OSCILLATION PROPERTIES OF EVEN-ORDER LINEAR DIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** The ordinary, self-adjoint differential equation

$$[r(x)y^{(n)}]^{(n)} + p(x)y = 0 \tag{A}$$

is considered for $r(x)$ and $p(x)$ continuous and $r(x) > 0$, $p(x) \neq 0$ on $[a, \infty)$. For $n = 1$, this equation has been the subject of many investigations over a number of years. For $n = 2$, J. H. Barrett [1]-[3] and W. Leighton and Z. Nehari [7] have recently considered equations of this type. For arbitrary n, similar equations have been investigated by W. T. Reid [8], by H. Kaufman [6], by H. M. and R. L. Sternberg [6], [9], and by the author [5].

In the present paper, a general boundary problem is considered for a solution $y(x)$ of (A); namely,

$$\begin{aligned} &y(a) = y'(a) = \cdots = y^{(n-1)}(a) = y_1(a) = \cdots = y_1^{(n-i-1)}(a) = 0, \\ &y(b) = y'(b) = \cdots = y^{(i-1)}(b) = 0 \end{aligned} \tag{B}$$

for $1 \leqq i \leqq n, b > a$, and $y_1(x) \equiv r(x)y^{(n)}(x)$. If $y(x)$ satisfies the first of these conditions, then $y(x)$ is said to be a solution of (A) with a zero of order $2n - i$ at $x = a$, i.e., after the $(n-1)$st derivative of $y(x)$, derivatives of $y_1(x)$ are used. For $i = n$, this is similar to boundary problems considered in [1]-[9]. In general, however, the techniques utilized in these latter investigations are dependent on the fact that $i = n$. In §2, a basic property of zeros of solutions of (A) is established. In §§3 and 4, separation and oscillation properties of solutions of (A) are established for $p(x) > 0$ on $[a, \infty)$.

2. **A basic property of zeros of solutions of** (A). Consider the equations

$$[r(x)y^{(n)}]^{(n)} + p(x)y = 0, \tag{1}$$

$$[r(x)y^{(n)}]^{(n)} - p(x)y = 0, \tag{2}$$

both of which have $r(x)$ and $p(x)$ continuous and positive on $[a, \infty)$. An

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important general property of zeros of solutions of (1) and (2) is then contained in the following theorem.

THEOREM 1. *If i is an even* [*odd*] *positive integer and $y(x)$ is a nontrivial solution of* (1) [(2)] *with a zero of order $\geqq 2n - i$ at $x = a$, then $y(x)$ cannot have a zero of order $\geqq i$ on (a, ∞), i.e., there does not exist a nontrivial solution satisfying the conditions* (B).

Proof. Suppose that, for $i \leqq n, y(x)$ has a zero of order $2n - i$ at $x = a$ and has a zero of order i at $x = b$ on (a, ∞) and that $y(x)$ has m zeros on (a, b), $m \geqq 0$. Let $c \in (a, b]$ be the first zero of $y(x)$ on $(a, b]$. It shall then first be established that $y_1^{(n-1)}(x)$ must have a zero on $(a, c]$, regardless of whether i is even or odd.

Note that $y(a) = y'(a) = \cdots = y_1(a) = y_1'(a) = \cdots = y_1^{(n-i-1)}(a) = 0$ (where, if $i = n$, $y_1^{(-1)}(a)$ is interpreted as $y^{(n-1)}(a)$) and $y(b) = y'(b) = \cdots = y^{(i-1)}(b) = 0$. Then successive application of Rolle's Theorem to $y(x)$, $y'(x), \cdots, y^{(i)}(x)$ gives that $y^{(i)}(x)$ has at least $m + i$ zeros on (a, b), a zero of order $2(n - i)$ at $x2\ a$, and does not vanish at $x = b$. Continuation of the process shows that $y_1(x)$ has at least $m + i$ zeros on (a, b), a zero of order $n - i$ at $x = a$, and does not vanish at $x = b$. Then finally, $y_1^{(n-1)}(x)$ must have at least $m + 1$ zeros on (a, b). Suppose that $y_1^{(n-1)}(x) \neq 0$ on $(a, c]$. Then $y_1^{(n-1)}(x)$ has at least $m + 1$ zeros on (c, b) and hence $y_1^{(n)}(x)$ has at least m zeros on (c, b). But $y_1^{(n)}(x) = \pm p(x)y(x)$ and thus $y(x)$ would have m zeros on (c, b), a contradiction. (If $m = 0$, the last three statements are not necessary.) Note that $y_1^{(n-1)}(x)$ has exactly $m + 1$ zeros on (a, b).

