ON EMBEDDING HIGHLY CONNECTED
MANIFOLDS IN EUCLIDEAN SPACE

BY

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CHAPTER I. INTRODUCTORY REMARKS AND DEFINITIONS

1. Introduction. Let M^k and T^k be closed k-dimensional differentiable manifolds. A homeomorphism $h: M^k \to T^k$ will be called an *almost diffeomorphism* and we will say that M^k and T^k are *almost diffeomorphic* if $h|M^k - \{x_0\}$ is a diffeomorphism for some point x_0 in M^k.

A closed k-dimensional differentiable manifold M^k will be said to be *almost differentiable embeddable* in Euclidean m space R^m if M^k is almost diffeomorphic to a manifold T^k which is differentiably embeddable in R^m.

The following example is illustrative: Let θ^k denote the group of differentiable manifolds homeomorphic to the standard k-sphere S^k. It is well known that every manifold $M^k \in \theta^k$ ($k \geqq 7$) is almost diffeomorphic to S^k (see Smale [13]) and therefore every such manifold is almost differentiably embeddable in R^{k+1}. It follows from [13, Theorem 1.1] that $M^k \in \theta^k$ is differentiably embeddable in R^{k+1} if and only if M^k is diffeomorphic to S^k ($k \geqq 7$).

In this paper proofs of the following facts will be given:

THEOREM 1. *Let n be an even integer, $n \geqq 6$. Let M^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold. Then M^{2n} is almost differentiably embeddable in R^{2n+1} if and only if the index $I(M^{2n})$ of M^{2n} is zero and M^{2n} is n-parallelizable.*

A differentiable manifold is called a π-manifold if the Whitney sum of its tangent bundle with a trivial line bundle is trivial.

COROLLARY 1. *A closed $(n-1)$-connected $2n$-dimensional differentiable manifold (n even, $n \geqq 6$) is almost differentiably embeddable in R^{2n+1} if and only if it is a π-manifold.*

THEOREM 2. *Let $n \equiv 3, 5, 7 \pmod 8$, $n \geqq 5$. Let M^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold. Then M^{2n} is almost differentiably embeddable in R^{2n+1} if and only if the Arf invariant $\Phi(M^{2n})$ of M^{2n} (defined below) is zero.*

It has been conjectured by various researchers that if n is odd the Arf invariant of a closed $(n-1)$-connected $2n$-dimensional differentiable manifold is always zero. Some additional results without any restrictions on

the Arf invariant are also obtained in this paper (see Theorem 3 below)(^{1}).

2. **The picture to keep in mind.** The proofs of Theorems 1 and 2 are generalizations of the following "proof" of the fact that the 2-dimensional torus T^2 is embeddable in 3-space. If we take out the interior of a small disk $D_0^2 \subset T^2$ then what is left, $N^2 = T^2 - \operatorname{int} D_0^2$, is a handlebody. More precisely N^2 is a disk D^2 with two handles $D_1^1 \times D^1$, $D_2^1 \times D^1$ attached along the tubular neighborhoods of two linked zero spheres, S_1^0, $S_2^0 \subset \partial D^2$ (see Milnor [8, Introduction]).
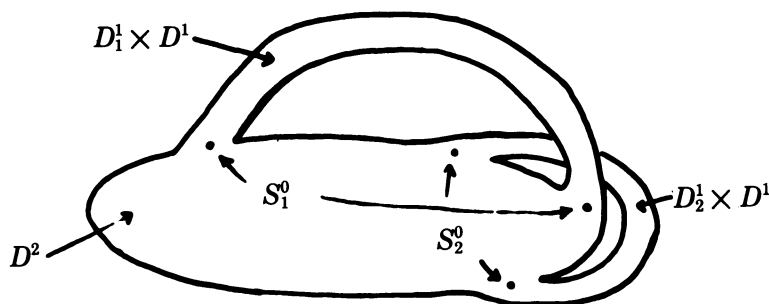

FIGURE 1

The handles are attached without twisting (that is, there is no Möbius strip embedded in Figure 1).

Let Euclidean 3-space R^3 be decomposed into the direct sum of two subspaces $R^3 = R^2 \oplus R^1$. Let

$$R_+^3 = \{ R^2 \oplus x \mid x > 0 \},$$
$$R_-^3 = \{ R^2 \oplus x \mid x < 0 \}.$$

Define an embedding f of $N^2 = D^2 \cup D_1^1 \times D^1 \cup D_2^1 \times D^1$ into R^3 as follows:

Look at D^2 as the set of all pairs of real numbers (x, y) such that $x^2 + y^2 \le 1$.

Let $f | D^2$ be defined by

$$f(x, y) = (x, y, x^2 + y^2 - 1).$$

Then f maps ∂D^2 into the unit circle in R^2 and it maps $\operatorname{int} D^2$ into R_-^3.

Now embed the handles $D_1^1 \times D^1$ and $D_2^1 \times D^1$ in R^2 by embedding

(int D_1^1) $\times D^1$ in the unbounded component of $R^2 - f(\partial D^2)$ and embedding (int D_2^1) $\times D^1$ in the bounded component. We have to do this to make sure that the handles do not intersect in R^2 since S_1^0 and S_2^0 are linked in ∂D^2. Note also that the fact that the handles are attached without twisting is important. We could not have embedded them in R^2 (compatibly with $f \mid D^2$) if they had been attached with a twist.
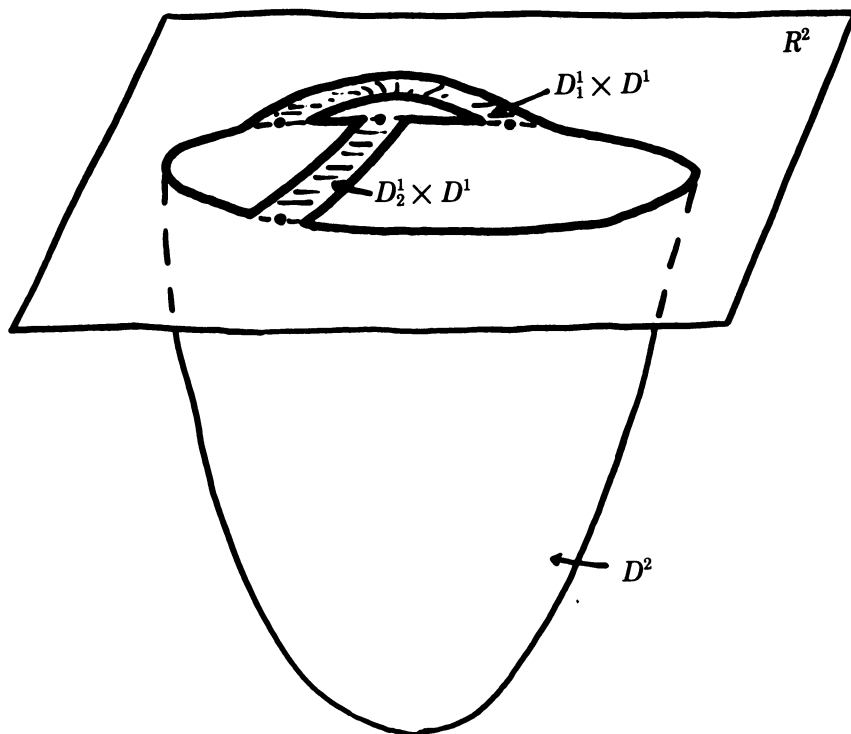

