POWER SERIES WHOSE SECTIONS HAVE ZEROS OF LARGE MODULUS

BY

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1. **Introduction.** Several theorems in the theory of polynomials deal with the problem of obtaining bounds for the modulus of one or more zeros of a polynomial, $a_0 + a_1 z + \cdots + a_n z^n$, when certain of the coefficients $a_0, a_1, \ldots, a_k$, are regarded as fixed, and the remaining are arbitrary (cf. [2, Chapter 8]). In the present paper we apply results of this nature to partial sums of a power series $\sum a_p z^p$. For each positive integer n, $\mathfrak{r}_n$ will denote the radius of the largest circle with center at $z = 0$ whose interior contains no zero of the nth section,

$$s_n(z) = \sum_{p=0}^{n} a_p z^p .$$

We shall be concerned primarily with growth properties of the sequence $\{\mathfrak{r}_n\}$. The most interesting case is that in which $\sum a_p z^p$ is the power series for an entire function which omits the value zero. It is not hard to show that this is equivalent to having $\lim \mathfrak{r}_n = \infty$. One can, however, construct other power series for which $\limsup \mathfrak{r}_n = \infty$.

Since nothing is lost by doing so, we shall always suppose that $a_0 = 1$. This assumption will be used freely and without explicit mention. For notational convenience, $\sum$ and $\sum'$ will denote sums taken over the nonnegative and positive integers, respectively.

In §2, upper bounds for $\mathfrak{r}_n$ are obtained from algebraic relations between the zeros of $s_n(z)$ and the first "few" of the numbers $a_1, a_2, a_3, \cdots$. From algebraic considerations alone, we show that

$$1 + \mathfrak{r}_n = n^{o(1)}, \tag{1.1}$$

except possibly for certain "exceptional" power series, and, further, that these exceptions must be power series for entire functions of the form $\exp\{P(z)\}$, where $P(z)$ is a polynomial.

In §3 we use analytic methods to obtain lower bounds for $\mathfrak{r}_n$ in case $\sum a_p z^p$ is an entire function which omits the value zero. We are able to show that the "apparent exceptions" to (1.1) are actual exceptions, and thus characterize entire functions of the form $\exp\{P(z)\}$ for $P(z)$ a polynomial.

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Taken together, the upper and lower bounds yield a number of asymptotic properties of the sequence $\{\mathfrak{r}_n\}$. For $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of order ρ, we show that

$$\limsup \frac{\log\log n}{\log \mathfrak{r}_n} = \rho .$$

This and similar results are discussed in §4.

In §5 we prove that

$$\limsup \mathfrak{r}_n |a_n|^{1/n} = 1, \tag{1.2}$$

provided that $\sum a_p z^p$ is an entire function of infinite order which has no zeros. Making use of (1.2) and the observation that $|a_n|^{-1/n}$ is the geometric mean of the zeros of $s_n(z)$, we deduce the following: *If $\sum a_p z^p$ is an entire function of infinite order without zeros, $\varepsilon > 0$, and $\varepsilon' > 0$, then, for infinitely many integers n, fewer than $n\varepsilon'$ zeros of $s_n(z)$ have moduli greater than $\mathfrak{r}_n(1+\varepsilon)$.* This result is of some interest in connection with theorems of F. Carlson [1] and P. Rosenbloom [6] on zeros of sections of entire series of infinite order.

2. **Upper bounds.** Let $\sum' b_p z^p$ be the power series obtained formally from the identity

$$\frac{\sum p a_p z^{p-1}}{\sum a_p z^p} = \sum' p b_p z^{p-1} .$$

THEOREM 2.1. *If k is a positive integer such that $b_k \neq 0$, and $n \geqq k$, then $s_n(z)$ has a zero in the disc*

$$|z| \leqq \left\{\frac{n}{k|b_k|}\right\}^{1/k}.$$

Proof. If one lets

$$\frac{s_n'(z)}{s_n(z)} = \sum' p b_p^{(n)} z^{p-1},$$

and observes that $b_p^{(n)} = b_p$ for $p \leqq n$, the result then follows from a theorem of G. Sz.-Nagy ([3], [4], and [2, Example 2, p. 43]).