Now let $x = x_0$ be the first zero of $y_1^{(n-1)}(x)$ on $(a, c]$ and assume, with no loss of generality, that $y(x) > 0$ on (a, c). Then $y_1^{(n-i)}(a) \geqq 0$ regardless of whether i is even or odd. From (1)

$$y_1^{(n-1)}(x_0) - y_1^{(n-1)}(a) = - y_1^{(n-1)}(a) = - \int_a^{x_0} p(x)y\, dx < 0,$$

and $y_1^{(n-1)}(a) > 0$. Similarly, using (2), $y_1^{(n-1)}(a) < 0$. If $i = 1$, the last statement is a contradiction and the special case of the theorem for $i = 1$ and equation (2) is established.

Suppose $i > 1$ and note that $y_1^{(n-2)}(x)$ must have a zero on (a, x_0) since the $m + 1$ zeros of $y_1^{(n-1)}(x)$ fall between the extreme zeros of $y_1^{(n-2)}(x)$ on (a, b). This comes from the first part of the proof. Also, $y_1^{(n-2)}(x)$ has exactly $m + 2$ zeros on (a, b). Similarly, $y_1^{(n-3)}(x)$ must have a zero before the first zero of $y_1^{(n-2)}(x)$ on (a, x_0); and $y_1^{(n-3)}(x)$ has exactly $m + 3$ zeros on (a, b). This same type of reasoning can then be followed to establish that $y_1^{(n-i)}(x)$ must have a zero on (a, x_0) before the first zero on (a, x_0) of $y_1^{(n-i+1)}(x)$ and

$y^{(n-i)}(x)$ has exactly $m+i$ zeros on (a,b). If $i=1$, of course, this entire procedure consists only of the statement that $y^{(n-1)}(x)$ vanishes at x_0 on $(a,c]$.

Then, using (1), $y_1^{(n-2)}(x)$ must begin at $x=a$ with positive slope and have a zero before its derivative has a zero. This is possible only if $y_1^{(n-2)}(a)<0$. Then $y_1^{(n-3)}(x)$ must begin at $x=a$ with negative slope and have a zero before its derivative does so, implying that $y_1^{(n-3)}(a)>0$. Successive applications of this argument yield the fact that $y_1^{(n-i)}(a)<0$ if i is an even integer. This is a contradiction since $y_1^{(n-i)}(a)>0$. Hence, the portion of the theorem regarding (1) and even values of i is established. In an analogous way, using (2), $y_1^{(n-i)}(a)<0$ if i is odd; and a contradiction results.

The theorem is now proved for $i \leqq n$. For $i>n$, it suffices to note that the change of variable $x=a-t+b$ leaves (1) and (2) unchanged while transforming $x=a$ into $t=b$ and $x=b$ into $t=a$. Thus, a solution $y(x)$ with a zero of order $2n-i$ at $x=a$ and order i at $x=b$ is transformed into a solution with a zero of order i at $t=a$ and order $2n-i$ at $t=b$. Then the case $i>n$ can be taken care of by the above proof. This completes the proof of the theorem.

Included in Theorem 1 is a generalization of Lemma 8.2 in [7, p. 358], which states that no nontrivial solution of (1) for $n=2$ has more than one double zero (i.e., a zero of order two). Theorem 1 includes the fact that no nontrivial solution of (1) has more than one nth-order zero. In fact, using the following definition, the following corollary holds.

Definition. The number $\eta_1(a)$ is the smallest number b on (a,∞) such that the boundary conditions

$$y(a)=y'(a)=\cdots=y^{(n-1)}(a)=0=y(b)=y'(b)=\cdots=y^{(n-1)}(b)$$

are satisfied nontrivially by a solution $y(x)$ of (1) or (2). This is the type of boundary problem considered in [1], [3], [5], [7], [8], [9].

Corollary 1. *If n is even [odd], then $\eta_1(a)$ does not exist for* (1) [(2)].