FIGURE 2

Thus $f: N^2 \to R^3$ is an embedding such that $f(N^2) \cap R^2 = f(N^2 - \text{int } D^2)$ and $f(\text{int } D^2) \subset R_-^3$ (see Figure 2). Note that ∂N^2 is a circle (as it must be since we started by taking out the interior of a disk $D^2 \subset T^2$) and that f embeds ∂N^2 in R^2 ("thickened" curve in Figure 2). Now $f(\partial N^2)$ is not the unit circle in R^2 but it bounds something homeomorphic to a disk $D_0^2 \subset R^2$. Lift int D_0^2 off R^2 into R_+^3 to get an embedding $T^2 \subset R^3$.

The above is the geometrical motivation for what follows. The proofs of Theorems 1 and 2 rely heavily on the fact that an $(n-1)$-connected $2n$-dimensional differentiable manifold M^{2n} becomes a handlebody if we remove the interior of a small disk from the manifold (Smale [12]). It will be shown that the hypotheses of these theorems are good enough to ensure that this handlebody can be embedded in R^{2n+1} in an analogous

fashion to $f: N^2 \subset R^3$. Smale's version of the Schoenflies theorem will then be used to "put the cap back" on this handlebody in R^{2n+1} giving an almost differentiable embedding $M^{2n} \subset R^{2n+1}$.

3. **Definitions and notation.** Much of the material in this section will be found in Smale [12], [14] and Wall [17].

Let M^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold, $n \geqq 3$.

Let D_0^{2n} be a differentiably embedded disk $D_0^{2n} \subset M^{2n}$. Smale has shown (see [12, Theorem G]) that $M^{2n} - \text{int } D_0^{2n}$ is a handlebody in $\mathscr{H}(2n, k, n)$ where $k = \text{rank } H_n(M^{2n}, Z)$ and $\mathscr{H}(2n, k, n)$ is defined as follows:

Let $D_k = \bigcup_{i=1}^k D_i^n \times D^n = $ disjoint union of k copies of $D^n \times D^n$, where D^m (or D_i^m) denotes the closed disk of dimension m.

Let $C_k = \bigcup_{i=1}^k \partial D_i^n \times D^n$ $(C_k \subset D_k)$.

Let $\lambda: C_k \rightarrow \partial D^{2n}$ be a differentiable embedding.

Let $N^{2n} = D^{2n} \cup_\lambda D_k$. There is a unique differentiable structure on N^{2n} compatible with the structures on D^{2n} and D^k. In Smale's terminology the differentiable manifold N^{2n} is a handlebody with presentation λ and the elements of $\mathscr{H}(2n, k, n)$ consist of the diffeomorphism classes of all manifolds obtained in this way as λ varies. Note that a given handlebody could conceivably have two different (that is, not diffeotopic) presentations.

Given a presentation λ of a handlebody N^{2n} in $\mathscr{H}(2n, k, n)$ we now define three new quantities: $L(\lambda)$, the linking matrix of λ; $B(\lambda)$, the canonical basis of $H_n(N^{2n}, Z)$ associated with λ; $\alpha(\lambda)$, the "twisting coefficients" of λ. $L(\lambda) = \|c_{ij}\|$ is the $k \times k$ integer-valued matrix defined by $c_{ii} = 0$, $i = 1, \cdots, k$, and

$$c_{ij} = \text{Lk}(\lambda(\partial D_i^n \times 0), \lambda(\partial D_j^n \times 0)), \qquad i \neq j,$$

where $\text{Lk}(\alpha, \beta)$ is the linking number of two $(n-1)$-dimensional cycles α, β in an oriented $(2n-1)$-sphere. $L(\lambda)$ is a symmetric matrix $(c_{ij} = c_{ji})$ if n is even and a skew symmetric matrix $(c_{ij} = -c_{ji})$ if n is odd. (See Siefert and Threlfall [11, pp. 277-282] for definitions and properties of $\text{Lk}(\alpha, \beta)$.)

$B(\lambda)$ is defined as follows: For each $i = 1, \cdots, k$ there is an embedding $g_i: D_i^n \rightarrow \partial D^{2n}$ such that $g_i | \partial D_i^n = \lambda | \partial D_i^n \times 0$ (see Lemma 2 below). The subset $S_i^n = g_i(D_i^n) \cup D_i^n \times 0 \subset N^{2n}$ is an embedded n-sphere in N^{2n}. The homology classes carried by $\{S_i^n\}$ define a free basis for $H_n(N^{2n}, Z)$ depending only on λ which we denote by $B(\lambda)$.

Note that the intersection number of the homology classes carried by S_i^n and S_j^n $(i \neq j)$ is obtained by looking at the geometrical intersection

$$S_i^n \cap S_j^n = g_i(D_i^n) \cap g_j(D_j^n).$$

(See Seifert and Threlfall [11, pp. 246-248] for definitions and properties

of intersection numbers.) Since the linking number of $\lambda(\partial D_i^n \times 0)$ and $\lambda(\partial D_j^n \times 0)$ is also obtained by looking at $g_i(D_i^n) \cap g_j(D_j^n)$ we have that the intersection number of $[S_i^n]$ and $[S_j^n]$ in $H_n(N^{2n}, Z)$ is equal to

$$\text{Lk}(\lambda(\partial D_i^n \times 0), \lambda(\partial D_j^n \times 0)), \qquad i \neq j.$$

This then gives another way of determining $L(\lambda)$ which will be useful later.

