is the sheaf generated by the presheaf $U \rightarrow H^q(U, U \sim Y; G)$; therefore, since Y is closed, by Theorem 4.9.1 of [5, p. 187], we have

$$\check{H}^p_\Phi(X; \mathscr{H}^q(\mathscr{L}^*)) \cong \check{H}^p_{\Phi|Y}(Y; \mathscr{T}^q_G(X, Y)).$$

By using Theorem 3.9.9 of [5, p. 159] and the standard facts about the singular theory, one can show that the cohomology of the complex $\Gamma_\Phi(\mathscr{L}^*)$ is $H^*_\Phi(X, X \sim Y; G)$.

   By Proposition 1.1, the following corollary is obvious.

1.3. **Corollary.** *In Theorem 1.2 assume, in addition, that Y is locally acyclic and $\mathscr{T}^q_G(X, Y)$ is locally simple. Then E_2 of the spectral sequence becomes: $E^{p,q}_2 = H^p_{\Phi|Y}(Y; \mathscr{T}^q_G(X, Y))$.*

1.4. **Examples.** [A] Let X be a topological space, and let Y be a closed subspace of X such that, for each point y of Y, there is an open (in X) neighborhood U of y and a homeomorphism ϕ of U onto $E^n \times (U \cap Y)$ with the property that $\phi[U \cap Y] = \{0\} \times (U \cap Y)$. (We note in passing that, if $B \rightarrow^i E \rightarrow^j B$ is a microbundle in the sense of Milnor [7], the pair (E, B) satisfies the condition stated above, where n is taken to be the fiber dimension of the microbundle.) Under the assumption, the stalk of $\mathscr{T}^q_G(X, Y)$ at y is isomorphic to $\lim \mathrm{dir}_{V \in \mathscr{V}_y} H^{q-n}(V; G)$, where $\mathscr{V}_y$ is the system of open (in Y) neighborhoods of y. In particular, $\mathscr{T}^q_G(X, Y) = 0$ for $q < n$. Therefore, if X is paracompact, $H^p(X, X \sim Y; G) = 0$ for $p < n$. If, in addition, Y is locally acyclic, $\mathscr{T}^q_G(X, Y) = 0$ for $q \neq n$, and $\mathscr{T}^n_G(X, Y)$ is a locally simple sheaf with each stalk isomorphic to G. By Corollary 1.3, we have an isomorphism

$$H^p_{\Phi|Y}(Y; \mathscr{T}^n_G(X, Y)) \cong H^{p+n}_\Phi(X, X \sim Y; G).$$

We may call this the Thom isomorphism.

[B] Let X be a topological space and let Y be a closed subspace of X. We say that Y is of *codimension* n ($n \geq 1$) in X if the following condition is satisfied: for each point y of Y, there is an open (in X) neighborhood U of y and a homeomorphism ϕ of U onto a locally convex linear topological space E_y (possibly finite dimensional!) such that $\phi[U \cap Y]$ is a closed subspace of E_y of codimension n. Now suppose that Y is of codimension n in X. Then $\mathcal{S}_G^q(X, Y) = 0$ for $q \neq n$ and $\mathcal{S}_G^n(X, Y)$ is a locally simple sheaf on Y with each stalk isomorphic to G. (This follows from 2(A) of Eells [2] and the following fact: if E is a linear topological space, F a subspace, y a point in F and U a convex open neighborhood of y in E, then the inclusion map $(U, U \sim F) \to (E, E \sim F)$ admits a homotopy inverse.) Therefore, if X is paracompact, by Corollary 1.3, we have an isomorphism

$$H_\Phi^{p+n}(X, X \sim Y; G) \cong H_{\Phi|Y}^p(Y; \mathcal{S}_G^n(X, Y)).$$

This isomorphism is the 'duality theorem' which is exploited by Eells [2]. For example, suppose M is a smooth $(n + k)$-manifold and let A be a k-dimensional submanifold. If (S, s_0) is a compact space with base point, then Eells shows that the space $(M, A)^{(S, s_0)}$(^{2}) is of codimension n in M^S(^{2}), and, therefore, the duality theorem applies to these spaces. As pointed out in the introduction, this approach puts a very strong restriction on M and A. In the sequel, we shall develop an alternative way to apply Theorem 1.2 to function spaces.

2. A continuous map $f : X \to X'$ is called a *local fiber map* if the following two conditions are satisfied:

(i) f is an open map, and,

(ii) for each point x of X and for each neighborhood V of x, there is an open neighborhood W of x such that $x \in W \subset V$ and $f | W : W \to f[W]$ is a Serre fiber map with an acyclic fiber.

The following proposition is an obvious consequence of the definition.

2.1. PROPOSITION. *Let* $f : X \to X'$ *be a local fiber map and let* Y' *be a subspace of* X'. *Then* $f | f^{-1}[Y'] : f^{-1}[Y'] \to Y'$ *is a local fiber map.*

In the proofs of the next two assertions, we shall use the following fact: If $p : E \to B$ is an *onto* Serre fiber map with an acyclic fiber then

$$p^* : H^q(B; G) \overset{\cong}{\Longrightarrow} H^q(E; G)$$

for any Abelian group G and any integer p. This will follow from Serre's spectral sequence [8] when B is pathwise connected. But then it is trivial to generalize it to nonpathwise connected B by decomposing B into path-

(^{2}) $(M, A)^{(S, s_0)}$ is the space of all continuous maps $f : S \to M$ such that $f(s_0) \in A$, the topology being the compact-open topology. $M^S = (M, M)^{(S, s_0)}$.

wise connected components (cf. [3, p. 210]). We wish to point out that there is a proof, due to Ganea, of this isomorphism without using Serre's spectral sequence([3]).