3. **Separation properties of solutions of (1).** First of all, Theorem 1 can be utilized to give the following generalization of Lemma 9.1 in [7, p. 361].

Theorem 2. *If $y(x)$ and $z(x)$ are linearly independent solutions of* (1) *which have zeros of order $\geqq 2n-2$ at $x=a$, then the zeros of $y(x)$ and $z(x)$ separate each other on (a,∞).*

Proof. If $y(x)$ has consecutive zeros between which $z(x)$ has no zero, a basic result contained in [7, Lemma 1.2, p. 327] gives the existence of a constant k such that $y(x)-kz(x)$ has a double zero in the interval between the zeros of $y(x)$ under consideration. But $y(x)-kz(x)$ is a solution of (1) with a zero of order $2n-2$ at $x=a$ and cannot have a zero of order two on (a, ∞) by Theorem 1. Also, $y(x)$ and $z(x)$ cannot have a zero in common for then a constant k could be chosen so that $y(x) - kz(x)$ has a double zero at the common zero, a contradiction. Therefore, the zeros of $y(x)$ and $z(x)$ must separate each other on (a, ∞).

Let $\{u_i(x)\}$ $(i = 1, 2, 3, \cdots, n)$ be a set of solutions of (1) which satisfy the boundary conditions

$$\begin{aligned} &[u_i(x)]^{(j-1)} = 0 \quad \text{for } x = a \quad \text{and } i, j = 1, 2, \cdots, n, \\ (4) \qquad &[u_{i,1}(x)]^{(j-1)} = \delta_{ij} \quad \text{for } x = a \quad \text{and } i, j = 1, 2, \cdots, n \\ &\text{and } u_{i,1}(x) = r(x) u_i^{(n)}(x). \end{aligned}$$

Also, let $\{U_i(x)\}$ $(i = 1, 2, \cdots, n)$ be a set of solutions of (1) satisfying the conditions

$$\begin{aligned} &[U_i(x)]^{(j-1)} = \delta_{ij} \quad \text{for } x = a \quad \text{and } i, j = 1, 2, \cdots, n, \\ (5) \qquad &[U_{i,1}(x)]^{(j-1)} = 0 \quad \text{for } x = a \quad \text{and } i, j = 1, 2, \cdots, n \\ &\text{and } U_{i,1}(x) = r(x) U_i^{(n)}(x). \end{aligned}$$

THEOREM 3. *Consider the sets* $\{u_i(x)\}$ *and* $\{U_i(x)\}$ $(i = 1, 2, \cdots, n)$ *of solutions of* (1) *satisfying the conditions of* (4) *and* (5), *respectively. Then the zeros of* $u_n(x)$ *and the zeros of any member of the set* $\{U_i(x)\}$ $(i = 1, 2, \cdots, n)$, *or of* $\{u_i(x)\}$ $(i = 1, 2, \cdots, n - 1)$, *separate each other on* (a, ∞).

Proof. The proof uses similar techniques to that of Theorem 1. Consider first, the set $\{U_i(x)\}$ $(i = 1, 2, \cdots, n)$, and let $U_i(x)$ be one of the members of the set and $y_i(x)$ any linear combination of $U_i(x)$ and $u_n(x)$. If it can be established that $y_i(x)$ cannot have a double zero on (a, ∞), it will follow as in the proof of Theorem 2 that the zeros of $u_n(x)$ and $U_i(x)$ must separate each other. Thus, suppose that $y_i(x)$ does have a double zero on (a, ∞) at $x = b$ and that $y_i(x)$ has m zeros on (a, b), $m \geqq 0$. Also, let $x = c$ on (a, b) be the first of these zeros to the right of $x = a$. It will first be established that, under these assumptions, $y_{i,1}^{(n-1)}(x)$ must have a zero on $(a, c]$. Recall $y_{i,1}(x) = r(x) y_i^{(n)}(x)$.