It remains to define $\alpha(\lambda)$. This is done as follows: Let $\bar{h}$ denote the inclusion $\bar{h}_i : S_i^n \subset N^{2n}$. Since N^{2n} is $(n-1)$-connected, $n \geq 3$, we have by results of Haefliger (see [3, Theorems 4.1 and 5.1]) that $\bar{h}_i$ is homotopic to a differentiable embedding $h_i : S^n \to N^{2n}$ and further any two such differentiable embeddings h_i and h_i' are diffeotopic. That is, $h_i(S^n)$ and $h_i'(S^n)$ have equivalent normal bundles in N^{2n}. Thus $\lambda \mid \partial D_i^n \times D^n$ determines a unique element of $\pi_{n-1}(\text{SO}(n))$ the elements of which are in one-to-one correspondence with n-dimensional vector bundles over S^n (see Steenrod [15, Theorem 18.6]). Let $\alpha(\lambda_i) \in \pi_{n-1}(\text{SO}(n))$ denote this element. Then

$$\alpha(\lambda) = (\alpha(\lambda_1), \alpha(\lambda_2), \cdots, \alpha(\lambda_k)) \in \sum_{i=1}^{k} \pi_{n-1}(\text{SO}(n)).$$

$\alpha(\lambda)$ is just a measure of how twisted the handles D_k are in $D^{2n} \cup_\lambda D_k$.

The following proposition will be needed in Chapter II. It is an easy consequence of Smale [12, Theorem M] and Smale [13, Theorem 1.4] (or Munkres [10, Theorem 6.10]).

PROPOSITION. *Let $k > 7$. Two closed k-dimensional differentiable manifolds M^k and T^k are almost diffeomorphic if and only if there exists $H^k \in \theta^k$ such that T^k is the connected sum of M^k and H^k ($T^k = M^k \# H^k$). (See Milnor [7] or Kervaire and Milnor [5] for the definition of connected sum.)*

Another piece of notation which will be used is the following: Let U be a square matrix. The symbol $\text{diag}(U, U, \cdots, U)$ will denote the matrix

$$\begin{pmatrix} U & & & \\ & U & & 0 \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \cdot \\ & & & & U \end{pmatrix}.$$

CHAPTER II. PROOFS OF THEOREMS 1 AND 2

1. **The main lemma.** Let Euclidean $(2n + 1)$-space be decomposed into the direct sum of a line and a hyperplane $R^{2n+1} = R^{2n} \oplus R^1$. Let

$$R_+^{2n+1} = \{ R^{2n} \oplus x \mid x > 0 \},$$
$$R_-^{2n+1} = \{ R^{2n} \oplus x \mid x \leq 0 \}.$$

The proofs of Theorems 1 and 2 depend upon the following lemma.

LEMMA 1. *Let N^{2n} be an element of $\mathscr{A}(2n, k, n)$, $n \geq 5$. Suppose N^{2n} has a presentation λ such that $\alpha(\lambda) = 0$ and $L(\lambda) = \mathrm{diag}(U, U, \cdots, U)$ where U is a 2×2 integer-valued matrix. Then there is a differentiable embedding $f \colon N^{2n} \to R^{2n+1}_{-}$ such that $f(\partial N^{2n}) \subset R^{2n}$.*

The proof of Lemma 1 depends in turn upon the following.

LEMMA 2. *Given a differentiable embedding $\lambda \colon C_k \to \partial D^{2n}$, $n \geq 5$, $k = 2s$, if $L(\lambda) = \mathrm{diag}(U, U, \cdots, U)$ for some 2×2 integer-valued matrix U then there exist differentiable embeddings $g_i \colon D^n_i \to \partial D^{2n}$, $i = 1, \cdots, k$, such that*

   (i) $g_i \mid \partial D^n_i = \lambda \mid \partial D^n_i \times 0$,

   (ii) $g_i(D^n_i) \cap g_j(D^n_j) = \emptyset$ *unless $i = 2l - 1$, $j = 2l$ for some $1 \leq l \leq s$. (In other words, these embedded disks intersect in pairs.)*

Note that Lemma 2 is false in the case $n = 2$. There are three circles disjointly embedded in the three spheres $S^1_1, S^1_2, S^1_3 \subset S^3$ such that $\mathrm{Lk}(S^1_i, S^1_j) = 0$, $i \neq j$, but the inclusion map $f \colon S^1_1 \to S^3 - (S^1_2 \cup S^1_3)$ gives a nonzero element of $\pi_1(S^3 - (S^1_2 \cup S^1_3))$ (see Figure 3).
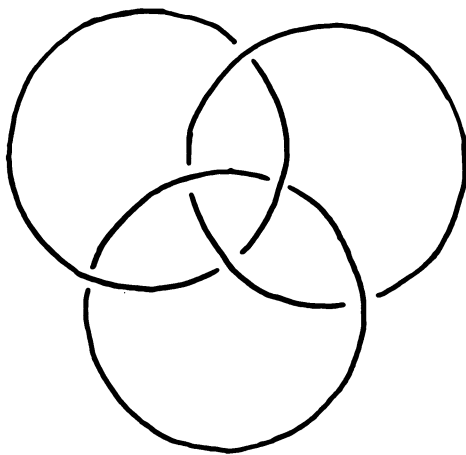

FIGURE 3

Proof of Lemma 2. Let $X_l = \partial D^{2n} - \bigcup_{i \neq 2l-1, 2l} \lambda(\partial D^n_i \times 0)$, $l = 1 \cdots, s$. Then

$$\lambda \mid \partial D^{2n}_{2l-1} \times 0 \colon \partial D^{2n}_{2l-1} \times 0 \to X_l, \quad \lambda \mid \partial D^{2n}_{2l} \times 0 \colon \partial D^{2n}_{2l} \times 0 \to X_l.$$