Again let $p: E \rightarrow B$ be an *onto* Serre fiber map with an acyclic fiber, let B' be a subset of B and let $E' = p^{-1}[B']$. Then, from the remark above and the Five Lemma, it follows that

$$p^*: H^q(B, B'; G) \stackrel{\cong}{\Longrightarrow} H^q(E, E'; G)$$

for arbitrary G and q.

2.2. PROPOSITION. *Let* $f: X \rightarrow X'$ *be a local fiber map. If* X' *is locally acyclic so is* X.

Proof. Let $x \in X$ and let V be a neighborhood of x in X. Then there is an open neighborhood W of x such that $W \subset V$ and $f | W: W \rightarrow f[W]$ is a Serre fiber map with an acyclic fiber. Since f is an open map, $f[W]$ is an open neighborhood of $f(x)$. Hence there is an acyclic open neighborhood U of $f(x)$ contained in $f[W]$. Let $W' = W \cap f^{-1}[U]$; then $f | W': W' \rightarrow U$ is a Serre fiber map with the fiber, and the base, acyclic; hence W' is acyclic. Clearly, W' is an open neighborhood of x contained in V; therefore X is locally acyclic.

The following theorem provides the reason for introducing the notion of local fiber maps.

2.3. THEOREM. *Let* $f: X \rightarrow X'$ *be a local fiber map, let* Y' *be a closed subspace of* X' *and let* $Y = f^{-1}[Y']$. *Then, for an arbitrary Abelian group* G *and an arbitrary integer* p, *the sheaf* $\mathcal{T}_G^p(X, Y)$ *is isomorphic to*

$$(f | Y)^* \ \mathcal{T}_G^p(X', Y'),$$

the reciprocal image of $\mathcal{T}_G^p(X', Y')$ *by the map* $f | Y$([4]).

Proof. Let $y \in Y$, let $\mathcal{W}_y$ be the directed family of open (in X) neighborhoods of y, and let $\mathcal{U}_{f(y)}$ be the directed family of open (in X') neighborhoods of $f(y)$. Then

$$(f | Y)^* \ \mathcal{T}_G^p(X', Y')(y) = \lim_{U \in \mathcal{U}_{f(y)}} \mathrm{dir} \ H^p(U, U \sim Y'; G),$$

and

$$\mathcal{T}_G^p(X, Y)(y) = \lim_{W \in \mathcal{W}_y} \mathrm{dir} \ H^p(W, W \sim Y; G).$$

([3]) Here is an outline of Ganea's proof: $p: E \rightarrow B$ can be written as the composition of $E \xrightarrow{i} E \cup CF \xrightarrow{\bar{p}} B$, where i is the inclusion and $\bar{p}$ is the extension of p by sending CF to the base point. The 'fiber' of $\bar{p}$ is $\Omega B * F$ (see [4]) which is acyclic and 1-connected hence aspherical; therefore, $\bar{p}$ induces isomorphisms of cohomology groups. Clearly, i induces cohomology isomorphisms too.

([4]) For a definition of reciprocal images, see [5, p. 120].

Now f induces the map

$$\alpha_y: \lim_{W \in \mathscr{W}_y} \mathrm{dir}\, H^p(f[W], f[W] \sim Y'; G) \to \lim_{W \in \mathscr{W}_y} \mathrm{dir}\, H^p(W, W \sim Y; G).$$

Let $\mathscr{W}_y'$ be a subfamily of $\mathscr{W}_y$ consisting of all those W such that $f|W: W \to f[W]$ is a Serre fiber map with an acyclic fiber. By the definition of local fiber map, $\mathscr{W}_y'$ is a cofinal subset of $\mathscr{W}_y$. From the remarks preceding Proposition 2.2, for each W in $\mathscr{W}_y'$,

$$(f|W)^*: H^p(f[W], f[W] \sim Y'; G) \overset{\cong}{=\!=} H^p(W, W \sim Y; G).$$

Therefore, we see that α_y is an isomorphism. Next, since the map f is open, the correspondence $W \to f[W]$ induces an order-preserving map $\mathscr{W}_y \to \mathscr{U}_{f(y)}$. Hence we have a homomorphism:

$$\beta_y: \lim_{W \in \mathscr{W}_y} \mathrm{dir}\, H^p(f[W], f[W] \sim Y'; G) \to \lim_{U \in \mathscr{U}_{f(y)}} \mathrm{dir}\, H^p(U, U \sim Y'; G),$$

and, by Theorem 4.13 of [3, p. 223], β_y is an isomorphism. It follows that

$$\alpha_y \beta_y^{-1}: (f|Y)^* \,\mathscr{T}_G^p(X', Y')(y) \overset{\cong}{=\!=} \mathscr{T}_G^p(X, Y)(y).$$

Therefore, the collection of isomorphisms $\{\alpha_y \beta_y^{-1}: y \in Y\}$ defines a one-to-one onto map $\gamma: (f|Y)^* \,\mathscr{T}_G^p(X', Y') \to \mathscr{T}_G^p(X, Y)$. To complete the proof it remains to show that γ is continuous, for then γ is necessarily an open map also. But the continuity of γ is immediate once all the relevant definitions are assembled.