Successive applications of Rolle's Theorem to $y_i(x), y_i'(x), \cdots, y_i^{(i-1)}(x)$ show that $y_i^{(i)}(x)$ must have at least $m + 1$ zeros on (a, b). Then, similarly $y_{i,1}^{(n-1)}(x)$ must have at least $m + 1$ zeros on (a, b). Suppose $y_{i,1}^{(n-1)}(x)$ does not have a zero on $(a, c]$. Then it has $m + 1$ zeros on (c, b) and hence $y_{i,1}^{(n)}(x)$, or $y_i(x)$, has m zeros on (c, b), a contradiction. Note that if $m = 0$ and hence $b = c$, the latter part of the argument is unnecessary. In any case, it follows that $y_{i,1}^{(n-1)}(x)$ has a zero on $(a, c]$. Note that $y_{i,1}^{(n-1)}(x)$ has exactly $m +$ 1 zeros on (a, b). Note, also, from the above argument that $y_i^{(i)}(x)$ must have exactly $m+1$ zeros on (a, b) and that these zeros lie between the first and last zeros of $y_i^{(i-1)}(x)$ on (a, b).

Now, let $x = x_0$ be the first zero of $y_{i,1}^{(n-1)}(x)$ on $(a, c]$ and assume, with no loss of generality, that $y(x) > 0$ on (a, c). This implies that $y^{(i-1)}(a) > 0$. Then

$$y_{i,1}^{(n-1)}(x_0) - y_{i,1}^{(n-1)}(a) = -y_{i,1}^{(n-1)}(a) = -\int_a^{x_0} p(x)y\,dx < 0,$$

and $y_{i,1}^{(n-1)}(a) > 0$. Thus $y_{i,1}^{(n-2)}(x)$ must be zero and have positive slope at $x = a$ which implies that $y_{i,1}^{(n-2)}(x) > 0$ in a right neighborhood of $x = a$. Next, $y_{i,1}^{(n-3)}(x)$ must be zero and have zero slope at $x = a$, with positive slope in a right neighborhood of $x = a$, so that $y_{i,1}^{(n-3)}(x) > 0$ in a right neighborhood of $x = a$. This same property must hold for the succeeding functions $y_{i,1}^{(n-4)}(x), \cdots, y_i^{(i)}(x)$. Then $y^{(i-1)}(x)$ must begin at $x = a$ with positive slope and have positive slope in a right neighborhood of $x = a$. Furthermore, by the comment following the first part of the proof, $y^{(i-1)}(x)$ must have a zero before its derivative $y^{(i)}(x)$ has a zero on (a, b). These properties are possible only if $y^{(i-1)}(a) < 0$, a contradiction. Thus, the zeros of $u_n(x)$ and those of any member of the set $\{U_i(x)\}$ $(i = 1, 2, \cdots, n)$, separate each other.

The proof of the latter part of the theorem follows the same lines as that above. Using the same notation, the proof that $y_{i,1}^{(n-1)}(x)$ must have a zero on $(a, c]$ is obtained as before by successive applications of Rolle's Theorem. Hence, in this case, $y_{i,1}^{(i)}(x)$ has at least $m+1$ zeros on (a, b); and, furthermore, $y_{i,1}^{(n-1)}(x)$ has at least $m+1$ zeros on (a, b). If $y_{i,1}^{(n-1)}(x)$ did not have a zero on $(a, c]$, a contradiction would result as before. Also, $y_{i,1}^{(i)}(x)$ must have exactly $m+1$ zeros on (a, b) and these lie between the first and last zeros of $y_{i,1}^{(i-1)}(x)$ on (a, b).

Finally, as before, it follows that upon assuming $y(x) > 0$ on (a, c), and hence that $y_{i,1}^{(i-1)}(a) > 0$, $y_{i,1}^{(i-1)}(x)$ must have slope zero at $x = a$ and positive slope in a right neighborhood of $x = a$. Furthermore, $y_{i,1}^{(i-1)}(x)$ has a zero before its derivative has a zero. This gives $y_{i,1}^{(i-1)}(a) < 0$, a contradiction. This completes the proof of the theorem.

4. **Conditions for oscillation of solutions of (1).** The main result of this section is contained in the following theorem.

THEOREM 4. *Suppose that*

$$\int^{\infty} \left({}^{x}I^{n}\frac{t^{n-2}}{r}\right)p(x)\,dx = \infty$$

and

$$\int^{\infty} \left({}^{x}I^{n}\frac{p}{r(x)}\right)dx = \infty$$

(where ${}^xI^n$ *denotes the nth iterated integral). Then there exists a set of* $2n$ *linearly independent, oscillatory solutions (infinitely many zeros on* $[a, \infty)$) of (1) *with one oscillatory solution whose zeros separate the zeros of each of the other oscillatory solutions* $(n > 1)$.

