BOCKSTEIN SPECTRA[1]

BY

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Introduction. If K and K' are chain complexes of abelian groups, one of which is torsion-free, then the well-known Künneth Theorem states that the additive group $H(K \otimes K')$ is completely determined by $H(K)$ and $H(K')$. However, if K and K' are differential graded rings, it is not necessarily true that the ring $H(K \otimes K')$ is completely determined by the rings $H(K)$ and $H(K')$ (cf. Palermo [8]).

It was originally stated by Bockstein [1] and later proved by Palermo [8] that for K and K' both torsion-free differential rings, the ring $H(K \otimes K')$ is completely determined by the so-called homology spectra of K and K'. The (multiplicative) homology spectrum of K consists of the rings $H(K)$ and $H(K \otimes Z_m)$ (for all $m > 0$), together with all the coefficient homomorphisms and Bockstein connecting homomorphisms. (This concept was first used by Bockstein and later by J. H. C. Whitehead [9].) In fact a stronger statement was proved: the (multiplicative) homology spectrum of $K \otimes K'$ is completely determined by the (multiplicative) homology spectra of K and K'. Unfortunately, however, the construction used in the proof was extremely involved and cumbersome.

In §1 the notion of an homology spectrum is generalized to the concept of (what we have called) a Bockstein spectrum and a structure-classification theorem is stated. In §2 the tensor product of Bockstein spectra is defined (it is not another spectrum, but just an abelian group). Some known theorems are rephrased in terms of spectra and several new theorems are proved. The most important of these (Theorem 2.2) states that if K, K', and K'' are chain complexes of torsion-free abelian groups, then the tensor product of the homology spectra of K, K', and K'' is precisely the group $H(K \otimes K' \otimes K'')$. If K, K', and K'' are differential graded rings, then it is possible to define a multiplication in the tensor product of the homology spectra so that the isomorphism of Theorem 2.2 becomes a ring isomorphism (Corollary 3.3). Another, though unrelated, consequence of Theorem 2.2 is a generalization of the Triple Künneth Theorem of MacLane [6]; this is omitted here and will be treated in a future paper.

In the last section, Theorem 2.2 is used to construct for each $m \geq 0$ a group $R_m(K, K')$ which, for K and K' torsion-free differential graded rings,

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is (ring) isomorphic to $H(K \otimes K')$ for $m = 0$ and to $H(K \otimes K' \otimes Z_m)$ for $m > 0$ and depends only on the (multiplicative) homology spectra of K and K'. This leads to a new proof of the fact that the (multiplicative) homology spectrum of $K \otimes K'$ is completely determined by the (multiplicative) homology spectra of K and K' (Theorem 4.2). The advantages of this proof are that it provides a (relatively) simple construction (especially with regard to products) and it treats both the case $m = 0$ and $m > 0$ in a more or less symmetric fashion.

1. Definition of Bockstein spectra.

DEFINITION 1.1. Let Z^+ denote the non-negative integers. A *Bockstein spectrum* is a collection of abelian groups $\{B_m | m \in Z^+\}$ such that for each pair $(m, k) \in Z^+ \times Z^+$ there are homomorphisms

$$\lambda_m^{mk} \colon B_{mk} \to B_m \quad \text{and} \quad \mu_{mk}^m \colon B_m \to B_{mk}$$

with the following properties:
(1) $\lambda_m^m = \mu_m^m$ is the identity map on B_m ($m \geqq 0$);
(2) $\lambda_m^{mk} \lambda_{mk}^{mhk} = \lambda_m^{mhk} \colon B_{mhk} \to B_m$ ($mhk \geqq 0$);
(3) $\mu_{mhk}^{mk} \mu_{mk}^m = \mu_{mhk}^m \colon B_m \to B_{mhk}$ ($mhk > 0$);
(4) $\mu_{mh}^m \lambda_m^{mk} = \lambda_{mh}^{mhk} \mu_{mhk}^{mk} \colon B_{mk} \to B_{mh}$ ($hk > 0$);
(5) $\lambda_m^{mk} \mu_{mk}^m$ is multiplication by k in B_m ($mk > 0$) and $\mu_0^m \lambda_m^0$ is the zero map on B_0;
(6) $\mu_0^m \lambda_m^{mk} = k\mu_0^{mk} \colon B_{mk} \to B_0$ ($m > 0$) and $\mu_{mk}^m \lambda_m^0 = k\lambda_{mk}^0 \colon B_0 \to B_{mk}$ ($m > 0$).

Some of the more useful examples of Bockstein spectra are the following. Let Z_0 denote the integers and Z_m ($m > 0$) the integers mod m. Let λ_m^{mk} be the canonical projection: $Z_{mk} \to Z_m$ and μ_{mk}^m ($k > 0$) the canonical injection: $Z_m \to Z_{mk}$. Define μ_0^m to be the zero map. Then it is easy to verify that (Z_m, λ, μ) is a Bockstein spectrum. It is a special case of the following example.

If K is a (not necessarily positive) chain complex of torsion-free abelian groups, let $H(K, 0) = H(K)$ denote the integral homology of K and let $H(K, m)$ ($m > 0$) denote the homology of the complex $K \otimes Z_m$ (with differential $\partial \otimes 1$). Define λ_m^{mk} ($mk \geqq 0$) and μ_{mk}^m ($k > 0$) to be the coefficient homomorphisms induced by the canonical projections and injections: $Z_{mk} \to Z_m$ and $Z_m \to Z_{mk}$. Let $\mu_0^m \colon H(K, m) \to H(K, 0) = H(K)$ be the Bockstein connecting homomorphism induced by the exact sequence:

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0.$$

This defines the homology spectrum of K; it is denoted by $\{H(K, m)\}$.

If G is any abelian group let $_mG = \{g \in G | mg = 0\}$ for $m > 0$ and let $_0G = 0$. Let $\lambda_m^{mk} \colon {}_{mk}G \to {}_mG$ be the map induced by multiplication by k in G and $\mu_{mk}^m \colon {}_mG \to {}_{mk}G$ the inclusion map. The result is a Bockstein spectrum

which we denote by $\{_mG\}$. $\{_mG\}$ can also be considered as the result of "composing" (in the obvious way) the spectrum $\{Z_m\}$ with the functor $- \otimes G$, (since $Z_m \otimes G = {_mG}$). Similarly any additive covariant functor from abelian groups to abelian groups can be composed with a spectrum to obtain a new spectrum.

Note that if we replace $_0G = 0$ with the group G and set $\lambda_m^0 = 0$, $\mu_0^m = $ inclusion, then we still obtain a Bockstein spectrum, although it is not the spectrum $\{_mG\}$ defined above. We denote this spectrum by $\{_mG^*\}$.

Still another Bockstein spectrum is obtained from G by taking the groups G/mG $(m \geq 0)$ and letting λ_m^{mk}: $G/mkG \to G/mG$ be the map induced on the quotients by the identity map on G and μ_{mk}^m: $G/mG \to G/mkG$ be the map induced by multiplication by k in G. As usual this spectrum is denoted by $\{G/mG\}$; it is the composition of the spectrum $\{Z_m\}$ with the functor $\mathrm{Hom}(-, G)$ (since $\mathrm{Hom}(Z_m, G) = G/mG$).