If $p: E \to B$ is a locally trivial fiber map with a locally acyclic fiber, then it is obvious that p is a local fiber map. However, many fiber maps of interest are not locally trivial. Nevertheless, in the next section we shall see that there is a large class of fiber maps involving function spaces which are also local fiber maps.

3. Let X and C be topological spaces and let C_0 be a closed subspace of C. In this section, we shall present a sufficient condition for the map $\rho: X^C \to X^{C_0}$ to be a local fiber map, where $\rho(f) = f|C_0$. (Recall that X^Y is the space of all continuous maps on Y into X with the compact-open topology.) A pair (C, C_0) is *finitely triangulable* if there is a finite simplicial complex K and a subcomplex L such that $(|K|, |L|)$ is homeomorphic to (C, C_0), i.e., there is a homeomorphism ϕ on $|K|$ onto C which takes $|L|$ onto C_0. If (C, C_0) is finitely triangulable, then C_0 is a neighborhood retract in C; hence any continuous map defined on C_0 can be extended to an open neighborhood of C_0 in C. If $A \subset Y$ and $B \subset X$ then $N(A, B)$[5] denotes the

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[5] When $A \subseteq Y \subset Z$, there is nothing to indicate in this notation whether $N(A, B) \subset X^Y$ or $N(A, B) \subset X^Z$. However, in our usage the context will determine the domain of the functions in $N(A, B)$.

subspace of X^Y consisting of all maps f such that $f[A] \subset B$.

3.1. **Lemma.** *Let X be a topological space, let (C, C_0) be a finitely triangulable pair, and let $\rho: X^C \to X^{C_0}$ be the map defined by $\rho(f) = f | C_0$. If $K_1, \cdots, K_n$ are compact subsets of C and $U_1, \cdots, U_n$ are open subsets of X, then*

$$\rho | N(K_1, U_1) \cap \cdots \cap N(K_n, U_n): N(K_1, U_1) \cap \cdots \cap N(K_n, U_n)$$

$$\to N(K_1 \cap C_0, U_1) \cap \cdots \cap N(K_n \cap C_0, U_n)$$

is a Serre fiber map.

Proof. For brevity, denote

$$N(K_1, U_1) \cap \cdots \cap N(K_n, U_n)$$

and

$$N(K_1 \cap C_0, U_1) \cap \cdots \cap N(K_n \cap C_0, U_n)$$

by Z and Z_0, respectively, and denote $\rho | Z$ by ρ_0. Let P be a finitely triangulable space, let F be a continuous map $I \times P \to Z_0$, and let G be a continuous map $\{0\} \times P \to Z$ such that $\rho_0(G(0,p)) = F(0,p)$ for each p in P. We must construct a continuous extension $H: I \times P \to Z$ of G so that $\rho_0 \circ H = F$.

Let $Y = I \times P \times C$ and let $Y_0 = I \times P \times C_0 \cup \{0\} \times P \times C$. Define a continuous map $H_0: Y_0 \to X$ by

   $H_0(t, p, x) = F(t, p)(x)$ for $(t, p, x) \in I \times P \times C_0$, and

   $H_0(0, p, x) = G(0, p)(x)$ for $(0, p, x) \in \{0\} \times P \times C$.

By the hypothesis, $H_0(t, p, x) \in U_i$ whenever $(t, p, x) \in Y_0$ and $x \in K_i$. Clearly (Y, Y_0) is finitely triangulable; therefore, H_0 can be extended to a continuous map H_1 on an open neighborhood V of Y_0 into X. For each i $(i = 1, 2, \cdots, n)$, let $O_i = \{(t, p, x): (t, p, x) \in V \text{ and } H_1(t, p, x) \in U_i\}$. Then O_i is an open subset of V, hence open in Y. Also $Y_0 \cap (I \times P \times K_i) \subset O_i$. Let $V_i = O_i \cup (V \sim (I \times P \times K_i))$; then V_i is an open subset of Y such that $Y_0 \subset V_i \subset V$ and $H_1(t, p, x) \in U_i$ whenever $(t, p, x) \in V_i$ and $x \in K_i$. Let $V_0 = V_1 \cap V_2 \cap \cdots \cap V_n$. The rest of the proof follows a familiar pattern. There exists an open neighborhood W of $P \times C_0$ in $P \times C$ such that $I \times W \subset V_0$. Let τ be a continuous function on $P \times C$ into I such that $\tau[P \times C_0] = \{1\}$ and $\tau[P \times C \sim W] = \{0\}$. Define a continuous map $\tilde{H}: Y \to X$ by

$$\tilde{H}(t, p, x) = H_1(\tau(p, x) \cdot t, p, x).$$

Then obviously $\tilde{H}$ is an extension of H_0 and, if $x \in K_i$, then $\tilde{H}(t, p, x) \in U_i$ because $(\tau(p, x) \cdot t, p, x) \in V_0 \subset V_i$. Finally let $H: I \times P \to Z$ be the continuous map defined by $H(t, p)(x) = \tilde{H}(t, p, x)$. Clearly H is an extension of G such that $\rho_0 \circ H = F$. This completes the proof of Lemma 3.1.