Proof. Assume that the theorem is false, and consider the solution $y(x) \equiv u_n(x)$ with a zero of order $2n-1$ at $x = a$. Let $x = b$ be the last zero of $y(x)$ on $[a, \infty)$, and let m be the number of zeros of $y(x)$ on (a, ∞), $m \geqq 0$. All zeros of $y(x)$ on (a, ∞) must be simple by Theorem 1. Assume, with no loss of generality, that $y(x) > 0$ on (b, ∞). Then if m is even, $y(x) > 0$ between $x = a$ and the first zero of $y(x)$; if m is odd, $y(x) < 0$ between $x = a$ and the first zero of $y(x)$.

By repeated application of Rolle's Theorem, it follows that in each interval between consecutive zeros of $y(x)$ there exists exactly one simple zero (in order, right to left) of each of the functions $y'(x), y''(x), \cdots, y_1(x), y_1'(x), \cdots, y_1^{(n-1)}(x)$; and none of these functions have zeros coinciding with those of $y(x)$. If one of these functions had more than one zero on such an interval or a zero coinciding with one of $y(x)$, further application of Rolle's Theorem would show that $y_1^{(n)}(x) = -p(x)y(x)$ had a zero on that interval, an impossibility. Furthermore, $y'(x), y''(x), \cdots, y_1(x), y_1'(x), \cdots, y_1^{(n-1)}(x)$ are all positive for $x = b$ (unless $b = a$ in which case the functions are all positive in a right neighborhood of $x = b = a$) since they are all positive in a right neighborhood of $x = a$ for m even and all negative in a right neighborhood of $x = a$ for m odd and all vanish m times on (a, b).

It will now be shown that $y'(x), y''(x), \cdots, y_1(x), \cdots, y_1^{(n-2)}(x)$ do not have any zeros at all on (b, ∞) and hence are all positive on (b, ∞). Assume that $y'(x)$ has a zero at $x = c$ on (b, ∞). Then, by repeated use of Rolle's Theorem, $y''(x)$, $y'''(x), \cdots, y_1(x), \cdots, y_1^{(n-1)}(x)$ all have zeros on (b, c) and are thus all negative on (c, ∞), as is $y'(x)$, since another zero of any of these functions on (c, ∞) would lead to another zero of $y(x)$ on (b, ∞). But $y(x)$ cannot remain positive on (c, ∞) if $y'(x)'$ and $y''(x)$ are negative on that interval and hence a contradiction results. By the same reasoning, it follows that $y''(x)$, $y'''(x), \cdots, y^{(n-1)}(x)$ cannot have zeros on (b, ∞).

Next, suppose that $y_1(x)$ has a zero at $x = c$ on (b, ∞). Then $y_1'(x), y_1''(x), \cdots, y_1^{(n-1)}(x)$ must all have zeros on (b, c) and remain negative on (c, ∞) and $y_1(x)$ is negative on (c, ∞). Now

$$y^{(n-1)}(x) = y^{(n-1)}(c) + \int_c^x y^{(n)}(t)\,dt$$

$$= y^{(n-1)}(c) + \int_c^x \frac{y_1(t)}{r(t)}\,dt.$$

Since $y(x)$ is an increasing function on (c, ∞) and $y_1^{(n-1)}(c) < 0$, it follows that

$$y_1^{(n-1)}(x) = y_1^{(n-1)}(c) - \int_c^x p(t)y(t)\,dt$$
$$< -\int_c^x p(t)y(t)\,dt < -y(c)\int_c^x p(t)\,dt.$$

Further integrations and the conditions on $y_1(x), y_1'(x), \cdots, y_1^{(n-2)}(x)$ at $x = c$ thus yield

$$y_1(x) < -y(c)\,{}_c^xI^np,$$

where ${}_c^xI^np$ is the nth iterated integral of $p(x)$.

Then

$$y^{(n-1)}(x) = y^{(n-1)}(c) + \int_c^x \frac{y_1(t)}{r(t)}\,dt$$
$$< y^{(n-1)}(c) - y(c)\int_c^x \frac{({}_c^tI^np)}{r(t)}\,dt$$

and, using the hypothesis, $y^{(n-1)}(x) \to -\infty$ as $x \to \infty$. Then $y^{(n-1)}(x)$ must have a zero on (c, ∞), a contradiction. By the same argument as used for $y'(x), y''(x), \cdots, y^{(n-1)}(x)$, it now follows that $y_1'(x), y_1''(x), \cdots, y_1^{(n-2)}(x)$ cannot have zeros on (b, ∞).