Finally there is the spectrum $\{mG\}$, where λ_m^{mk} is the inclusion map: $mkG \to mG$ and μ_{mk}^m: $mG \to mkG$ is induced by multiplication by k in G.

G also gives rise to a "constant" Bockstein spectrum by defining $B_m = G$ for all m and letting λ_m^{mk} be the identity and μ_{mk}^m multiplication by k (or vice-versa).

A map of Bockstein spectra f: $B \to B'$ is a collection of homomorphisms f_m: $B_m \to B'_m$ which commute with the various λ's and μ's in the obvious way. It is not difficult to show that the Bockstein spectra form an abelian category.

If B is a Bockstein spectrum, the primary part of B is the collection of groups $\{B_0, B_{p^i}\}$ for all prime integers p and $i \geq 0$, together with the corresponding λ's and μ's. Any Bockstein spectrum B is completely determined by its primary part, as is shown in the structure theorem below. In the case where the group $B_1 = 0$ (as in the spectrum $\{Z_m\}$ and most of the other examples above) this theorem essentially states the fact that $B_m = \sum_{i=1}^t B_{p_i^{m_i}}$, where m has a prime decomposition $m = p_1^{m_1} \cdots p_t^{m_t}$. This of course, is well known for most of the examples given above. Since we do not have any applications of this theorem at present we shall not prove it here. The proof is not particularly deep, but does get somewhat complicated.

We use the following notation. If A is an abelian group, then $\sum_t A$ denotes the direct sum of t copies of A. $\triangle_A$: $A \to \sum_t A$ is the map given by $\triangle_A(a) = (a, \cdots, a)$; ∇_A: $\sum_t A \to A$ is the map given by $\nabla_A(a_1, \cdots, a_t) = a_1 + \cdots + a_t$ $(a, a_i \in A)$. If B is a Bockstein spectrum and $m > 0$ has a prime decomposition $m = p_1^{m_1} \cdots p_t^{m_t}$, then the map ϕ_m: $\sum_{i=1}^t B_{p_i^{m_i}} \to B_m$ is the composition $\nabla_{B_m}(\sum_{i=1}^t \mu_m^{p_i^{m_i}})$. ϕ_m^* is the map induced by ϕ_m on $\sum_{i=1}^t B_{p_i^{m_i}} / \ker \phi_m$. Note that this notation is somewhat ambiguous since m has other prime decompositions (differing from this one by factors of

the form p^0) which (if $B_0 \neq 0$) give rise to different groups $\sum B_{p_i^{m_i}}$ and maps ϕ_m.

THEOREM 1.1. *If B is a Bockstein spectrum, then B is completely determined by its primary part. In fact if $m, k > 0$ have prime decompositions $m = p_1^{m_1} \cdots p_t^{m_t}$, $k = p_1^{k_1} \cdots p_t^{k_t}$, then:*

(1) *for $m > 0$, $\sum_{i=1}^{t} B_{p_i^{m_i}} / \ker \phi_m$ (for convenience call this group SB_m) is independent of the choice of prime decomposition of m and $\phi^* = \phi_m^*$: $SB_m \cong B_m$;*

(2) *under the isomorphism ϕ^*, the maps $\lambda_m^{mk}: B_{mk} \to B_m$, $\mu_{mk}^m: B_m \to B_{mk}$ $(mk > 0)$, $\lambda_m^0: B_0 \to B_m$, and $\mu_0^m: B_m \to B_0$ correspond to the maps induced on the respective quotients by:*

$$\sum_{i=1}^{t} \hat{k}_i \lambda_{p_i^{m_i}}^{p_i^{m_i+k_i}} : \sum_{i=1}^{t} B_{p_i^{m_i+k_i}} \to \sum_{i=1}^{t} B_{p_i^{m_i}} \qquad (where \ \hat{k}_i = k/p_i^{k_i}),$$

$$\sum_{i=1}^{t} \mu_{p_i^{m_i+k_i}}^{p_i^{m_i}} : \sum_{i=1}^{t} B_{p_i^{m_i}} \to \sum_{i=1}^{t} B_{p_i^{m_i+k_i}},$$

$$\left(\sum_{i=1}^{t} r_i \lambda_{p_i^{m_i}}^{0} \right) \triangle_{B_0} : B_0 \to \sum_{i=1}^{t} B_{p_i^{m_i}} \qquad (where \ \sum_{i=1}^{t} r_i(m/p_i^{m_i}) = 1),$$

$$\nabla_{B_0} \left(\sum_{i=1}^{t} \mu_0^{p_i^{m_i}} \right) : \sum_{i=1}^{t} B_{p_i^{m_i}} \to B_0.$$

2. **The tensor product of Bockstein spectra.** In the following definitions we shall allow the possibility that for any given Bockstein spectrum B, B_m is a graded abelian group and λ and μ are maps of graded groups. We should remark, however, that in actual practice the only map which may possibly have nonzero degree will be μ_0^m (in the spectrum $\{H(K, m)\}$, where K is a complex of torsion-free abelian groups). The degree of a group element u or a map f will be denoted by $|u|$ and $|f|$, respectively. The signs in the definitions below are those which arise from the usual sign convention.

DEFINITION 2.1. If B and $\bar{B}$ are Bockstein spectra, then $B \otimes \bar{B}$ is the abelian group $[\sum_{m \geq 0} (B_m \otimes \bar{B}_m)]/S$, where S is the subgroup generated by all elements of the form:

(1) $\qquad \lambda_m^{mk} u_{mk} \otimes v_m - (-1)^{|\bar{\mu}| \cdot |u_{mk}|} u_{mk} \otimes \bar{\mu}_{mk}^m v_m \qquad (mk \geq 0),$

(2) $\qquad \mu_{mk}^m u_m \otimes v_{mk} - (-1)^{|\bar{\lambda}| \cdot |u_m|} u_m \otimes \bar{\lambda}_m^{mk} v_{mk} \qquad (mk \geq 0),$

(3) $\qquad \lambda_m^0 \mu_0^m u_m \otimes v_m + (-1)^{(|\bar{\lambda}| + |\bar{\mu}|) |u_m|} u_m \otimes \bar{\lambda}_m^0 \bar{\mu}_0^m v_m \qquad (m > 0),$

where $u_i \in B_i$ and $v_i \in \bar{B}_i$.