As a consequence of the above, we have

$$\mathfrak{r}_n = O(n^{1/k}) \tag{2.1}$$

for every value of k for which $b_k \neq 0$. If $\sum a_p z^p = \exp\{P(z)\}$ for some polynomial $P(z)$, then (2.1) holds with k equal to the degree of $P(z)$. If $\sum a_p z^p$ is not a power series of this form, then (2.1) holds for infinitely many integers k. This establishes (1.1).

COROLLARY 2.2.

$$\liminf \frac{\log \mathfrak{r}_n}{\log\log n} \leqq \liminf \frac{\log(1/|b_k|)}{k \log k} . \tag{2.2}$$

Proof. From Theorem 2.1 we have

$$\frac{\log \mathfrak{r}_n}{\log k} \leqq \frac{\log n}{k \log k} - \frac{1}{k} + \frac{\log(1/|b_k|)}{k \log k}.$$

Choose $n = n(k)$ so that $\log n \sim k$, and let $k \to \infty$.

The above result is of particular interest if $\sum' b_p z^p = g(z)$, where $g(z)$ is an entire function of order ρ. We then have $\sum a_p z^p = \exp\{g(z)\}$, and

$$\limsup \frac{\log\log n}{\log \mathfrak{r}_n} \geqq \rho, \tag{2.3}$$

since the right-hand side of (2.2) is $1/\rho$. Later we shall see that equality holds in (2.3).

For certain entire functions $g(z)$, Theorem 2.1 can be used to obtain an extremely good upper bound for $\mathfrak{r}_n$. For this purpose we make use of the *maximum term* function, $\mu(r) = \mu(r, g)$, defined by

$$\mu(r) = \max_p \{|b_p| r^p\}, \tag{2.4}$$

and the *central index*, $\nu(r)$, which is the largest integer m such that

$$\mu(r) = |b_m| r^m.$$

THEOREM 2.3. *Let $\sum a_p z^p = \exp\{g(z)\}$, where $g(z) = \sum' b_p z^p$ is an entire function of finite order. For each n, let β_n be the positive number such that $\mu(\beta_n) = n$, where $\mu(r)$ is defined by* (2.4). *Then for all sufficiently large n, $s_n(z)$ has a zero in the disc $|z| \leqq \beta_n$.*

Proof. From Theorem 2.1,

$$\mathfrak{r}_n \leqq \left\{\frac{n}{k|b_k|}\right\}^{1/k} \leqq \left\{\frac{n}{|b_k|}\right\}^{1/k}.$$

Let $k = \nu(\beta_n)$. Then $|b_k| \beta_n^k = \mu(\beta_n) = n$. Therefore $\mathfrak{r}_n \leqq \beta_n$.

It remains to show that $k \leqq n$, or equivalently, that $\nu(\beta_n) \leqq \mu(\beta_n)$. This is true provided the inequality

$$\nu(r) < \mu(r) \tag{2.5}$$

holds for all sufficiently large r. A proof of (2.5) follows easily from the relation [10, p. 34]

$$\limsup \frac{\log \nu(r)}{\log r} = \rho,$$

where ρ is the order of $g(z)$. The hypothesis that ρ is finite can, therefore, be replaced by (2.5).

3. **Lower bounds.** We obtain lower bounds for the numbers $\mathfrak{r}_n$ under the assumption that $\sum a_p z^p = \exp\{g(z)\}$, where $g(z) = \sum' b_p z^p$ is an entire function.

$G(z)$ will denote the majorant of $g(z)$ defined by

(3.1) $$G(z) = \Sigma' |b_p| z^p.$$

We note for future use that the order of $G(z)$ is the same as that of $g(z)$; we shall also need the inequality

(3.2) $$\Sigma |a_p| r^p \leqq \exp\{G(r)\} \qquad \text{if } r \geqq 0.$$

A proof of (3.2) follows from expanding $\exp\{G(r)\}$ as a power series in r and observing that the coefficient of r^p is at least as great as $|a_p|$.

THEOREM 3.1. *Let* $\Sigma a_p z^p = \exp\{g(z)\}$, *where* $g(z)$ *is an entire function with majorant* $G(z)$ *given by* (3.1). *If* n *is a positive integer, then*

(3.3) $$\mathfrak{r}_n > r \exp\{-2G(r)/n\} \qquad \text{for all } r \geqq 0.$$

In particular, if α_n *is the positive number such that* $G(\alpha_n) = n$, *then*

(3.4) $$\mathfrak{r}_n > \alpha_n / e^2.$$

Furthermore, if $g(z)$ *is not a polynomial, then for* $\varepsilon > 0$, *one has*

(3.5) $$\mathfrak{r}_n > \alpha_n(1-\varepsilon)$$

for all sufficiently large n.