Since $L(\lambda) = \mathrm{diag}(U, U, \cdots, U)$ for some 2×2 matrix U we know that $\lambda(\partial D^n_{2l-1} \times 0)$ and $\lambda(\partial D^n_{2l} \times 0)$ are linked with none of the other $\lambda(\partial D^n_i \times 0)$. It will now be shown that this implies that the two maps written above are nullhomotopic in X_l. To see this observe that

$$\mathrm{Lk}\left(\lambda(\partial D^n_{2l-1} \times 0), \bigcup_{i \neq 2l-1, 2l} \lambda(\partial D^n_i \times 0)\right)$$
$$= \sum_{i \neq 2l-1, 2l} \mathrm{Lk}(\lambda(\partial D^n_{2l-1} \times 0), \lambda(\partial D^n_i \times 0)) = 0$$

(see [11, p. 278]). Similarly,

$$\text{Lk}\left(\lambda(\partial D_{2l}^n \times 0), \bigcup_{i \neq 2l-1, 2l} \lambda(\partial D_i^n \times 0)\right) = 0.$$

Thus by the definition of linking number $\lambda(\partial D_{2l-1}^n \times 0)$ and $\lambda(\partial D_{2l}^n \times 0)$ both bound chains in the complement of $\bigcup_{i \neq 2l-1, 2l} \lambda(\partial D_i^n \times 0)$. In other words, they both bound chains in X_l.

Further, X_l is simply connected if $n \geq 3$. To see this let $S^1 \subset X_l$ be a differentiably embedded circle. This circle is nullhomotopic in ∂D^{2n} and so $S^1 \subset \partial D^{2n}$ bounds a differentiably embedded disk $D^2 \subset \partial D^{2n}$. Now $(n-1) + 2 < 2n - 1$ since $n \geq 3$. Therefore since $\bigcup_{i \neq 2l-1, 2l} \lambda(\partial D_i^n \times 0)$ is $(n-1)$-dimensional we can assume this disk does not intersect $\bigcup_{i \neq 2l-1, 2l} \lambda(\partial D_i^n \times 0)$ as desired.

Now, by the Alexander Duality Theorem,

$$H^p\left(\bigcup_{i \neq 2l-1, 2l} \lambda(\partial D_i^n \times 0), Z\right) \approx \widetilde{H}_{2n-p-2}(X_l, Z).$$

Therefore $H_i(X_l, Z) = 0$, $i \leq n - 2$. Therefore, by the Hurewicz Isomorphism Theorem, $\pi_{n-1}(X_l) \approx H_{n-1}(X_l, Z)$.

But we have shown that $\lambda \mid \partial D_{2l-1}^n \times 0$ and $\lambda \mid \partial D_{2l}^n \times 0$ are both nullhomologous in X_l. So they are both nullhomotopic in X_l, $l = 1, \cdots, s$.

Thus we have proven the existence of a continuous map $\bar{g}_1 \colon D_1^n \to X_1$ such that $\bar{g}_1 \mid \partial D_1^n = \lambda \mid \partial D_1^n \times 0$. To find a differentiable embedding g_1 with the same property proceed as follows: Since $n \geq 5$ we can, using Haefliger [3], approximate $\bar{g}_1$ by a differentiable embedding $h \colon D_1^n \to X_1$. Then using results of Thom and Haefliger (see Introduction to [3]) we construct a diffeomorphism $\phi \colon X_1 \to X_1$ such that $\bar{g}_1 \mid \partial D_1^n = \phi \circ (h \mid \partial D_1^n)$. The differentiable embedding $g_1 \colon D_1^n \to X_1$ is now defined by $g_1(x) = \phi \circ h(x)$ for all $x \in D_1^n$. Clearly $g_1 \mid \partial D_1^n = \lambda \mid \partial D_1^n \times 0$.

In the same way from the existence of a continuous map $\bar{g}_2 \colon D_2^n \to X_1$ satisfying $\bar{g}_2 \mid \partial D_2^n = \lambda \mid \partial D_2^n \times 0$ the existence of a differentiable embedding $g_2 \colon D_2^n \to X_1$ with the same property can be proven.

A differentiable embedding $g_3 \colon D_3^n \to X_2$ such that $g_3 \mid \partial D_3^n = \lambda \mid \partial D_3^n \times 0$ can be defined as above. Now, however, we must make sure that $g_3(D_3^n)$ does not intersect $g_1(D_1^n) \cup g_2(D_2^n)$ so that condition (ii) of the lemma will be satisfied. This is done as follows: We know that $\lambda \mid \partial D_3^n \times 0 \colon \partial D_3^n \times 0 \to X_2$ is nullhomotopic. Claim further that $\lambda \mid \partial D_3^n \times 0 \colon \partial D \to X_2 - (g_1(D_1^n) \cup g_2(D_2^n))$ is nullhomotopic. To see this pick a point $p_2 \in g_1(D_1^n) \cap g_2(D_2^n)$ (which must be nonempty if $\text{Lk}(\lambda(\partial D_1^n \times 0), \lambda(\partial D_2^n \times 0)) \neq 0$ as it will be in the application of this lemma; for completeness, however, if $g_1(D_1^n) \cap g_2(D_2^n) = \emptyset$ pick two points $p_2 \in g_1(D_1^n)$, $p_2' \in g_2(D_2^n)$). Then $X_2 - (g_1(D_1^n) \cup g_2(D_2^n))$ is a deformation retract of $X_2 - \{p_2\}$ (or $X_2 - \{p_2, p_2'\}$). But $\lambda \mid \partial D_3^n \times 0 \colon 2D_3^n \times 0 \to X_2 - \{p_2\}$ is nullhomotopic and thus $\lambda \mid \partial D_3^n \times 0 \colon \partial D_3^n \times 0 \to X_2 - (g_1(D_1^n) \cup g_2(D_2^n))$ is nullhomotopic.

Therefore, as before, we can find a differentiable embedding

$$g_3\colon D_3^n \to X_2 - (g_1(D_1^n) \cup g_2(D_2^n)) \quad \text{with } g_3 \mid \partial D_3^n = \lambda \mid \partial D_3^n \times 0.$$

Continue in this way to define $g_4, \cdots, g_k$ and Lemma 2 is proved.

Proof of Lemma 1. Recall that $N^{2n} = D^{2n} \cup_\lambda D_k$. A topological embedding $f'\colon N^{2n} \to R^{2n+1}$ having the desired properties will now be constructed.

Look at D^{2n} as $2n$-tuples of real numbers $(x_1, \cdots, x_{2n})$ such that $\sum_{i=1}^{2n} x_i^2 \leqq 1$.