DEFINITION 2.2. If B, $\bar{B}$, and $\widetilde{B}$ are Bockstein spectra, then $B \otimes \bar{B} \otimes \widetilde{B}$ is the abelian group $[\sum_{m \geq 0} (B_m \otimes \bar{B}_m \otimes \widetilde{B}_m)]/S$, where S is the subgroup

generated by all elements of the form:

(1)
$$(-1)^{|\bar{\lambda}||u_{mk}|}\lambda_m^{mk}u_{mk}\otimes\overline{\lambda}_m^{mk}v_{mk}\otimes w_m$$
$$-(-1)^{(|u_{mk}|+|v_{mk}|)|\bar{\mu}|}u_{mk}\otimes v_{mk}\otimes\widetilde{\mu}_{mk}^m w_m \qquad (mk\geq 0),$$

(2)
$$(-1)^{(|u_{mk}|+|v_m|)|\bar{\lambda}|}\lambda_m^{mk}u_{mk}\otimes v_m\otimes\widetilde{\lambda}_m^{mk}w_{mk}$$
$$-(-1)^{|u_{mk}||\bar{\mu}|}u_{mk}\otimes\overline{\mu}_{mk}^m v_m\otimes w_{mk} \qquad (mk\geq 0),$$

(3)
$$(-1)^{|u_m||\bar{\lambda}|+(|u_m|+|v_{mk}|)|\bar{\lambda}|}u_m\otimes\overline{\lambda}_m^{mk}v_{mk}\otimes\widetilde{\lambda}_m^{mk}w_{mk}$$
$$-\mu_{mk}^m u_m\otimes v_{mk}\otimes w_{mk} \qquad (mk\geq 0),$$

(4)
$$\lambda_m^\cup\mu_0^m u_m\otimes v_m\otimes w_m+(-1)^{|u_m|(|\bar{\lambda}|+|\bar{\mu}|)}u_m\otimes\overline{\lambda}_m^\cup\overline{\mu}_0^m v_m\otimes w_m$$
$$+(-1)^{(|u_m|+|v_m|)(|\bar{\lambda}|+|\bar{\mu}|)}u_m\otimes v_m\otimes\widetilde{\lambda}_m^\cup\widetilde{\mu}_0^m w_m \qquad (m>0),$$

where $u_i\in B_i$, $v_i\in\overline{B}_i$, $w_i\in\widetilde{B}_i$.

Although it is possible to combine these definitions into a single statement, it is convenient here to have separate statements. The tensor product of spectra can be used to formulate several concepts, as in the following theorems.

PROPOSITION 2.1. *If A, B, and C are abelian groups, then*

$$\mathrm{Tor}_1(A,B)\cong\{_mA\}\otimes\{_mB\}$$

and

$$\mathrm{Trip}_2(A,B,C)\cong\{_mA\}\otimes\{_mB\}\otimes\{_mC\}.$$

THEOREM 2.1. *If K is a chain complex of torsion-free abelian groups while G is any abelian group, then*

$$H_*(K\otimes G)\cong\{H(K,m)\}\otimes\{_mG^*\}.$$

Proposition 2.1 follows immediately from the description of Tor and Trip by generators and relations given in MacLane [6]. (N. B. In this paper $\mathrm{Trip}_2(A,B,C)$ is called $\mathrm{Tor}(A,B,C)$.) The complete proof of Theorem 2.1 is in Bockstein [4] (in Russian); a partial proof (in French) is in Bockstein [2]. In Bockstein [3], [5] the concept of the tensor product of (two) spectra is used to prove both the Universal Coefficient and Künneth Theorems.

Hereafter we shall use the following notation. If K is a chain complex and $u\in K_p$ is a cycle, then $\eta(u)$ denotes the homology class of u. If $\partial u = mu'$, then $\eta(u\otimes 1_m)$ denotes the homology class of the m-cycle $u\otimes 1_m$ in $K\otimes Z_m$. If K, K', and K'' are torsion-free chain complexes we introduce a grading in the spectra tensor product

$$\{H(K,m)\}\otimes\{H(K',m)\}\otimes\{H(K'',m)\}$$

by defining the degree of the (coset of the) element $\eta(u\otimes 1_m)\otimes\eta(v\otimes 1_m)$

$\otimes \, \eta(w \otimes 1_m)$ to be $|u| + |v| + |w|$ if $m = 0$ and $|u| + |v| + |w| - 1$ if $m > 0$. We now state the main theorem of this section.

THEOREM 2.2. *If K, K', and K'' are chain complexes of torsion-free abelian groups, then there is an isomorphism of graded groups:*

$$\{ H(K,m) \} \otimes \{ H(K',m) \} \otimes \{ H(K'',m) \} \cong H_*(K \otimes K' \otimes K'').$$

Proof. For each $m \geq 0$ define a map

$$\overline{F}_m \colon \; H(K,m) \otimes H(K',m) \otimes H(K'',m) \to H(K \otimes K' \otimes K'')$$

as follows. $\overline{F}_0 = \alpha \colon \, H(K) \otimes H(K') \otimes H(K'') \to H(K \otimes K' \otimes K'')$, where α is the homology product given by

$$\alpha \big[\eta(u) \otimes \eta(v) \otimes \eta(w) \big] = \eta(u \otimes v \otimes w).$$

For $m > 0$, $\overline{F}_m$ is the composition

$$H(K,m) \otimes H(K',m) \otimes H(K'',m) \xrightarrow{\;\alpha\;} H(K \otimes K' \otimes K'',m) \xrightarrow{\;\delta_0^m\;} H(K \otimes K' \otimes K'')$$

where α is the homology product (mod m) and δ_0^m is the Bockstein connecting homomorphism induced by the exact sequence

$$0 \to Z \xrightarrow{m} Z \to Z_m \to 0.$$

Explicitly, we have:

$$\overline{F}_m \big[\eta(u \otimes 1_m) \otimes \eta(v \otimes 1_m) \otimes \eta(w \otimes 1_m) \big]$$
$$= \eta \big[u' \otimes v \otimes w + (-1)^{|u|} u \otimes v' \otimes w + (-1)^{|u|+|v|} u \otimes v \otimes w' \big],$$

where $\partial u = m u'$, $\partial v = m v'$, $\partial w = m w'$.

Let

$$\overline{F} = \nabla \Big[\sum_{m \geq 0} \overline{F}_m \Big] \colon \; \sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m) \to H(K \otimes K' \otimes K'').$$

A direct calculation shows that every generator of the subgroup S (of Definition 2.2) is contained in the kernel of $\overline{F}$. Hence $\overline{F}(S) = 0$ and $\overline{F}$ induces a homomorphism

$$F \colon \; \{ H(K,m) \} \otimes \{ H(K',m) \} \otimes \{ H(K'',m) \} \to H(K \otimes K' \otimes K'').$$

To show that F is an isomorphism it suffices, as usual, to take K, K', K'' to be elementary complexes (cf. Palermo [8]). To avoid unnecessary trouble with signs we shall assume that all elementary complexes have non-zero groups only in dimensions 1 and 0, or 0 only, as the case may be. $Z(a)$ and $Z_m(u)$ will denote a free group on the generator a and a cyclic group of order m with generator u, respectively. We shall state the proof only for the case in which

$$K = Z(a) \xrightarrow{\partial} Z(a'), \qquad \partial a = ra', \ r \neq 0;$$

$$K' = Z(b) \xrightarrow{\partial} Z(b'), \qquad \partial b = r'b', \ r' \neq 0;$$

$$K'' = Z(c) \xrightarrow{\partial} Z(c'), \qquad \partial c = r''c', \ r'' \neq 0.$$

The other cases are similar (and easier). Furthermore, we shall assume $r = p^k$, $r' = p^m$, $r'' = p^n$ (p a prime, $k \leqq m \leqq n$). The justification for this is given in the Appendix.