Proof. Suppose $r > 0$ and let $f(z) = \Sigma a_p z^p$. We shall establish (3.3) by showing that

(3.6) $$|z| \leqq r \exp\{-2G(r)/n\}$$

implies

$$|1 - s_n(z)/f(z)| < 1,$$

and therefore that $s_n(z) \neq 0$. The latter is obviously true if $z = 0$; suppose z satisfies (3.6) and $z \neq 0$. Then $0 < |z| < r$, and $|1/f(z)| = |\exp\{-g(z)\}| < \exp\{G(r)\}$. Also,

$$\begin{aligned} |f(z) - s_n(z)| &\leqq |z/r|^n \sum_{p=n+1}^{\infty} |a_p| r^p |z/r|^{p-n} \\ &< |z/r|^n \sum_{p=n+1}^{\infty} |a_p| r^p \\ &< |z/r|^n \exp\{G(r)\} \end{aligned}$$

by virtue of (3.2). Therefore

$$|1 - s_n(z)/f(z)| < \{|z/r| \exp\{2G(r)/n\}\}^n \leqq 1,$$

by (3.6). This proves (3.3). If $r = \alpha_n$, we have (3.4).

The proof of (3.5) depends on the following property of $G(r)$ (cf. [5, Vol. 2, p. 4]): If $g(z)$ is not a polynomial, and $0 < c < 1$, then

$$\lim_{r\to\infty} \frac{G(cr)}{G(r)} = 0. \tag{3.7}$$

We now make use of (3.7) and the sequence $\{\alpha_n\}$ to construct a sequence $\{c_n\}$ such that

$$\lim c_n = 1 \quad \text{and} \quad \lim \frac{G(c_n\alpha_n)}{G(\alpha_n)} = 0.$$

In (3.3) let $r = c_n\alpha_n$ and replace n by $G(\alpha_n)$. Then

$$\mathfrak{r}_n > c_n\alpha_n \exp\{-2G(c_n\alpha_n)/G(\alpha_n)\}.$$

Observing that

$$\lim c_n \exp\{-2G(c_n\alpha_n)/G(\alpha_n)\} = 1$$

establishes (3.5).

4. **Asymptotic properties.** In a number of cases, fairly precise information about the sequence $\{\mathfrak{r}_n\}$ can be obtained by comparing the upper bounds of §2 with the lower bounds developed in §3.

THEOREM 4.1. *If* $\sum a_p z^p = \exp\{P(z)\}$, *where* $P(z)$ *is a polynomial of degree* k, *then there are positive numbers* A *and* B *such that*

$$An^{1/k} < \mathfrak{r}_n < Bn^{1/k}, \qquad n = 1,2,3,\cdots.$$

The proof, which is omitted, follows easily from (2.1) and (3.4).

In view of (1.1), one sees that Theorem 4.1 characterizes power series for entire functions of the form $\exp\{P(z)\}$. Among all power series, the exponential series (more accurately, the series for ae^{bz}) is the only one for which $\mathfrak{r}_n$ increases as rapidly as a linear function of n. Zeros of sections and remainders of this series have been investigated by G. Szegö [8].

Our next theorem is similar in some respects to a theorem of M. Tsuji [9] on the maximum modulus of zeros of sections of an entire series. If we let R_n denote the largest modulus of a zero of $s_n(z)$ (with the convention that $R_n = \infty$ if $a_n = 0$), Tsuji's theorem asserts that

$$\limsup \frac{\log n}{\log R_n}$$

is equal to the order of $\sum a_p z^p$. For an entire function $\sum a_p z^p$ which omits zero we obtain an analogous result involving $\mathfrak{r}_n$ and the order of the logarithm of $\sum a_p z^p$.