A topological space X is said to be *aspherical*(^{6}) if $\pi_n(X) = 0$ for $n = 1, 2, \cdots$
A topological space X is *locally aspherical* if the family of all aspherical open
subsets of X forms a base for the topology. Obviously a (locally) aspherical
space is (locally) acyclic. Since contractible spaces are aspherical, an arbi-
trary CW-complex is locally aspherical because of property (M) proved in
J. H. C. Whitehead's paper [10, p. 230]. Any (infinite- or finite-dimensional)
manifold modeled on a locally convex linear topological space is also locally
aspherical. If X is aspherical and (A, B) is triangulable, then any continuous
map on B into X can be extended to a continuous map of A into B. When
we speak of a simplex of a simplicial complex, we mean a simplex in the
sense of Hilton-Wylie [6] (i.e., open simplex in the sense of [3]).

3.2. LEMMA. *Let X be a locally aspherical space, let C be a finitely triangu-
lated space, let f be a member of X^C, and let O be an open neighborhood of f in
X^C. Then there is a triangulation K of C which is the result of a repeated sub-
division of the original triangulation of C, and a family $\{U_\sigma : \sigma \in K\}$ of
aspherical open subsets of X, indexed by simplices of K, such that*
   (a) *if $\tau \subset \bar{\sigma}$ (i.e., if τ is a face of σ), $U_\tau \subset U_\sigma$, and*
   (b) *$f \in \bigcap_{\sigma \in K} N(\bar{\sigma}, U_\sigma) \subset O(^7).$*

Proof. Without loss of generality, one can assume that $O = N(C_1, U_1) \cap$
$\cdots \cap N(C_n, U_n)$ where $C_1, \cdots, C_n$ are compact subsets of C, $C_1 \cup \cdots \cup C_n$
$= C$, and $U_1, \cdots, U_n$ are open subsets of X. Let d denote a metric for C.
Set $V_i = f^{-1}[U_i]$, and let $\epsilon = \min_i d(C_i, C \sim V_i) > 0$. Then it is possible to
subdivide the triangulation of C sufficiently many times to obtain a tri-
angulation K_0 of C such that $\operatorname{diam}(\sigma) < \epsilon$ for each simplex σ of K_0. From
our choice of ϵ, it is clear that $\bar{\sigma} \cap C_i \neq \emptyset$ implies that $\bar{\sigma} \subset V_i$, i.e., $f[\bar{\sigma}]$
$\subset U_i$. Now, for each $\bar{\sigma}$ in K_0, choose U_σ^0 as follows: if $\{i_1, \cdots, i_k\}$ is the com-
plete set of indices i for which $\bar{\sigma} \cap C_i \neq \emptyset$, we put $U_\sigma^0 = U_{i_1} \cap \cdots \cap U_{i_k}$.
Obviously $f \in \bigcap_{\sigma \in K_0} N(\bar{\sigma}, U_\sigma^0) \subset O$.

Let the dimension of C be p, and let $\mathscr{U}$ be the family of all aspherical
open subsets of X. By induction on m we shall choose triangulations K_m
$(m = 0, 1, \cdots)$ of C and a family $\{U_\sigma^m : \sigma \in K_m\}$ of open subsets of X with
the following properties:
   (i) for $m > 0$, K_m is obtained from K_{m-1} by a repeated subdivision;
   (ii) if $\dim \sigma \geq p - (m - 1)$, then $U_\sigma^m \in \mathscr{U}$;
   (iii) if $\tau \subset \bar{\sigma}$ and $\dim \tau \geq p - (m - 1)$, then $U_\tau^m \subset U_\sigma^m$; and
   (iv) $f \in \bigcap_{\sigma \in K_m} N(\bar{\sigma}, U_\sigma^m) \subset O.$
For $m = 0$, the triangulation K_0 and the family $\{U_\sigma^0 : \sigma \in K_0\}$ constructed

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(^{6}) *Added in proof.* The use of the word 'aspherical' here is, unfortunately, nonstandard
in that we require $\pi_1(X) = 0$.
   (^{7}) We shall not distinguish a simplex σ of K and the image of σ in C under the homeo-
morphism $|K| \to C$ given by the triangulation.

above satisfy all these properties. (In fact, (i), (ii) and (iii) are satisfied vacuously.)

Assume that a triangulation K_m of C and the family $\{U_\sigma^m : \sigma \in K_m\}$ have been constructed so as to satisfy (i)-(iv). Let σ be a simplex of K_m of dimension $p - m$, and let $V_\sigma = \bigcap \{U_\tau^m : \tau \in K_m, \sigma \subset \overline{\tau}\}$. Then from (iv), $f[\overline{\sigma}] \subset V_\sigma$. Now subdivide K_m sufficiently many times to obtain K_{m+1} so that the following requirement is fulfilled. First, let us denote by $\sigma(\tau)$ a unique simplex in K_m which contains a simplex τ of K_{m+1}. We then require that, whenever τ is a simplex of K_{m+1} such that $\dim \tau = \dim \sigma(\tau) = p - m$, for some member U_τ^{m+1} of $\mathcal{U}$, $f[\overline{\tau}] \subset U_\tau^{m+1} \subset V_{\sigma(\tau)}$. This is surely possible because $\mathcal{U}$ is a base for the topology of X. For a simplex τ of K_{m+1} such that either $\dim \tau \neq p - m$ or $\dim \sigma(\tau) \neq p - m$, we set $U_\tau^{m+1} = U_{\sigma(\tau)}^m$. We must verify the properties (i)-(iv) (with m replaced by $m + 1$) for K_{m+1} and the family $\{U_\tau^{m+1} : \tau \in K_{m+1}\}$.