Define $f' \mid D^{2n}\colon D^{2n} \to R_-^{2n+1}$ by the formula

$$f'(x_1, \cdots, x_{2n}) = \left(x_1, \cdots, x_{2n}, \sum_{i=1}^{2n} (x_i^2 - 1) \right).$$

Note that $f(\partial D^{2n})$ is the unit $(2n-1)$-sphere $S^{2n-1} \subset R^{2n}$.

Extend f' to $D^{2n} \cup_\lambda \bigcup_{i=1}^k D_i^n \times 0$ as follows: Since by hypothesis $L(\lambda) = \mathrm{diag}(U, U, \cdots, U)$ we can by Lemma 2 find differentiable embeddings $g_i\colon D_i^n \to \partial D^{2n}$ such that (i) $g_i \mid \partial D_i^n = \lambda \mid \partial D_i^n \times 0$ and (ii) the $\{g_i(D_i^n)\}$ intersect in pairs. The points $y \in f'(g_i(D_i^n)) \subset S^{2n-1} \subset R^{2n}$, $i = 1, \cdots, k$, can be looked at as unit vectors in R^{2n} and so it makes sense to multiply them by scalars. Let $\gamma\colon [0,1] \to R$ be a real-valued function satisfying

  (i) γ is C^∞ on $[0,1)$,
  (ii) $d\gamma(0)/dt = 0$, $d\gamma(1)/dt = -\infty$,
  (iii) $\gamma(0) = 1/2$, $\gamma(1) = 0$,
  (iv) γ is decreasing on $[0,1]$.

Let x be a point in $D_i^n \times 0 \subset N^{2n}$ and let $\|x\|$ be the norm of x looked at as a vector in the unit ball $D_i^n \times 0$. Define $f' \mid D_i^n \times 0\colon D_i^n \times 0 \to R^{2n}$ by the formula

$$f'(x) = \begin{cases} (1 + \gamma(\|x\|))f'(g_i(x)), & i \text{ odd}, \\ (1 - \gamma(\|x\|))f'(g_i(x)), & i \text{ even}. \end{cases}$$

(This is just a way of saying embed half of the $\{D_i^n \times 0\}$ in the bounded component of $R^{2n} - S^{2n-1}$ and half in the unbounded component.)

Since $g_i \mid \partial D_i^n = \lambda \mid \partial D_i^n \times 0$ we know that f' is defined on $D^{2n} \cup_\lambda \bigcup_{i=1}^k D_i^n \times 0$. Since the $\{g_i(D_i^n)\}$ intersect in pairs we know that

$$f'\colon D^{2n} \cup_\lambda \bigcup_{i=1}^k D_i^n \times 0 \to R_-^{2n+1}$$

is an embedding. Further

$$f\left(\partial D^{2n} \cup_\lambda \bigcup_{i=1}^k D_i^n \times 0 \right) \subset R^{2n}.$$

It remains to extend f' to the rest of D_k (in other words, we want to "thicken the cores" $D_i^n \times 0$ of the handles $D_i^n \times D^n$ in R^{2n}). This is where

the condition $\alpha(\lambda) = 0$ comes in.

Let $E(\nu_i^n) \subset R^{2n}$ be the total space of the normal tube ν_i^n of $f'(D_i^n \times 0)$ in R^{2n} whose radius is chosen so that $E(\nu_i^n) \cap f'(\partial D^{2n}) = f'(\lambda(\partial D_i^n \times D^n))$. (We can assume without loss of generality that the radius of D^n is small enough so that $E(\nu_i^n) \cap E(\nu_j^n) = \emptyset$ $(i \neq j)$.)

Let $p_i: D_i^n \times D^n \to E(\nu_i^n)$ be a diffeomorphism defining a product structure on ν_i^n (in particular, $p_i \mid D_i^n \times 0 = f' \mid D_i^n \times 0$). Look at $\overline{p}_i = p_i \mid \partial D_i^n \times D^n: \partial D_i^n \times D^n \to E(\nu_i^n \mid \partial D_i^n)$. Note that $E(\nu_i^n \mid \partial D_i^n) = f'(\lambda(\partial D_i^n \times D^n))$. Therefore $(f')^{-1} \circ \overline{p}_i$ and $\lambda \mid \partial D_i^n \times D^n$ both define product structures on the tubular neighborhood $\lambda(\partial D_i^n \times D^n)$ of $\lambda(\partial D_i^n \times 0) \subset \partial D^{2n}$. Now by the "tubular neighborhood theorem" (see Wall [17, bottom of p. 164]) there is a diffeotopy

$$F: (\partial D_i^n \times D^n) \times I \to \lambda(\partial D_i^n \times D^n)$$

between $(f')^{-1} \circ \overline{p}_i$ and some product structure

$$t_i: \partial D_i^n \times D^n \to \lambda(\partial D_i^n \times D^n),$$

where t_i is related to $\lambda \mid \partial D_i^n \times D^n$ by a map $s_i: \partial D_i^n \to SO(n)$ such that

$$t_i(x, y) = \lambda(x, s_i(x) \circ y).$$

(Essentially this says that the product structure $(f')^{-1} \circ \overline{p}_i$ differs from $\lambda \mid \partial D_i^n \times D^n$ by the action of $SO(n)$.)

It is easily seen that the homotopy class $[s_i] \in \pi_{n-1}(SO(n))$ is precisely the characteristic class $\alpha(\lambda_i)$ of the normal bundle of the embedded sphere $h(S_i^n) \subset N^{2n}$ defined in Chapter I and so

$$\alpha(\lambda) = ([s_1], [s_2], \cdots, [s_k]) \in \pi_{n-1}(SO(n)).$$

Now by hypothesis, $\alpha(\lambda) = 0$. Therefore each s_i is homotopic to the constant map. This homotopy supplies us with a diffeotopy between t_i and $\lambda \mid \partial D_i^n \times D^n$ and so with a diffeotopy betweem $(f')^{-1} \circ \overline{p}_i$ and $\lambda \mid \partial D_i^n \times D^n$. Let $\overline{p}$ be the union of the maps $\overline{p}_i$. Then $(f')^{-1} \circ \overline{p}$ is diffeotopic to λ and therefore the handlebody $D^{2n} \cup_{(f')^{-1} \circ \overline{p}} D_k$ is diffeomorphic to $N^{2n} = D^{2n} \cup_\lambda D_k$. Further since

$$(f')^{-1} \circ \overline{p} \mid \partial D_i^n \times 0 = \lambda \mid \partial D_i^n \times 0, \qquad i = 1, \cdots, k,$$

the map

$$f': D^{2n} \cup_{(f')^{-1} \circ \overline{p}} \bigcup_{i=1}^{k} D_i^n \times 0 \to R_-^{2n+1}$$

is defined. Now f' can be extended to the rest of D_k by $f' \mid D_i^n \times D^n = p_i$. Thus we have constructed a topological embedding $f': N^{2n} \to R_-^{2n+1}$ such that $f'(\partial N^{2n}) \subset R^{2n}$. Smooth f' at the "corners" to obtain a differentiable embedding f proving Lemma 1.