THEOREM 4.2. *If* $\sum a_p z^p = \exp\{g(z)\}$, *where* $g(z)$ *is an entire function of order* ρ $(0 \leqq \rho \leqq \infty)$, *then*

$$\limsup \frac{\log\log n}{\log \mathfrak{r}_n} = \rho.$$

Proof. Since the order of $G(z)$ is also ρ, we have

$$\rho = \limsup_{r\to\infty} \frac{\log \log G(r)}{\log r} \geqq \limsup_{n\to\infty} \frac{\log \log n}{\log \alpha_n}$$

$$\geqq \limsup \frac{\log\log n}{\log(\mathfrak{r}_n e^2)}$$

by (3.4). Since $\log(\mathfrak{r}_n e^2) \sim \log \mathfrak{r}_n$, we can neglect the factor e^2. The other half of the proof follows from (2.3).

If $g(z)$ is of finite positive order ρ and of type τ $(0 \leqq \tau \leqq \infty)$, one can prove a sharper result, namely, that

$$\limsup \frac{\log n}{\mathfrak{r}_n^{\rho}} = \tau.$$

For this one needs (3.5) in place of (3.4); the "$\geqq$" half of the result is obtained from Theorem 2.1 by a procedure similar to the proof of Corollary 2.2. In this case one chooses $n = n(k)$ so that $\log n \sim k/\rho$.

If $g(z)$ is of finite order, the asymptotic relation [5, Vol. 2, p. 8]

$$\log G(r) \sim \log \mu(r, G) = \log \mu(r, g) \tag{4.1}$$

yields information about the relative sizes of α_n and β_n. From (4.1) and the definitions of α_n and β_n, we have

$$\log G(\alpha_n) \sim \log G(\beta_n). \tag{4.2}$$

This supplies a connecting link between our upper and lower bounds for $\mathfrak{r}_n$.

THEOREM 4.3. *If $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of finite order, then*

$$\log \alpha_n \sim \log \mathfrak{r}_n \sim \log \beta_n.$$

Proof. Since $\alpha_n/e^2 < \mathfrak{r}_n \leqq \beta_n$ for large n, it suffices to prove that $\log \alpha_n \sim \log \beta_n$. If we note that $\alpha_n \leqq \beta_n$, the result follows from (4.2) and the Hadamard three circle theorem (applied to circles with radii 1, α_n, and β_n).

THEOREM 4.4. *Let $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function of finite order. If, for some $\delta > 0$, there is a nondecreasing function $H(r)$ such that* $\log G(r) \sim r^{\delta} H(r)$, *then*

$$\alpha_n \sim \mathfrak{r}_n \sim \beta_n.$$

Proof. Suppose $\varepsilon > 0$. The condition on $G(r)$ implies that $g(z)$ is not a polynomial; therefore $\alpha_n(1-\varepsilon) < \mathfrak{r}_n \leqq \beta_n$ for large n, and we need only prove that $\alpha_n \sim \beta_n$.

From (4.2) and the condition on $G(r)$, we have

$$1 \geqq \left[\frac{\alpha_n}{\beta_n}\right]^{\delta} \sim \frac{H(\beta_n)}{H(\alpha_n)} \geqq 1,$$

since $\alpha_n \leqq \beta_n$ and $H(r)$ is nondecreasing. Hence $\alpha_n \sim \beta_n$.

As a special case of the above, we note that the condition $\log G(r) \sim \tau r^{\rho}$ for positive numbers ρ and τ implies that

$$\mathfrak{r}_n \sim \left[\frac{\log n}{\tau}\right]^{1/\rho}.$$

For $g(z)$ of finite order, $\log G(r) \sim \log M_g(r)$, where $M_g(r)$ is the maximum modulus of $g(z)$ on $|z| = r$. Therefore the condition on $G(r)$ in the hypothesis of Theorem 4.4 is equivalent to the corresponding condition on $M_g(r)$. In addition, we note that Theorems 4.3 and 4.4 remain valid if α_n is replaced by α_n', where α_n' is defined by $M_g(\alpha_n') = n$.

5. **Comparison with the coefficients.** In this section we restrict our attention to the case $\sum a_p z^p = \exp\{g(z)\}$, where $g(z)$ is an entire function which is not a polynomial. This is equivalent to requiring that $\sum a_p z^p$ be an entire function of infinite order without zeros. We shall compare the lower bound (3.5) for $\mathfrak{r}_n$ with the elementary upper bound

$$\mathfrak{r}_n \leqq |a_n|^{-1/n} \qquad \text{if } a_n \neq 0. \tag{5.1}$$

(The right-hand side of (5.1) is the geometric mean of the moduli of zeros of $s_n(z)$.) Our principal result is the following:

THEOREM 5.1. *If $\sum a_p z^p$ is an entire function of infinite order without zeros, then*

$$\limsup \alpha_n |a_n|^{1/n} = \limsup \mathfrak{r}_n |a_n|^{1/n} = 1.$$

Before proving Theorem 5.1 we shall consider two of its corollaries.