PROPERTY (i). This is obviously satisfied.

PROPERTY (ii). Assume that $\tau \in K_{m+1}$ and $\dim \tau \geqq p - m$. Then $\dim \sigma(\tau) \geqq p - m$ also. If $\dim \sigma(\tau) = p - m$ then, by our choice, $U_\tau^{m+1} \in \mathcal{U}$. Otherwise, $\dim \sigma(\tau) \geqq p - (m - 1)$; hence, $U_\tau^{m+1} = U_{\sigma(\tau)}^m \in \mathcal{U}$ by our inductive hypothesis (ii).

PROPERTY (iii). Assume that $\tau_1, \tau_2 \in K_{m+1}$, $\tau_1 \subset \overline{\tau_2}$ and $\dim \tau_1 \geqq p - m$. If $\tau_1 = \tau_2$ then, trivially, $U_{\tau_1}^{m+1} \subset U_{\tau_2}^{m+1}$. Assume now that $\tau_1 \neq \tau_2$. Then $\dim \tau_1 < \dim \tau_2$. It follows that $\dim \sigma(\tau_2) \geqq \dim \tau_2 \geqq p - m + 1$. Hence by the definition $U_{\tau_2}^{m+1} = U_{\sigma(\tau_2)}^m$. First consider the case where $\dim \tau_1 = \dim \sigma(\tau_1) = p - m$. In this case, since $\tau_1 \subset \overline{\tau_2}$ implies $\sigma(\tau_1) \subset \overline{\sigma(\tau_2)}$, we have

$$U_{\tau_1}^{m+1} \subset V_{\sigma(\tau_1)} \subset U_{\sigma(\tau_2)}^m = U_{\tau_2}^{m+1}.$$

Next assume that it is not the case that $\dim \tau_1 = \dim \sigma(\tau_1) = p - m$. Since $\dim \tau_1 \geqq p - m$ and $\dim \sigma(\tau_1) \geqq \dim \tau_1$, we have $\dim \sigma(\tau_1) \geqq p - (m - 1)$. Hence, by inductive hypothesis (iii),

$$U_{\tau_1}^{m+1} = U_{\sigma(\tau_1)}^m \subset U_{\sigma(\tau_2)}^m = U_{\tau_2}^{m+1}.$$

PROPERTY (iv). Take $\tau \in K_{m+1}$. If $\dim \tau = \dim \sigma(\tau) = p - m$ then, by our choice, $f[\overline{\tau}] \subset U_\tau^{m+1}$. Otherwise, $f[\overline{\tau}] \subset f[\overline{\sigma(\tau)}] \subset U_{\sigma(\tau)}^m = U_\tau^{m+1}$ by inductive hypothesis (iv). It follows that $f \in \bigcap_{\tau \in K_{m+1}} N(\overline{\tau}, U_\tau^{m+1})$. Finally, let $g \in \bigcap_{\tau \in K_{m+1}} N(\overline{\tau}, U_\tau^{m+1})$. We shall show that $g \in \bigcap_{\sigma \in K_m} N(\overline{\sigma}, U_\sigma^m)$. Let $\sigma \in K_m$ and let σ be the union of simplices $\tau_1, \cdots, \tau_k$ of K_{m+1}. Then, for each i $(i = 1, \cdots, k)$, $\sigma(\tau_i) = \sigma$. Notice that for any τ in K_{m+1}, we have $U_\tau^{m+1} \subset U_{\sigma(\tau)}^m$; therefore, $U_{\tau_i}^{m+1} \subset U_\sigma^m$ for $i = 1, \cdots, k$. It follows that $g[\overline{\tau_i}] \subset U_\sigma^m$ and hence $g[\overline{\sigma}] = g[\overline{\tau_1} \cup \cdots \cup \overline{\tau_k}] \subset U_\sigma^m$. Since σ was arbitrary, we have $g \in \bigcap_{\sigma \in K_m} N(\overline{\sigma}, U_\sigma^m)$, which in turn implies that $g \in O$ by inductive hypothesis (iv). This completes the proof of the induction.

In order to conclude the proof of the lemma, it is only necessary to let $K = K_{p+1}$ and, for $\sigma \in K$, let $U_\sigma = U_\sigma^{p+1}$.

The following theorem is the main result of this section.

3.3. THEOREM. *Let the pair (C, C_0) be finitely triangulable and let X be a locally aspherical space. Then the map $\rho: X^C \to X^{C_0}$ defined by $\rho(f) = f \,|\, C_0$ is a local fiber map.*

Proof. Let f be an arbitrary member of X^C and let O be an arbitrary open neighborhood of f in X^C. Then by Lemma 3.2, we can find a finite simplicial pair (K, L) and a family $\{U_\sigma: \sigma \in K\}$ of aspherical open subsets of X such that the pair $(|K|, |L|)$ is homeomorphic to the pair (C, C_0) and the family satisfies conditions (a) and (b) of the lemma. For the rest of the proof we shall use $(|K|, |L|)$ instead of (C, C_0). Let $W = \bigcap_{\sigma \in K} N(\bar{\sigma}, U_\sigma)$; then, by Lemma 3.1, the map $\rho \,|\, W: W \to \bigcap_{\sigma \in K} N(\bar{\sigma} \cap |L|, U_\sigma)$ is a Serre fiber map. First let us show that the fiber of $\rho \,|\, W$ through f, that is,

$$F = \{g: g \in W \ and \ f \,|\, |L| = f \,|\, |L|\},$$

is aspherical and hence acyclic.