Now $y_1^{(n)}(x) = -p(x)y(x)$ so that $y_1^{(n)}(x)$ is negative on (b, ∞) and hence $y_1^{(n-1)}(x)$ is a decreasing function on (b, ∞) and can have at most one zero on (b, ∞). Assume that $y_1^{(n-1)}(x)$ has no zeros on (b, ∞). Note that $y_1(x) \geqq x^{n-2}$ on (x_1, ∞) for some $x_1 \geqq b$ since $y_1, y_1', \cdots, y_1^{(n-1)}$ are all positive functions on (b, ∞). Furthermore,

$$y^{(n-1)}(x) = y^{(n-1)}(b) + \int_b^x \frac{y_1(t)}{r(t)}\,dt > \int_{x_1}^x \frac{t^{n-2}}{r(t)}\,dt.$$

Then, because of the conditions on $y(x), y'(x), \cdots, y^{(n-2)}(x)$ at $x = x_1$, $n-1$ integrations give

$$y(x) > {}_{x_1}^xI^n\frac{t^{n-2}}{r(t)}.$$

From these results,

$$\int_{x_1}^\infty y_1^{(n)}(x)\,dx = \lim_{x\to\infty} y_1^{(n-1)}(x) - y_1^{(n-1)}(x_1)$$
$$= -\int_{x_1}^\infty p(x)y(x)\,dx \leqq -\int_{x_1}^\infty \left({}^xI^n\frac{t^{n-2}}{r(t)}\right) p(x)\,dx = -\infty,$$

and hence $y_1^{(n-1)}(x)$ does have a zero for some value $x = c$ on (x_1, ∞). But then $y_1^{(n-2)}(x)$ is positive on (c, ∞) while $y_1^{(n-1)}(x)$ and $y_1^{(n)}(x)$ are negative on (c, ∞), an impossibility. Thus, $y(x) \equiv u_n(x)$ cannot have a "last" zero on (a, ∞) and must be oscillatory. From Theorem 3, the zeros of $u_n(x)$ separate those of any member of the sets $\{U_i(x)\}$ $(i = 1, 2, \cdots, n)$, and

$\{u_i(x)\}$ $(i = 1, 2, \cdots, n-1)$; and thus each of these solutions of (1) must oscillate. This completes the proof of the theorem.

COROLLARY 2. *If* $\int^\infty x^{2n-2}p(x)\,dx = \infty$ *and* $\int^\infty (x^{n-1}/r(x))\,dx = \infty$, *then there exists a set of* $2n$ *linearly independent solutions of* (1) *as in Theorem* 4.

Proof. $\int^\infty ({}^xI^n(t^{n-2}/r(t)))p(x)dx = \infty$ implies $\int^\infty x^{2n-2}p(x)\,dx = \infty$ and $\int^\infty ({}^xI^n p/r(t))\,dx = \infty$ implies that $\int^\infty (x^{n-1}/r(x))\,dx = \infty$. The result of the corollary then follows.

For $n = 2$, the following corollary generalizes Theorem 11.4 of [7] which Leighton and Nehari have for the case $r(x) \equiv 1$.

COROLLARY 3. *If* $\int^\infty ({}^xI^2 r)p(x)\,dx = \infty$ *and* $\int^\infty (({}^xI^2 p)/r(x))\,dx = \infty$, *then* (1) (*for* $n = 2$) *is oscillatory.*

Proof. By Corollary 9.10 of [7], the solutions of (1) for $n = 2$ are either all oscillatory or all nonoscillatory. Application of Theorem 4 then proves the desired result.

COROLLARY 4. *If* $\int^\infty ({}^xI^n(t^{n-2}/r(t)))p(x)\,dx = \infty$ *and* $\int^\infty ({}^xI^n p)/r(x)\,dx = \infty$, *then any solution of* (1) *with a zero of order* $2n-2$ *at* $x = a$ *is oscillatory.*

Proof. Theorem 2 and Theorem 4 combine to give this result.

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