In this situation we have:

$$H_1(K) = 0;$$

$$H_0(K) = Z_r(\eta(a'));$$

$$H_0(K, m) = Z_{(m,r)}(\eta(a' \otimes 1_m));$$

$$H_1(K, m) = Z_{(m,r)}(\eta[m/(m,r)a \otimes 1_m]).$$

In dimension 1, λ_m^{mk} is the map $\bar{k}\pi_{(m,r)}^{(mk,r)} \colon Z_{(mk,r)} \to Z_{(m,r)}$, where $k \geq 0$, $\bar{k} = k/(k, r/(m,r))$, and π is the canonical projection of cyclic groups. μ_{mk}^m is the canonical injection $\iota_{(mk,r)}^{(m,r)} \colon Z_{(m,r)} \to Z_{(mk,r)}$ ($k > 0$). μ_0^m is the injection $\iota_r^{(m,r)} \colon Z_{(m,r)} \to Z_r$; it has degree -1; all other maps are of degree 0. In dimension 0, $\lambda_m^{mk} = \pi_{(m,r)}^{(mk,r)} \colon Z_{(mk,r)} \to Z_{(m,r)}$ for $k \geq 0$ and $\mu_{mk}^m = \bar{k}\iota_{(mk,r)}^{(m,r)} \colon Z_{(m,r)} \to Z_{(mk,r)}$, where $k > 0$ and $\bar{k}$ is as above; $\mu_0^m = 0$.

$H(K', m)$ and $H(K'', m)$ are the same, mutatis mutandis. Furthermore, $H_i(K \otimes K' \otimes K'') = 0$ if $i \geqq 3$, and

$$H_0(K \otimes K' \otimes K'') = Z_{p^k}[\eta(a' \otimes b' \otimes c')];$$

$$H_1(K \otimes K' \otimes K'') = Z_{p^k}[\eta(a' \otimes b \otimes c' - p^{m-k}a \otimes b' \otimes c')]$$
$$+ Z_{p^k}[\eta(a' \otimes b' \otimes c - p^{n-k}a \otimes b' \otimes c')];$$

$$H_2(K \otimes K' \otimes K'') = Z_{p^k}[\eta(a' \otimes b \otimes c - p^{m-k}a \otimes b' \otimes c + p^{n-k}a \otimes b \otimes c')].$$

A direct calculation shows that $\bar{F}$ induces an isomorphism of the group

$$H_1(K, p^k) \otimes H_1(K', p^k) \otimes H_1(K'', p^k)$$

(i)
$$+ H_1(K, p^k) \otimes H_1(K', p^k) \otimes H_0(K'', p^k)$$
$$+ H_1(K, p^k) \otimes H_0(K', p^k) \otimes H_1(K'', p^k)$$
$$+ H_0(K) \otimes H_0(K') \otimes H_0(K'')$$

to $H(K \otimes K' \otimes K'')$. In particular this implies that no element of the group (i) is identified with zero (modulo S) in the group $\sum_{m \geq 0} H(K, m) \otimes H(K', m) \otimes H(K'', m)$ (since $\bar{F}(S) = 0$). To complete the proof that $\bar{F}$ induces an isomorphism on

$$\frac{\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)}{S}$$

$$= \{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\}$$

we need only show that every element of $\sum_{m \geq 0} H(K,m) \otimes H(K',m)$ $\otimes H(K'',m)$ is identified (modulo S) with an element of the group (i). A proof of this fact is given in the Appendix. This completes the proof of Theorem 2.2.

3. Products.

DEFINITION 3.1. A *differential graded ring* K over Z is a chain complex of abelian groups on which an associative and distributive product is defined, so that:

(1) for $u \in K_p$, $v \in K_q$, $uv \in K_{p+q}$;

(2) $\partial(uv) = (\partial u)v + (-1)^{|u|} u(\partial v)$;

(3) there is a unit element $1 \in K_0$ such that $1u = u1 = u$ for every $u \in K$.

The following propositions are well known. We state them without proof, mainly for reference purposes.

PROPOSITION 3.1. *If* K, K', *and* K'' *are differential graded rings, then* $H(K \otimes K' \otimes K'')$ *is a (differential) graded ring, with product defined by:*

$$\eta(u \otimes v \otimes w)\eta(\overline{u} \otimes \overline{v} \otimes \overline{w}) = (-1)^{(|v|+|w|)|\overline{u}|+|\overline{v}||w|}\eta(u\overline{u} \otimes v\overline{v} \otimes w\overline{w}).$$

PROPOSITION 3.2. *If* K, K', *and* K'' *are differential graded rings, then* $H(K,m) \otimes H(K',m) \otimes H(K'',m)$ *is a (differential) graded ring for each* $m \geq 0$, *with product defined by:*

$$[\eta(u \otimes 1_m) \otimes \eta(v \otimes 1_m) \otimes \eta(w \otimes 1_m)] \cdot [\eta(\overline{u} \otimes 1_m) \otimes \eta(\overline{v} \otimes 1_m) \otimes \eta(\overline{w} \otimes 1_m)]$$

$$= (-1)^{(|v|+|w|)|\overline{u}|+|\overline{v}||w|}\eta(u\overline{u} \otimes 1_m) \otimes \eta(v\overline{v} \otimes 1_m) \otimes \eta(w\overline{w} \otimes 1_m)$$

(with the obvious modification for $m = 0$*).*

If K, K', and K'' are torsion-free differential graded rings, we would like to put a ring structure on $\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\}$ in such a way that the isomorphism

$$F: \{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\} \to H(K \otimes K' \otimes K'')$$

(of Theorem 2.2) becomes a ring isomorphism. Since $\{H(K,m)\} \otimes \{H(K',m)\}$ $\otimes \{H(K'',m)\}$ is a quotient group of $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$, we shall begin by defining a product on this group. We denote by $\hat{\lambda}_m^{mk}$ the map $\lambda_m^{mk} \otimes \lambda_m^{mk} \otimes \lambda_m^{mk}$: $H(K,mk) \otimes H(K', mk) \otimes H(K'', mk) \to H(K, m) \otimes H(K',m) \otimes H(K'',m)$.

DEFINITION 3.2. Let K, K', and K'' be differential graded rings; let $x \in H(K,i) \otimes H(K',i) \otimes H(K'',i)$ and $y \in H(K,j) \otimes H(K',j) \otimes H(K'',j)$;

then a product $\circ$ is given in $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ by:

$$x \circ y = (\hat{\lambda}_c^i x)(\hat{\lambda}_c^j y),$$

where $(i,j) = c$ and the product on the right is as in Proposition 3.2.