Let ϕ be a continuous map $S^n \to F$ such that $\phi(*) = f$. We must show that ϕ is homotopic to the trivial map relative to the base point $*$. But this is equivalent to showing the existence of a map $\Phi: S^n \times I \times |K| \to X$ such that

(i) $\Phi(x, 0, y) = \phi(x)(y)$ for $(x, 0, y) \in S^n \times \{0\} \times |K|$,

(ii) $\Phi(*, t, y) = f(y)$ for $(*, t, y) \in \{*\} \times I \times |K|$,

(iii) $\Phi(x, 1, y) = f(y)$ for $(x, 1, y) \in S^n \times \{1\} \times |K|$,

(iv) $\Phi(x, t, y) = f(y)$ for $(x, t, y) \in S^n \times I \times |L|$, and

(v) $\Phi(x, t, y) \in U_\sigma$ for $(x, t, y) \in S^n \times I \times \bar{\sigma}$.

We shall define, by induction on m, a continuous map

$$\Phi_m: S^n \times I \times (|K^m| \cup |L|) \to X$$

satisfying (i)-(iv) for $y \in |K^m| \cup |L|$ and (v) for simplices σ of dim $\leq m$. (K^m denotes the m-skeleton of K.) Once this is done, the existence of Φ satisfying (i)-(v) is immediate. First define $\Phi_{-1}: S^n \times I \times |L| \to X$ by $\Phi_{-1}(x, t, y) = f(y)$. Now assume that Φ_m $(m \geq -1)$ has been defined. Let σ be an $(m+1)$-simplex of K which is not in L, and let

$A = S^n \times I \times \sigma$, and

$B = (S^n \times I \times \dot{\sigma}) \cup (S^n \times \{0\} \times \bar{\sigma}) \cup (\{*\} \times I \times \bar{\sigma}) \cup (S^n \times \{1\} \times \bar{\sigma}).$

Define a continuous map $g_\sigma: B \to X$ as follows: $g_\sigma \,|\, S^n \times I \times \dot{\sigma} = \Phi_m \,|\, S^n \times I \times \dot{\sigma}$, $g_\sigma(x, 0, y) = \phi(x)(y)$, $g_\sigma(*, t, y) = f(y)$ and $g_\sigma(x, 1, y) = f(y)$ for $x \in S^n$, $t \in I$, $y \in \bar{\sigma}$. If $\tau_1, \cdots, \tau_k$ are m-simplices of K such that $\bar{\tau}_1 \cup \cdots \cup \bar{\tau}_k = \dot{\sigma}$, then $\Phi_m[S^n \times I \times \bar{\tau}_i] \subset U_{\tau_i} \subset U_\sigma$, for $i = 1, \cdots, k$, by (a) of Lemma 3.2. Therefore, $g_\sigma[S^n \times I \times \dot{\sigma}] \subset U_\sigma$. Since $\phi(x)$ and f are in W,

$$g_\sigma[(S^n \times \{0\} \times \bar{\sigma}) \cup (\{*\} \times I \times \bar{\sigma}) \cup (S^n \times \{1\} \times \bar{\sigma})] \subset U_\sigma.$$

In short, $g_\sigma[B] \subset U_\sigma$. Since U_σ is aspherical and the pair (A, B) is triangulable, g_σ can be extended to $\bar{g}_\sigma \colon A \to U_\sigma$. Now extend Φ_m to Φ_{m+1} by agreeing that $\Phi_{m+1} | S^n \times I \times \bar{\sigma} = \bar{g}_\sigma$ for each $(m+1)$-simplex σ of K which is not in L. Clearly Φ_{m+1} satisfies (i)-(iv) for $y \in |K^{m+1}| \cup |L|$ and (v) for simplices σ of dim $\leq m + 1$. This completes the proof of the induction, and that F is aspherical is now established.

By letting $L = \emptyset$, we see that the set of the form $\bigcap_{\sigma \in K} N(\bar{\sigma}, U_\sigma)$ is aspherical. Therefore the open subset $\bigcap_{\sigma \in K} N(\bar{\sigma} \cap |L|, U_\sigma) = \bigcap_{\sigma \in L} N(\bar{\sigma}, U_\sigma)$ of $X^{|L|}$ is also aspherical, hence pathwise connected. Since

$$\rho | W \colon W \to \bigcap_{\sigma \in K} N(\bar{\sigma} \cap |L|, U_\sigma)$$

is a Serre fiber map, it follows that $\rho[W] = \bigcap_{\sigma \in K} N(\bar{\sigma} \cap |L|, U)$ and that ρ is open. This completes the proof of the theorem.

3.4. REMARK. In the course of the proof of Theorem 3.3, we have proved the following fact which is of independent interest: *If X is a locally aspherical space and if C is a finitely triangulable space, then X^C is also locally aspherical.* It is likely that the requirement of triangulability of C can be relaxed.