2. **Auxiliary lemmas.** In this section we investigate the conditions under which the hypotheses of Lemma 1 will be satisfied.

LEMMA 3. *Let M^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold $(n$ even, $n \neq 2)$ and suppose the index $I(M^{2n}) = 0$ and M^{2n} is n-parallelizable. Let $D_0^{2n} \subset M^{2n}$ be a differentiably embedded disk and let $N^{2n} = M^{2n} - \mathrm{int} D_0^{2n}$. Then N^{2n} has a presentation λ such that*

$$\alpha(\lambda) = 0 \quad and \quad L(\lambda) = \mathrm{diag}(U, U, \cdots, U).$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This result is essentially due to Milnor. Using [9, Lemma 9], [17, Theorem 1], and the fact that the linking matrix of a presentation is the same as a matrix of intersection numbers of homology classes in $H_n(N^{2n}, Z)$ $(\approx H_n(M^{2n}, Z))$, we can find a presentation λ of N^{2n} such that

$$L(\lambda) = \mathrm{diag}(U, U, \cdots, U), \qquad \text{where } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The fact that $\alpha(\lambda) = 0$ then follows immediately from [9, Lemma 7].

Before stating the corresponding result for n odd, some remarks are in order. It is well known (see Albert [1, p. 52]) that every skew symmetric bilinear form of determinant ± 1 has a matrix representation of the form $\mathrm{diag}(U, U, \cdots, U)$, where

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since the linking matrix of a presentation of an element of $\mathscr{A}(2n, k, n)$ is skew symmetric for n odd and since this matrix will (by Poincaré Duality) have determinant ± 1 if the handlebody arises by taking the interior of a disk out of a closed $(n-1)$-connected $2n$-dimensional manifold we know by Wall [17, Theorem 1]: If M^{2n} is a closed $(n-1)$-connected $2n$-dimensional differentiable manifold, n odd, and if D_0^{2n} is an embedded disk then $N^{2n} = M^{2n} - \mathrm{int} D_0^{2n}$ has a presentation λ such that

$$L(\lambda) = \mathrm{diag}(U, \cdots, U), \qquad \text{where } U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It remains to find the right condition on M^{2n} to ensure that $\alpha(\lambda) = 0$. Assume $n \equiv 3, 5, 7 \pmod 8$, $n \neq 3, 7$. In these dimensions it follows from results of Bott and Kervaire (see Wall [17, p. 171]) that $\pi_{n-1}(\mathrm{SO}(n)) \approx Z_2$. Thus if $N^{2n} = M^{2n} - \mathrm{int} D_0^{2n}$ and λ is a presentation of N^{2n} then

$$\alpha(\lambda) \in Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2 \qquad (k \text{ terms}).$$

Assume

$$L(\lambda) = \operatorname{diag}(U, U, \cdots, U), \qquad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The *Arf invariant* $\Phi(M^{2n}) \in Z_2$ of M^{2n} is now defined by the formula

$$\Phi(M^{2n}) = \sum_{j=1}^{s} \alpha(\lambda_{2j-1})\alpha(\lambda_{2j}), \qquad \text{where } k = 2s.$$

It can be shown (see Wall [17, p. 172, Case 6]) that $\Phi(M^{2n})$ does not depend on λ, i.e., $\Phi(M^{2n})$ is a topological invariant of M^{2n}. Further, it is known that if $\Phi(M^{2n}) = 0$, then N^{2n} has a presentation λ such that

$$L(\lambda) = \operatorname{diag}(U, U, \cdots, U), \quad \text{where } U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha(\lambda) = 0$$

(again see [17]).

The above discussion can be summarized as follows:

LEMMA 4. *Let $n \equiv 3, 5, 7 \pmod{8}$. Let M^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold and suppose $\Phi(M^{2n}) = 0$. Let D_0^{2n} be a differentiably embedded disk and let $N^{2n} = M^{2n} - \operatorname{int} D_0^{2n}$. Then N^{2n} has a presentation λ such that*

$$\alpha(\lambda) = 0 \quad and \quad L(\lambda) = \operatorname{diag}(U, U, \cdots, U),$$

where

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that in the statement of Lemma 4 we have included the cases $n = 3, 7$ although they were excluded from the preceding discussion. This is because $\pi_2(\mathrm{SO}(3)) = \pi_6(\mathrm{SO}(7)) = 0$ and so $\alpha(\lambda)$ is always zero in these dimensions.

3. **Proofs of the main theorems.** The proof of Theorem 1 is now completed as follows. Suppose M^{2n} is a closed $(n-1)$-connected $2n$-dimensional n-parallelizable manifold, $I(M^{2n}) = 0$, n even, $n \geqq 6$, and $N^{2n} = M^{2n} - \operatorname{int} D_0^{2n}$. Then (combining Lemmas 3 and 1) there exists a differentiable embedding $f \colon N^{2n} \to R_-^{2n+1}$ such that $f(\partial N^{2n})$ is contained in R^{2n}. Note that $f \mid \partial N^{2n}$ is a differentiable embedding of a $(2n-1)$-sphere in $2n$-space and so by the Generalized Schoenflies Theorem (see Smale [12, Theorem M]) the closure of the bounded component of $R^{2n} - f(\partial N^{2n})$ is diffeomorphic to a disk $\overline{D}_0^{2n}$. Lift the interior of this disk off R^{2n} into R_+^{2n+1} and smooth along $f(\partial N^{2n})$. The result is a differentiable embedding of a closed

manifold T^{2n} in R^{2n+1}. In general, T^{2n} is not diffeomorphic to the manifold M^{2n} that we started with. T^{2n} is the union of N^{2n} with a disk D_0^{2n} attached along $\partial N^{2n} = S^{2n-1}$ by a diffeomorphism of S^{2n-1} onto itself. That is, T^{2n} is diffeomorphic to the connected sum of M^{2n} with an element $H^{2n} \in \theta^{2n}$ (see Wall [17, bottom of p. 169]). Thus M^{2n} is almost diffeomorphic to T^{2n} and so M^{2n} is almost differentiably embeddable in R^{2n+1} proving half of Theorem 1.