It is easy to verify that under this product $\sum_{m \geq 0} H(K,m) \otimes H(K',m)$ $\otimes H(K'',m)$ is an (associative) ring. Unfortunately, however, the map

$$\bar{F} : \sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m) \to H(K \otimes K' \otimes K'')$$

(cf. proof of Theorem 2.2) does not preserve the product. (It does not even behave properly with respect to degrees.) Consequently we shall use the $\circ$ product to define a new product on $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ in such a way that $\bar{F}$ becomes a ring homomorphism. If a generator $\eta(u \otimes 1_m) \otimes \eta(v \otimes 1_m) \otimes \eta(w \otimes 1_m)$ of $H(K,m) \otimes H(K',m) \otimes H(K'',m)$ is such that u, v, and w are homogeneous elements of K, K', and K'', then it will be called a homogeneous generator.

DEFINITION 3.3. Let K, K', and K'' be chain complexes of torsion-free abelian groups; then for each $m > 0$, the map

$$D_m : H(K,m) \otimes H(K',m) \otimes H(K'',m) \to H(K,m) \otimes H(K',m) \otimes H(K'',m)$$

is given on a homogeneous generator $\eta(u \otimes 1_m) \otimes \eta(v \otimes 1_m) \otimes \eta(w \otimes 1_m)$ by

$$D_m = \lambda_m^0 \mu_0^m \otimes 1 \otimes 1 + (-1)^{|u|} 1 \otimes \lambda_m^0 \mu_0^m \otimes 1 + (-1)^{|u|+|v|} 1 \otimes 1 \otimes \lambda_m^0 \mu_0^m.$$

(Note that the map D_m has degree -1.)

DEFINITION 3.4. Let K, K', and K'' be torsion-free differential graded rings; let x be a homogeneous generator of $H(K,i) \otimes H(K',i) \otimes H(K'',i)$ and y a homogeneous generator of $H(K,j) \otimes H(K',j) \otimes H(K'',j)$. Then a product $*$ is given in $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ by:

$$x * y = x \circ y \qquad\qquad\qquad \text{if } j = 0;$$

$$x * y = (-1)^{|x|} x \circ y \qquad\qquad \text{if } j > 0 \text{ and } i = 0;$$

$$x * y = ax \circ D_j y + (-1)^{|D_i x|} b D_i x \circ y \quad \text{if } j > 0 \text{ and } i > 0,$$

where $ai + bj = (i,j)$.

The $*$ product behaves somewhat strangely with respect to degrees. Also it depends on the choice of a and b for each pair (i,j). Under this product $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ is a (nonassociative) ring. The interesting result is

THEOREM 3.1. *If K, K', and K'' are torsion-free differential graded rings, then the map*

$$\bar{F} : \sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m) \to H(K \otimes K' \otimes K'')$$

is a ring homomorphism with respect to the $*$ product.

The proof of the theorem follows directly from

LEMMA 3.1. *If K, K', and K'' are torsion-free differential rings, then:*
(1) for each $m \geqq 0$, the homology product

$$\alpha\colon H(K,m) \otimes H(K',m) \otimes H(K'',m) \to H(K \otimes K' \otimes K'', m)$$

is a ring homomorphism;
(2) for $x \in H(K \otimes K' \otimes K'', i)$ $(i > 0)$ and $y \in H(K \otimes K' \otimes K'')$,

$$\delta_0^i(x \cdot \lambda_i^0 y) = (\delta_0^i x) y;$$

(3) for $x \in H(K \otimes K' \otimes K'')$ and $y \in H(K \otimes K' \otimes K'', j)$ $(j > 0)$,

$$\delta_0^j(\lambda_j^0 x \cdot y) = (-1)^{|x|} x(\delta_0^j y);$$

(4) for all m, $k \geqq 0$, the following diagram is commutative:

$$
\begin{array}{ccc}
H(K,mk) \otimes H(K',mk) \otimes H(K'',mk) & \xrightarrow{\alpha} & H(K \otimes K' \otimes K'', mk) \\
\Big\downarrow \hat{\lambda}_m^{mk} & & \Big\downarrow \lambda_m^{mk} \\
H(K,m) \otimes H(K',m) \otimes H(K'',m) & \xrightarrow{\quad\alpha\quad} & H(K \otimes K' \otimes K'', m);
\end{array}
$$

(5) for each $m > 0$, the following diagram is commutative:

$$
\begin{array}{ccc}
H(K,m) \otimes H(K',m) \otimes H(K'',m) & \xrightarrow{\alpha} & H(K \otimes K' \otimes K'', m) \\
\Big\downarrow & & \Big\downarrow \delta_0^m \\
D_m & & H(K \otimes K' \otimes K'') \\
\Big\downarrow & & \Big\downarrow \lambda_m^0 \\
H(K,m) \otimes H(K',m) \otimes H(K'',m) & \xrightarrow{\alpha} & H(K \otimes K' \otimes K'', m).
\end{array}
$$

The proof is straightforward and is omitted.

COROLLARY 3.1. *The subgroup S of $\sum_{m \geqq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ (cf. Definition 2.2) is an ideal with respect to the $*$ product.*

Proof. By Theorem 2.2, $S = \mathrm{Kernel}\ \overline{F}$.

COROLLARY 3.2. *The product induced by $*$ in $\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\}$ is associative and is independent of the choice of a and b in Definition 3.4; hence $\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\}$ is a ring under the product induced by $*$.*

Proof. Since $\overline{F}(x * y) = (\overline{F}x)(\overline{F}y)$ the $*$ product is associative and unique modulo kernel $\overline{F} = S$. But by definition,

$$\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\}$$

$$= \frac{\displaystyle\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)}{S}.$$

COROLLARY 3.3. *If K, K', and K'' are torsion-free differential graded rings, then*

$$F: \{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\} \to H(K \otimes K' \otimes K'')$$

is a ring isomorphism.

Proof. This is immediate from the theorem and preceding corollaries.

4. The homology spectrum of $K \otimes K'$.

PROPOSITION 4.1. *If G is an abelian group, $X \to {}^{\prime}G$ is a free resolution of G, K and K' are chain complexes of torsion-free abelian groups, then there is an isomorphism*:

$$\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(X,m)\} \cong H(K \otimes K' \otimes G).$$

Proof. By Theorem 2.2 we have

$$F: \{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(X,m)\} \cong H(K \otimes K' \otimes X).$$

An application of the Künneth Theorem and the Five Lemma shows that $(1 \otimes 1 \otimes \epsilon)_*\colon H(K \otimes K' \otimes X) \to H(K \otimes K' \otimes G)$ is an isomorphism.

DEFINITION 4.1. Let K and K' be complexes of torsion-free abelian groups; let X_m be a free resolution of Z_m (for each $m \geq 0$, $Z_0 = Z$). Then for each $m \geq 0$, $R_m(K,K')$ is the group $\{H(K,k)\} \otimes \{H(K',k)\} \otimes \{H(X_m,k)\}$.