4. We may assemble various facts of the first three sections to obtain the following theorem on function spaces.

4.1. THEOREM. *Let E be a locally aspherical space, let (C, C_0) be a finitely triangulable pair, let $\rho \colon E^C \to E^{C_0}$ be the map defined by $\rho(f) = f | C_0$, and let E^C be paracompact. For any subset A of E^{C_0}, let us denote $\rho^{-1}[A]$ by A^*. Let X and Y be two closed subsets of E^{C_0} such that $Y \subset X$ and Y is locally acyclic. Then there is a spectral sequence such that $E_2^{p,q} = H^p(Y^*; (\rho | Y^*)^* (\mathscr{T}_G^q(X, Y)))$ and E_∞ is the graded group associated with $H^*(X^*, X^* \sim Y^*; G)$ suitably filtered.*

Proof. By Theorem 3.3 and Proposition 2.1, we see that $\rho | X^* \colon X^* \to X$ and $\rho | Y^* \colon Y^* \to Y$ are local fiber maps. It follows from Theorem 2.3 that $\mathscr{T}_G^q(X^*, Y^*) = (\rho | Y^*)^* \mathscr{T}_G^q(X, Y)$. Since E^C is paracompact, its closed subspace X^* is also paracompact. Furthermore, Y^* is locally acyclic because Y is locally acyclic (Proposition 2.2). Therefore, from Corollary 1.3, by taking Φ to be the family of all closed subsets of X^*, we obtain the spectral sequence described in the theorem.

The following corollary is now immediate.

4.2. COROLLARY. *In addition to the hypotheses of Theorem 4.1, assume that for $n > 0$, $\mathscr{T}_G^q(X, Y) = 0$ for $q \neq n$, and $\mathscr{T}_G^n(X, Y)$ is the simple sheaf of an Abelian group H over Y. Then there is an isomorphism*

$$H^p(Y^*; H) \xrightarrow{\cong} H^{p+n}(X^*, X^* \sim Y^*; G) \text{ for all } p.$$

In applications, C_0 is usually a finite set. If this is the case, E^{C_0} is simply the product $(E)^k$, where k is the cardinality of C_0, and the facts concerning $\mathscr{T}_G^q(X, Y)$ are relatively easy to obtain. On the other hand, to decide whether E^C is paracompact or not poses an interesting question. If E is metrizable and C is compact, then E^C is metrizable, hence paracompact. It follows that, if E is a locally finite CW-complex and C is a compact space, then E^C is paracompact. E. Michael remarked (orally) that, if E is a countable CW-complex and C is a compact *metrizable* space, then E^C is paracompact. For, if E is a countable CW-complex, then $E = \bigcup \{ K_i : i = 1, 2, \cdots \}$, where K_i is a finite subcomplex of E and $K_i \subset K_{i+1}$. Since each K_i is compact and metrizable, K_i^C is Lindelöf. It is easy to see that $E^C = \bigcup \{ K_i^C : i = 1, 2, \cdots \}$, and hence E^C is also Lindelöf. Obviously E^C is regular; therefore E^C is paracompact. This leads us to a conjecture that, if K is a CW-complex and C is a compact *metrizable* space, then K^C is paracompact. We remark that there are examples of compact spaces E such that E^I is not paracompact (see [9]).

Using Corollary 4.2, it is possible to restate many of Eells' results on function spaces without smoothness assumptions. In the following, we shall present an application of the spectral sequence of Theorem 4.1.

Let (E, p, B) be a sphere bundle. By the *Thom space* of the bundle, we mean the space $CE \cup_p B$, where CE is the unreduced cone over E.

4.3. THEOREM. *Assume that (E, p, B) is a sphere bundle with the fiber S^n ($n \geq 1$) such that B is 1-connected, compact, metrizable and locally aspherical. Let X be the Thom space of the bundle. Then, for any Abelian group G, $H^0(\Omega X; G) \cong H^n(\Omega X; G) \cong G$, and $H^q(\Omega X; G) = 0$ for $0 < q < n$. Furthermore, there is a spectral sequence such that $E_2^{p,q} = H^p(\Omega X; \check{H}^q(E; G))$ and, for $r > 0$, $\sum \{ E_\infty^{p,q} : p + q = r \}$ is the graded group associated with $H^{r+n}(\Omega X; G)$ suitably filtered. (The differential operator d_k maps $E_k^{p,q}$ into $E_k^{p+k, q-k+1}$.)*

Proof. Clearly E is compact, metrizable and locally aspherical; therefore the same is true of $X = CE \cup_p B$ also. (Note that a quotient space of a compact metrizable space is metrizable if it is Hausdorff.) Using the local product structure of E, it can be seen that $\mathscr{T}_G^q(X, B) = 0$ if $q \neq n + 1$ and $\mathscr{T}_G^{n+1}(X, B)$ is a locally simple sheaf with each stalk isomorphic to G. Since we are assuming that B be 1-connected, it follows that $\mathscr{T}_G^{n+1}(X, B)$ is the simple sheaf of the group G over B. Let us denote by x_0 the vertex of the cone CE and by X^* the space of paths in X emanating from x_0. Let $\rho : X^* \to X$ be the usual map defined by $\rho(l) = l(1)$, and, for each subset A of X, let $A^* = \rho^{-1}[A]$. Let $C = I$ and $C_0 = \{0, 1\}$, and we apply Corollary 4.2 to two subspaces $\{x_0\} \times B \subset \{x_0\} \times X$ of X^{C_0}. It follows that $H^p(B^*; G) \to^\cong H^{p+n+1}(X^*, X^* \sim B^*; G)$ for all p. However, X^* is contractible to a point and $X^* \sim B^* = \rho^{-1}[X \sim B]$, where $X \sim B$ is contractible to a point over itself. Therefore