To go in the other direction, suppose M^{2n} is almost differentiably embeddable in R^{2n+1}. Then there exists a differentiable manifold T^{2n}, an almost diffeomorphism $h: M^{2n} \to T^{2n}$, and a differentiable embedding $f: T^{2n} \to R^{2n+1}$. By the Jordan Curve Theorem, $R^{2n+1} - f(T^{2n})$ has two components. Let ω^{2n+1} denote the closure of the bounded component. Then ω^{2n+1} is a differentiable manifold with boundary T^{2n}. Therefore, $I(T^{2n}) = 0$ (see Thom [16, Corollary V. 11] or Milnor [6, Theorem 39]) and so $I(M^{2n}) = 0$ since the index is a topological invariant.

Further, since ω^{2n+1} is a $(2n + 1)$-dimensional manifold differentiably embedded in a $(2n + 1)$-space, the tangent bundle τ^{2n+1} of ω^{2n+1} (being just the restriction of the tangent bundle of R^{2n+1} to ω^{2n+1}) is trivial. Since $\partial \omega^{2n+1} = T^{2n}$, we know that the restriction of τ^{2n+1} to T^{2n} is equal to the Whitney sum of $\tau^{2n}(T^{2n})$ (the tangent bundle of T^{2n}) and a trivial line bundle θ' (the normal bundle of T^{2n} in ω^{2n+1}). Therefore, T^{2n} is a π-manifold. Therefore, T^{2n} is almost parallelizable (see Milnor [7, Lemma 1.2]). Since $h \mid M^{2n} - \{x_0\}$ is a diffeomorphism for some point $x_0 \in M^{2n}$ we know that $M^{2n} - \{x_0\}$ is almost parallelizable. Therefore, M^{2n} is almost parallelizable and so M^{2n} is n-parallelizable, completing the proof of Theorem 1.

Proof of Corollary 1. Suppose M^{2n} is a closed $(n - 1)$-connected $2n$-dimensional π-manifold. Then M^{2n} is almost parallelizable and so n-parallelizable. Since all the Pontrjagin classes of the tangent bundle of a π-manifold vanish, we known by the Hirzebruch Index Theorem (Milnor [6, Theorem 39]) that $I(M^{2n}) = 0$. Therefore, by Theorem 1, M^{2n} is almost differentiably embeddable in R^{2n+1}.

To prove the converse, suppose M^{2n} is almost differentiably embeddable in R^{2n+1}. Then M^{2n} is almost diffeomorphic to a π-manifold T^{2n} (the manifold which is differentiably embeddable in R^{2n+1}). Thus $M^{2n} = T^2 \# H^{2n}$ for some $H^{2n} \in \theta^{2n}$. But the elements of θ^{2n} are all π-manifolds (see Kervaire and Milnor [5, Theorem 3.1]). The desired result now follows from the fact that the connected sum of two π-manifolds is again a π-manifold.

Proof of Theorem 2. Let M^{2n} be a closed $(n - 1)$-connected $2n$-dimensional differentiable manifold with $\Phi(M^{2n}) = 0$. The fact that M^{2n} is almost differentiably embeddable in R^{2n+1} ($n \equiv 3, 5, 7$, (mod 8), $n \geq 5$) now follows from Lemmas 1 and 4.

To prove the converse, suppose M^{2n} is almost differentiably embedd-

able in R^{2n+1}. Then M^{2n} is almost diffeomorphic to a differentiable manifold T^{2n} which is differentiably embeddable in R^{2n+1}. It will be shown that $\Phi(T^{2n}) = 0$ from which it follows that $\Phi(M^{2n}) = 0$ since the Arf invariant is a topological invariant.

Note that T^{2n} bounds a parallelizable manifold ω^{2n+1} (the closure of the bounded component of the complement of T^{2n} in R^{2n+1}). Therefore, by Milnor [9, Corollary to Theorem 2] T^{2n} bounds an $(n-1)$-connected differentiable manifold. (In fact, using the Alexander Duality Theorem, it can be shown that ω^{2n+1} is itself $(n-1)$-connected.)

The following result due to Kervaire and Milnor (unpublished—see [5] for closely related material) completes the proof of Theorem 2.

LEMMA 5. *Let T^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold (n odd, $n \geq 3$). If T^{2n} bounds an $(n-1)$-connected $(2n+1)$-dimensional differentiable manifold, then T^{2n} has Arf invariant zero.*

A sketch of the proof of Lemma 5 in the particular case in which we are interested proceeds as follows: Let D^{2n+1} be a small disk in the interior of ω^{2n+1}. Then $N^{2n+1} = \omega^{2n+1} - \text{int } D^{2n+1}$ is a parallelizable manifold with boundary $T^{2n} - S^{2n}$. Let $f: N^{2n+1} \to R$ be a "nice function" (see Smale [12]). Embed N^{2n+1} in a high-dimensional Euclidean space such that the last coordinate is given by the nice function f. Since N^{2n+1} is parallelizable, we have normal frames over N^{2n+1}. The nice function can then be looked at as a sequence of framed spherical modifications on n-spheres passing from T^{2n} to S^{2n}. It follows that $\Phi(T^{2n}) = 0$ since the Arf invariant is invariant under such modifications (see [5]).

Note that as a corollary to Theorem 2 and Lemma 5 we have the following.

COROLLARY 2. *A closed $(n-1)$-connected $2n$-dimensional differentiable manifold M^{2n} ($n \equiv 3, 5, 7 \pmod 8$, $n \geq 5$) is almost differentiably embeddable in R^{2n+1} if and only if M^{2n} is almost diffeomorphic to the boundary of a parallelizable manifold.*

This result also holds for n even, $n \geq 6$, by Theorem 1.

The problem of extending Corollary 1 to include the cases $n \equiv 3, 5, 7$ $\pmod 8$, $n \geq 5$, now reduces to answering the following question in the affirmative: Is every $(n-1)$-connected $2n$-dimensional π-manifold almost diffeomorphic to the boundary of a parallelizable manifold? To put it another way, what is called for is the converse to the following well-known consequence of Theorem 2.