The fact that $R_m(K,K')$ is independent of the choice of the resolution X_m is clear since for each $m \geq 0$, $R_m(K,K') \cong H(K \otimes K',m)$ (a special case of Proposition 4.1).

We shall now make $\{R_m(K,K')\}$ into a Bockstein spectrum. For each $m \geq 0$, let X_m be a free resolution of Z_m ($Z_0 = Z$). Then the canonical projections $\pi_m^{mk}\colon Z_{mk} \to Z_m$ ($k \geq 0$) and injections $\iota_{mk}^m\colon Z_m \to Z_{mk}$ ($k > 0$) can be lifted to chain transformations $\bar{\lambda}_m^{mk}\colon X_{mk} \to X_m$ ($k \geq 0$) and $\bar{\mu}_{mk}^m\colon X_m \to X_{mk}$ ($k > 0$).

For $m > 0$, let E_m be the chain complex augmented over Z_m:

$$0 \to Z(c_m) \xrightarrow{\partial} Z(c_m') \xrightarrow{\epsilon} Z_m \to 0,$$

where $\partial c_m = m c_m'$ and $\epsilon(c_m') = 1_m$. E_m is a free resolution of Z_m and together with Z_m can be considered as a short exact sequence. Let X_0^m be the complex $(X_0 \to {}^{\prime}Z) \circ (E_m \to {}^{\prime}Z_m)$ (obtained by "splicing together" exact sequences in the usual manner). X_0^m is a free resolution of Z_m such that $(X_0^m)_i = (X_0)_{i-1}$ for $i \geq 1$ and $(X_0^m)_0 = Z$. The identity map on Z_m lifts to a chain trans-

formation I_m: $X_m \to X_0^m$. We consider X_0 to be 0 in dimension -1 and define a chain transformation $\bar{\mu}_0^m$: $X_m \to X_0$ of degree -1 by $\bar{\mu}_0^m = I_m$ in positive dimensions and $\bar{\mu}_0^m = 0$ in dimension 0. Finally let $\bar{\lambda}_0^0 = \bar{\mu}_0^0$: $X_0 \to X_0$ be the lifting of the identity map on Z.

Since the chain complexes X_m are free resolutions, the maps $\bar{\lambda}_m^{mk}$ and $\bar{\mu}_{mk}^m$ ($k \geq 0$) are unique up to chain homotopy. Consequently they induce well-defined maps (i.e., independent of the choice of $\bar{\lambda}, \bar{\mu}$):

$$\bar{\lambda}_m^{*mk}\colon\ H(X_{mk}, i) \to H(X_m, i)$$
$$\bar{\mu}_{mk}^{*m}\colon\ H(X_m, i) \to H(X_{mk}, i)$$
for all $m, k, i \geq 0$.

If K and K' are chain complexes of torsion-free abelian groups then the maps:

$$1 \otimes 1 \otimes \bar{\lambda}_m^{*mk}\colon\ H(K, i) \otimes H(K', i) \otimes H(X_{mk}, i)$$
$$\to H(K, i) \otimes H(K', i) \otimes H(X_m, i),$$

$$1 \otimes 1 \otimes \bar{\mu}_{mk}^{*m}\colon\ H(K, i) \otimes H(K', i) \otimes H(X_m, i)$$
$$\to H(K, i) \otimes H(K', i) \otimes H(X_{mk}, i)$$

induce maps:

$$\lambda_m^{mk}\colon\ \{H(K, i)\} \otimes \{H(K', i)\} \otimes \{H(X_{mk}, i)\}$$
$$\to \{H(K, i)\} \otimes \{H(K', i)\} \otimes \{H(X_m, i)\},$$

$$\mu_{mk}^m\colon\ \{H(K, i)\} \otimes \{H(K', i)\} \otimes \{H(X_m, i)\}$$
$$\to \{H(K, i)\} \otimes \{H(K', i)\} \otimes \{H(X_{mk}, i)\}.$$

This depends essentially on the fact that $\bar{\lambda}$ and $\bar{\mu}$ are chain transformations, and therefore the induced maps λ^* and $\bar{\mu}^*$ commute with coefficient and connecting homomorphisms.

PROPOSITION 4.2. *If K and K' are chain complexes of torsion-free abelian groups, then $(R_m(K, K'), \lambda, \mu)$ is a Bockstein spectrum.*

Proof. Since $\bar{\mu}_{mh}^m \bar{\lambda}_m^{mk}$ and $\bar{\lambda}_{mh}^{mhk} \bar{\mu}_{mhk}^{mk}$ are both chain maps lifting the same map:

$$\iota_{mh}^m \pi_m^{mk} = \pi_{mh}^{mhk} \iota_{mhk}^{mk}\colon\ Z_{mk} \to Z_m,$$

and are therefore chain homotopic, the induced maps are the same, i.e., $\mu_{mh}^m \lambda_m^{mk} = \lambda_{mh}^{mhk} \mu_{mhk}^{mk}$: $R_{mk}(K, K') \to R_{mh}(K, K')$. The rest of the proof follows in a similar fashion.

We shall hereafter assume that the resolution X_m of Z_m ($m \geq 0$) is a differential graded ring and that ϵ: $X_m \to Z_m$ is a ring homomorphism. Such resolutions do exist; for example the resolution E_m of Z_m mentioned above can be made into a differential graded ring by defining the following multiplication on the generators:

$$c_m c_m = 0; \qquad c_m c_m' = c_m = c_m' c_m; \qquad c_m' c_m' = c_m'.$$

A Bockstein spectrum in which each B_m is a ring will be called a multiplicative Bockstein spectrum. A map of multiplicative spectra is a spectra map which is a ring homomorphism for each $m \geq 0$. Note that it is not required that the maps λ and μ be ring homomorphisms. If K and K' are torsion-free differential graded rings, then $R_m(K, K')$ is a ring for each $m \geq 0$ by the preceding remarks; hence $\{R_m(K, K')\}$ is a multiplicative spectrum.

THEOREM 4.1. *If K and K' are chain complexes of torsion-free abelian groups, then there is an isomorphism of Bockstein spectra:*

$$\{R_m(K, K')\} \cong \{H(K \otimes K', m)\}.$$

If K and K' are torsion-free differential graded rings, then this is an isomorphism of multiplicative spectra.

Proof. For each $m \geq 0$, we have by Proposition 4.1

$$(1 \otimes 1 \otimes \epsilon)_* F \colon R_m(K, K') \cong H(K \otimes K', m).$$

If K and K' are differential graded rings, then F is a ring isomorphism by Corollary 3.3 and it is easily verified that $(1 \otimes 1 \otimes \epsilon)_*$ is also a ring isomorphism.