$$\tilde{H}^{p+n}(\Omega X; G) \cong \tilde{H}^{p+n}(X^* \sim B^*; G) \underset{\delta}{\overset{\cong}{\longrightarrow}} H^{p+n+1}(X^*, X^* \sim B^*; G),$$

and we obtain an isomorphism:

$$H^p(B^*; G) \cong \tilde{H}^{p+n}(\Omega X; G) \quad \text{for all } p.$$

Since B is 1-connected, so is X. Hence both B^* and ΩX are pathwise connected. Therefore, $H^0(\Omega X; G) \cong G$, $H^q(\Omega X; G) = 0$ for $0 < q < n$ and $H^n(\Omega X; G) \cong G$.

Now we shall repeat a similar argument with a different closed subset of X, namely $\{x_0\}$. Obviously $\mathcal{T}_G^q(X, \{x_0\}) = \tilde{H}^{q-1}(E; G)$. Applying Theorem 4.1 to two subspaces $\{x_0\} \times \{x_0\}$ and $\{x_0\} \times X$ of X^{C_0}, we obtain a spectral sequence such that

$$'E_2^{p,q} = H^p(\Omega X; \tilde{H}^{q-1}(E; G)),$$

and $\sum_{p+q=r} 'E_\infty^{p,q}$ is the graded group associated with $H^r(X^*, X^* \sim \{x_0\}^*; G)$. But, since B is a deformation retract of $X \sim \{x_0\}$, the inclusion

$$B^* \to X^* \sim \{x_0\}^*$$

is a homotopy equivalence. Therefore we have, for $r > 1$,

$$H^r(X^*, X^* \sim \{x_0\}^*; G) \underset{\delta}{\overset{\cong}{\longrightarrow}} H^{r-1}(X^* \sim \{x_0\}^*; G)$$

$$\overset{\cong}{\longrightarrow} H^{r-1}(B^*; G)$$

$$\cong H^{r+n-1}(\Omega X; G).$$

The spectral sequence of the theorem is obtained by putting $E_k^{p,q} = {}'E_k^{p,q+1}$.

4.4. EXAMPLE. To illustrate what sort of results one gets from the spectral sequence of Theorem 4.3, we shall consider the following simple example: let n be an even integer $(n \geq 2)$, $E = V_{n+1,2}$, and let $p: V_{n+1,2} \to S^n$ be the usual fiber map with the fiber S^{n-1}. We know that $H_0(E; Z) \cong H_{2n-1}(E; Z) \cong Z$, $H_{n-1}(E; Z) \cong Z_2$ and the rest of the homology groups are zero. Let us first consider the case where $G = Q$, the group of rational numbers. Let $X = CV_{n+1,2} \cup_p S^n$; then, by Theorem 4.3, we have

$$H^0(\Omega X; Q) \cong H^{n-1}(\Omega X; Q) \cong Q \quad \text{and} \quad H^q(\Omega X; Q) = 0$$

for $0 < q < n$. Also, in the spectral sequence of Theorem 4.3, $E_2^{p,q} = 0$ if $q \neq 2n - 1$ and $E_2^{p,2n-1} = H^p(\Omega X; Q)$. Therefore we see that $H^p(\Omega X; Q) = 0$ for $n - 1 < p < 3n - 2$ and that $H^{p+3n-2}(\Omega X; Q) \cong H^p(\Omega X; Q)$ for all $p \geq 0$. Hence we conclude that

$$H^q(\Omega X; Q) \cong \begin{cases} Q, & \text{if } q = k(3n - 2) \quad (k \geq 0), \\ Q, & \text{if } q = k(3n - 2) + (n - 1) \quad (k \geq 0), \\ 0, & \text{otherwise.} \end{cases}$$

The situation, when $G = Z_p$ (p an odd prime), is entirely analogous to the above case.

Next let us consider the case $G = Z$. In this case the E_2-terms of the spectral sequence are given by:

$$E_2^{p,q} = \begin{cases} H^p(\Omega X; Z_2), & \text{if } q = n, \\ H^p(\Omega X; Z), & \text{if } q = 2n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Using the fact that $H^0(\Omega X; Z) \cong H^{n-1}(\Omega X; Z) = Z$ and that $H^p(\Omega X; Z) = 0$ for $0 < p < n - 1$, we can get started in getting E_∞-terms which, in turn, yield additional terms of E_2. When $n \geq 4$ and $q \leq 6n - 5$, one can read off from the spectral sequence the following nonzero groups $H^q(\Omega X; Z)$:

q	0	$n-1$	$2n-1$	$3n-2$	$4n-3$	$4n-2$	$5n-4$
$H^q(\Omega X; Z)$	Z	Z	Z_2	$Z + Z_2$	$Z + Z_2$	0 or Z_2	Z_2

q	$5n-3$	$6n-5$	$\cdots$
$H^q(\Omega X; Z)$	$Z_2 + Z_2 + Z_2$ or $Z_2 + Z_4$	Z_2	$\cdots$

,

where

$$H^{4n-2}(\Omega X; Z) = \operatorname{coker}(d_n : E_n^{n-1,2n-1} \to E_n^{2n-1,n}).$$

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