COROLLARY 3. *If $n \equiv 3, 5, 7 \pmod 8$, $n \geq 5$, and M^{2n} is a closed $(n-1)$-connected $2n$-dimensional differentiable manifold of Arf invariant zero then M^{2n} is a π-manifold.*

Proof. The proof proceeds exactly as does half of the proof of Corollary 1; that is, $M^{2n} \# H^{2n}$ is a π-manifold for some $H^{2n} \in \theta^{2n}$ and so M^{2n} is itself a π-manifold.

4. Some more consequences and final remarks. The following facts are almost immediate consequences of Corollary 1 and the fact that almost diffeomorphic manifolds differ by homotopy spheres.

COROLLARY 4. *Let n be even, $n \geq 6$. Suppose $\theta^{2n} = 0$ (for example, $n = 6$). Then a closed $(n-1)$-connected $2n$-dimensional differentiable manifold is differentiably embeddable in R^{2n+1} if and only if it is a π-manifold.*

Proof. $\theta^{2n} = 0$ means that the $2n$-sphere has a unique differentiable structure. In this case, every almost diffeomorphism is a diffeomorphism and every almost differentiable embedding a differentiable embedding.

COROLLARY 5. *Let n be even, $n \geq 6$. Let $R^{m(n)}$ be the smallest-dimensional Euclidean space in which every homotopy $2n$-sphere is differentiably embeddable. Then every closed $(n-1)$-connected $2n$-dimensional π-manifold is differentiably embeddable in $R^{m(n)}$.*

Proof. If M^{2n} is a closed $(n-1)$-connected π-manifold, then there exists $H^{2n} \in \theta^{2n}$ such that $M^{2n} \# H^{2n}$ is differentiably embeddable in $R^{2n+1} \subset R^{m(n)}$. Now $(M^{2n} \# H^{2n}) \# (-H^{2n})$ is diffeomorphic to M^{2n}, where $-H^{2n}$ is the group inverse of H^{2n}. The desired result now follows from the fact that if each of two manifolds is differentiably embeddable in a given Euclidean space, then so is their connected sum.

Analogous results hold for $n \equiv 3, 5, 7 \pmod 8$, $n \geq 5$.

The following is worth noting here.

PROPOSITION. *If M^{2n} is a closed $(n-1)$-connected $2n$-dimensional differentiable manifold and $n \equiv 3, 5, 6, 7 \pmod 8$, then M^{2n} is n-parallelizable.*

Proof. Let $\tau: M^{2n} \to \mathrm{BSO}(2n)$ be the map which defines the tangent bundle of M^{2n}. Let $K^n \subset M^{2n}$ be the n skeleton. Then M^{2n} is n-parallelizable if and only if $\tau \mid K^n$ is homotopic to the constant map. Since M^{2n} is $(n-1)$-connected, K^n is a wedge of spheres. Therefore $\tau \mid K^n$ can be looked at as an element of $\pi_n(\mathrm{BSO}(2n)) \approx \pi_{n-1}(\mathrm{SO}(2n)) = 0$ (see Bott [2]) and so M^{2n} is n-parallelizable.

Thus n-parallelizability is implicit in Theorem 2 and redundant in the statement of Theorem 1 for $n \equiv 6 \pmod 8$.

The techniques described in this paper can be used to give a new proof of a result due to Hirsch (see [4, Corollary 4.3]).

THEOREM 3. *Let M^{2n} be a closed $(n-1)$-connected $2n$-dimensional differentiable manifold $n \equiv 3, 5, 7, \pmod 8$, $n \geq 5$. Let $x_0 \in M^{2n}$. Then $M^{2n} - \{x_0\}$ can be differentiably embedded in R^{2n+1}.*

Proof. As always, let $D_0^{2n} \subset M^{2n}$ be a small disk with center x_0 and $N^{2n} = M^{2n} - \text{int}\,D_0^{2n}$. It will be shown that N^{2n} (and therefore $M^{2n} - \{x_0\}$) is differentiably embeddable in R^{2n+1}.

Let λ be a presentation of N^{2n} such that

$$L(\lambda) = \text{diag}(U, U, \cdots, U), \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad N^{2n} = D^{2n} \cup_\lambda D_k.$$

As in Lemma 1, we can find an embedding

$$f' : D^{2n} \cup_\lambda \bigcup_{i=1}^{k} D_i^n \times 0 \to R_-^{2n+1}$$

such that $f'(\partial D^{2n} \cup_\lambda \bigcup_{i=1}^{k} D_i^n \times 0) \subset R^{2n}$.

Now, however, we can no longer thicken the cores $\{f'(D_i^n \times 0)\}$ of the handles in R^{2n} since we have removed the restriction $\alpha(\lambda) = 0$. We can, however, embed D_k in R^{2n+1} so that the embedding extends $f' \mid C_k \cup \bigcup_{i=1}^{k} D_i^n \times 0$. This is done as follows: Observe that the obstruction to thickening $f'(D_i^n \times 0)$ in R^{2n+1} (i.e., the obstruction to extending the field of n frames in the normal tube over $f'(\lambda(\partial D_i^n \times 0))$ given by $f'(\lambda(\partial D_i^n \times D^n))$) to a field of n frames in the normal tube of $f'(D_i^n \times 0)$ in R^{2n+1}) is an element of $\pi_{n-1}(V_{n+1,n}) \approx \pi_{n-1}(\text{SO}(n+1)) \approx \pi_{n-1}(\text{SO}(2n))$ which is the zero group if $n \equiv 3, 5, 6, 7 \pmod 8$. Thus N^{2n} is embeddable in R^{2n+1}.

ADDENDA. (1) Since this writing, I have been informed that by using results of Wu rather than those of Haefliger, the bound in Lemma 2 can be lowered to $n \geq 3$. Therefore, the bound in Theorem 1 can be lowered to $n \geq 4$ and in Theorem 2 to $n \geq 3$.

(2) If $n \equiv 1 \pmod 8$ we have that $\pi_{n-1}(\text{SO}(n))$ is isomorphic to $Z_2 \oplus Z_2$. The extra Z_2 necessitates the introduction of second invariant X (see Wall [17, bottom of p. 172]). Results similar to Theorem 2 can be obtained in this case involving Φ and X.

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