Thus we need only show that $(1 \otimes 1 \otimes \epsilon)_* F$ is in fact a map of spectra, i.e., we must show that the following diagram is commutative (for $k \geq 0$):

$$
\begin{array}{ccccc}
R_{mk}(K, K') & \xrightarrow{F} & H(K \otimes K' \otimes X_{mk}) & \xrightarrow{(1 \otimes 1 \otimes \epsilon)_*} & H(K \otimes K', mk) \\
\downarrow{\lambda_m^{mk}} & & \downarrow{(1 \otimes 1 \otimes \bar{\lambda}_m^{mk})_*} & & \downarrow{\lambda_m^{mk}} \\
R_m(K, K') & \xrightarrow{F} & H(K \otimes K' \otimes X_m) & \xrightarrow{(1 \otimes 1 \otimes \epsilon)_*} & H(K \otimes K', m) \\
\downarrow{\mu_{mk}^m} & & \downarrow{(1 \otimes 1 \otimes \bar{\mu}_{mk}^m)_*} & & \downarrow{\mu_{mk}^m} \\
R_{mk}(K, K') & \xrightarrow{F} & H(K \otimes K' \otimes X_{mk}) & \xrightarrow{(1 \otimes 1 \otimes \epsilon)_*} & H(K \otimes K', mk).
\end{array}
$$

The maps λ and μ in the right-hand column are induced by the maps π and ι. But the chain transformations $\bar{\lambda}$ and $\bar{\mu}$ in the center column are maps which lift π and ι, respectively. Therefore the right-hand squares are commutative.

The maps λ and μ in the left-hand column are induced by the chain transformations $\bar{\lambda}$ and $\bar{\mu}$ in the center column. The map F is induced by the map $\delta \alpha$ (α is the homology product and δ the connecting homomorphism; cf. Theorem 2.2); but on the image of α the map δ is essentially the map $\delta \otimes 1 \otimes 1 \pm 1 \otimes \delta \otimes 1 \pm 1 \otimes 1 \otimes \delta$. Since maps induced by chain transformations commute with α and the various connecting homomorphisms, it follows

that the left-hand squares are commutative. Hence $(1 \otimes 1 \otimes \epsilon)_* F$ is an isomorphism of (multiplicative) spectra.

THEOREM 4.2. *If K and K' are torsion-free differential graded rings, then the multiplicative homology spectrum of $K \otimes K'$ is completely determined by the multiplicative homology spectra of K and K'.*

Proof. This is an immediate consequence of Theorem 4.1 since by Definition 4.1, $R_m(K, K')$ depends only on the (multiplicative) homology spectra of K and K' (and, of course, on Z_m).

Appendix. We must first show that in Theorem 2.2 it suffices for the proof to assume $r = p^k$, $r' = p^m$, $r'' = p^n$ (p a prime, $k \leq m \leq n$). To do so, we need the following lemmas.

LEMMA 1. *If K, K', L, L', M, M' are chain complexes of abelian groups, $s: f \simeq f': K \to K'$, $t: g \simeq g': L \to L'$, and $u: h \simeq h': M \to M'$ are chain homotopies and chain maps as indicated, then there is a chain homotopy*

$$v: f \otimes g \otimes h \simeq f' \otimes g' \otimes h': K \otimes L \otimes M \to K' \otimes L' \otimes M'$$

given by

$$v(x \otimes y \otimes z)$$
$$= (s \otimes g \otimes h + (-1)^{|x|} f' \otimes t \otimes h + (-1)^{|x|+|y|} f' \otimes g' \otimes u)(x \otimes y \otimes z).$$

The proof is straightforward and is omitted.

LEMMA 2. *Let K be the chain complex (in dimensions 1 and 0 for convenience): $Z(a) \to^\partial Z(a')$, $\partial a = ra'$, $r \neq 0$; suppose r has a prime decomposition $r = p_1^{r_1} \cdots p_t^{r_t}$; let K_i ($i = 1, \cdots, t$) be the chain complex (in dimensions 1 and 0): $Z(a_i) \to^\partial Z(a_i')$, $\partial a_i = p_i^{r_i} a_i'$; then K is chain equivalent to the complex $\sum_{i=1}^t K_i$.*

Proof. Define $f: \sum_{i=1}^t K_i \to K$ and $g: K \to \sum_{i=1}^t K_i$ by: $f_1(a_i) = a$; $f_0(a_i') = (r/p_i^{r_i})a'$; $g_1(a) = \sum_{i=1}^t s_i(r/p_i^{r_i})a_i$, where $\sum_{i=1}^t s_i(r/p_i^{r_i}) = 1$; $g_0(a') = \sum_{i=1}^t s_i a_i'$. It is easy to verify that f and g are chain maps and that $fg = 1$. Define a chain homotopy $S: (\sum_{i=1}^t K_i)_0 \to (\sum_{i=1}^t K_i)_1$ by

$$S(a_j') = \sum_{i \neq j} s_i(r/p_i^{r_i}p_j^{r_j})a_i - \sum_{i \neq j} s_i(r/p_i^{r_i}p_j^{r_j})a_j.$$

A direct calculation shows that $\partial S + S\partial = gf - 1$.

LEMMA 3. *Let K, K', and K'' be chain complexes as in the proof of Theorem 2.2 (see p. 231) and suppose that $(r, r', r'') = 1$; then $\{H(K, m)\} \otimes \{H(K', m)\} \otimes \{H(K'', m)\} = 0 = H(K \otimes K' \otimes K'')$.*

Proof.

$$H_i(K,m) \otimes H_j(K',m) \otimes H_k(K'',m) = Z_{(r,m)} \otimes Z_{(r',m)} \otimes Z_{(r'',m)} \cong Z_{(r,r',r'',m)} = 0;$$

hence $\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\} = 0$. The complexes K, K', K'' are free resolutions of Z_r, $Z_{r'}$, $Z_{r''}$, respectively. Hence by definition $H_i(K \otimes K' \otimes K'') = \mathrm{Trip}_i(Z_r, Z_{r'}, Z_{r''})$. But $\mathrm{Trip}_0 = Z_r \otimes Z_{r'} \otimes Z_{r''} = 0$. Trip_2 is zero by Proposition 2.1 and Trip_1 is zero by Definition 4.1 of MacLane [6].

Now if K, K', and K'' are the chain complexes as in the proof of Theorem 2.2 (see p. 231) with any nonzero r, r', r'', then there are chain equivalences $f\colon \overline{K} = \sum_{i=1}^{t} K_i \to K$, $g\colon \overline{K}' = \sum_{i=1}^{t} K_i' \to K'$, $h\colon \overline{K}'' = \sum_{i=1}^{t} K_i'' \to K''$, where the K_i, K_i', K_i'' are the complexes corresponding to prime decompositions of r, r', r'' as in Lemma 2. It follows from Lemma 1 that

$$f_*\colon \{H(\overline{K},m)\} \cong \{H(K,m)\},$$

$$g_*\colon \{H(\overline{K}',m)\} \cong \{H(K',m)\},$$

$$h_*\colon \{H(\overline{K}'',m)\} \cong \{H(K'',m)\},$$

and

$$(f \otimes g \otimes h)_*\colon H(\overline{K} \otimes \overline{K}' \otimes \overline{K}'') \cong H(K \otimes K' \otimes K'').$$

Consider the diagram:

(i)
$$
\begin{array}{ccc}
\{H(\overline{K},m)\} \otimes \{H(\overline{K}',m)\} \otimes \{H(\overline{K}'',m)\} & \xrightarrow{F} & H(\overline{K} \otimes \overline{K}' \otimes \overline{K}'') \\
\downarrow{\scriptstyle f_* \otimes g_* \otimes h_*} & & \downarrow{\scriptstyle (f \otimes g \otimes h)_*} \\
\{H(K,m)\} \otimes \{H(K',m)\} \otimes \{H(K'',m)\} & \xrightarrow{F} & H(K \otimes K' \otimes K'').
\end{array}
$$

The vertical maps are isomorphisms and all the functors in the diagram commute with direct sums. If Theorem 2.2 is true in the case $r = p^k$, $r' = p^m$, $r'' = p^n$, it follows from Lemma 3 that the map F in the top row is an isomorphism; (since all the functors involved are symmetric in K, K', K'' it is permissible to assume $k \leq m \leq n$). Thus in order to prove that the map F in the bottom row is an isomorphism, we need only show that the diagram is commutative. To do this it suffices by Lemma 3 to show that for each i ($i = 1, \cdots, t$) the diagram (i) with $\overline{K}$, $\overline{K}'$, $\overline{K}''$ replaced by K_i, K_i', K_i'' is commutative. This fact is readily verified. Consequently, it suffices for the proof of Theorem 2.2 to assume $r = p^k$, $r' = p^m$, $r'' = p^n$ (p prime, $k \leq m \leq n$).

Under these hypotheses we must show that every element of $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ is identified (modulo the subgroup S) with an element of the group (i) on p. 232. We shall indicate how this is done in a few cases and leave the rest to the reader. The maps λ and μ are as described on p. 232

The group $H_1(K,m) \otimes H_1(K',m) \otimes H_1(K'',m)$ is zero unless $(r,r',r'',m) > 1$. Let $c = (r,r',r'')$; then $(c,m) = (r,r',r'',m)$. Let m be such that (c,m)

> 1 and use the notation: $\eta(m/(r,m)a \otimes 1_m) = \overline{a}$, $\eta(m/(r',m)b \otimes 1_m) = \overline{b}$, $\eta(m/(r'',m)c \otimes 1_m) = \overline{c}$, $\eta(a \otimes 1_{(m,c)}) = \tilde{a}$, $\eta(b \otimes 1_{(m,c)}) = \tilde{b}$, $\eta(c \otimes 1_{(m,c)}) = \tilde{c}$. Then we have: $H_1(K,m) = Z_{(m,r)}(\overline{a}) = Z_{(m,c)d}(\overline{a})$, where $d = (m/(m,c), r/(m,c))$, $H_1(K,(m,c)) = Z_{(m,c)}(\tilde{a})$, and $\mu_m^{(m,c)}(\tilde{a}) = d\overline{a}$. Similarly we have

$$H_1(K',m) = Z_{(r',m)}(\overline{b}) = Z_{(m,c)d'}(\overline{b}), \quad H_1(K',(m,c)) = Z_{(m,c)}(\tilde{b}),$$

where $d' = (m/(m,c), r'/(m,c))$ and $\mu_m^{(m,c)}(\tilde{b}) = d'\overline{b}$; $H_1(K'',m) = Z_{(r'',m)}(\overline{c}) = Z_{(m,c)d''}(\overline{c})$, $H_1(K'',(m,c)) = Z_{(m,c)}(\tilde{c})$, where $d'' = (m/(m,c), r''/(m,c))$ and $\mu_m^{(m,c)}(\tilde{c}) = d''\overline{c}$.

Since $(d,d',d'') = 1$, there exist integers e, e', e'' such that $ed + e'd' + e''d'' = 1$. This implies that every element of $H_1(K,m) \otimes H_1(K',m) \otimes H_1(K'',m)$ can be written as a multiple of the element $ed\overline{a} \otimes \overline{b} \otimes \overline{c} + \overline{a} \otimes e'd'\overline{b} \otimes \overline{c} + \overline{a} \otimes \overline{b} \otimes e''d''\overline{c}$. The first element of this sum can also be written as

$$ed\overline{a} \otimes \overline{b} \otimes \overline{c} = e\mu_m^{(m,c)}(\tilde{a}) \otimes \overline{b} \otimes \overline{c} \equiv \pm e\tilde{a} \otimes \lambda_{(m,c)}^m(\overline{b}) \otimes \lambda_{(m,c)}^m(\overline{c}),$$

by reason of (3) in Definition 2.2 ($\equiv$ means congruent modulo the subgroup S); the latter element is an element of $H_1(K,(m,c)) \otimes H_1(K',(m,c)) \otimes H_1(K'',(m,c))$; similarly for other elements of the sum.

Thus we have shown that every element of the group $H_1(K,m) \otimes H_1(K',m) \otimes H_1(K'',m)$ (for $(m,c) > 1$) is identified (mod S) with an element of some $H_1(K,h) \otimes H_1(K',h) \otimes H_1(K'',h)$, where $h|c$. In this case the map λ_h^c (for either K or K') is just the canonical projection $\pi_h^c \colon Z_c \to Z_h$. Hence every generator of $H_1(K,h) \otimes H_1(K',h) \otimes H_1(K'',h)$ can be written in the form $\lambda_h^c x \otimes \lambda_h^c y \otimes z \equiv \pm x \otimes y \otimes \mu_c^h z$ (by (1) of Definition 2.2); the latter is an element of $H_1(K,c) \otimes H_1(K',c) \otimes H_1(K'',c)$. Thus we have shown that for $m > 0$ every element of the group $H_1(K,m) \otimes H_1(K',m) \otimes H_1(K'',m)$ is identified (mod S) with an element of the group $H_1(K,p^k) \otimes H_1(K',p^k) \otimes H_1(K'',p^k)$ (for $c = (r,r',r'') = (p^k, p^m, p^n) = p^k$).

Using the same general method—but now for two factors instead of three—one shows that for $m > 0$ every element of $H_1(K,m) \otimes H_1(K',m) \otimes H_0(K'',m)$ is identified (mod S) with an element of $H_1(K,(r,r')) \otimes H_1(K',(r,r')) \otimes H_0(K'',(r,r'))$. Repeating the procedure for $H_1(K,m) \otimes H_0(K',m) \otimes H_1(K'',m)$ and $H_0(K,m) \otimes H_1(K',m) \otimes H_1(K'',m)$ and using relation (4) of Definition 2.2 we conclude that every element of total degree 2 in $\sum_{m \geq 0} H(K,m) \otimes H(K',m) \otimes H(K'',m)$ is identified (mod S) with an element of the group:

$$H_1(K,p^k) \otimes H_1(K',p^k) \otimes H_0(K'',p^k) + H_1(K,p^k) \otimes H_0(K',p^k) \otimes H_1(K'',p^k).$$

The other identifications are made in a similar (but usually less complicated) fashion and are omitted here. This completes the proof of Theorem 2.2.

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