COROLLARY 5.2. *Let $\sum a_p z^p$ satisfy the hypothesis of Theorem 5.1. If $\varepsilon > 0$, then*

$$\alpha_n(1-\varepsilon) < \mathfrak{r}_n \quad \text{and} \quad \mathfrak{r}_n |a_n|^{1/n} \leqq 1$$

for all sufficiently large n, and

$$\alpha_n(1+\varepsilon) > \mathfrak{r}_n > (1-\varepsilon)|a_n|^{-1/n}$$

for infinitely many n.

Proof. Theorem 5.1 and inequalities (3.5) and (5.1).

COROLLARY 5.3. *If* $\sum a_p z^p$ *is an entire function of infinite order without zeros,* $\varepsilon > 0$ *and* $\varepsilon' > 0$, *then for infinitely many integers* n, *fewer than* $n\varepsilon'$ *zeros of* $s_n(z)$ *have moduli greater than* $\mathfrak{r}_n(1+\varepsilon)$.

Proof. Choose δ so that

$$0 < \delta < 1 - (1+\varepsilon)^{-\varepsilon'}.$$

If n is a positive integer for which $\mathfrak{r}_n > (1-\delta)|a_n|^{-1/n}$, then an easy calculation shows that fewer than $n\varepsilon'$ zeros of $s_n(z)$ have moduli greater than $\mathfrak{r}_n(1+\varepsilon)$.

Proof of Theorem 5.1. From (3.5) and (5.1) it follows that

$$\limsup \alpha_n |a_n|^{1/n} \leqq \limsup \mathfrak{r}_n |a_n|^{1/n} \leqq 1;$$

consequently, we have only to prove that

$$\limsup \alpha_n |a_n|^{1/n} \geqq 1. \tag{5.2}$$

To facilitate the proof of (5.2) we first establish two lemmas. These will enable us to obtain a lower bound for α_n in terms of the maximum modulus of $\sum a_p z^p$.

LEMMA 5.1a. *For all* $r > 0$,

$$\alpha_n > r \exp\{-2G(r)/n\}. \tag{5.3}$$

Proof. The function $u(r) = r\exp\{-2G(r)\}$ assumes its maximum at the number $r = \gamma_n$ such that $2\gamma_n G'(\gamma_n) = n$. Since $2rG'(r) > G(r)$ for all $r > 0$, we have

$$\alpha_n > \gamma_n > \gamma_n \exp\{-2G(\gamma_n)/n\}.$$

LEMMA 5.1b. *If* $\varepsilon > 0$, *then*

$$\alpha_n > \frac{r}{1+\varepsilon}[M(r)]^{-8/n\varepsilon} \qquad \textit{for all } r > 0,$$

where

$$M(r) = \max_{|z|=r} \left|\sum a_p z^p\right|. \tag{5.4}$$

Proof. The proof depends on the following variant of the Borel-Carathéodory inequality (proved, but not explicitly stated, in [10, pp. 17–20]): If $0 < r < R$, then

$$G(r) = \sum{}' |b_p| r^p \leqq \frac{4r}{R-r} A(R), \tag{5.5}$$

where

$$A(R) = \max_{|z|=R} \{\mathrm{Re} \sum{}' b_p z^p\}.$$

If we rewrite (5.3) in the form

$$\alpha_n > \frac{r}{1+\varepsilon}\exp\left\{-\frac{2}{n}G\left(\frac{r}{1+\varepsilon}\right)\right\}$$

and in (5.5) replace r and R by $r/(1+\varepsilon)$ and r respectively, we have

$$\alpha_n > \frac{r}{1+\varepsilon}\exp\left\{\frac{-8A(r)}{n\varepsilon}\right\}.$$

Since $A(r) = \log M(r)$, the result follows.

It is worth noting that the inequality of Lemma 5.1b remains valid if α_n is replaced by $\mathfrak{r}_n$. To see this one uses (3.3) in place of (5.3).

We are now in a position to establish (5.2). Let $\mu(r)$ and $\nu(r)$ denote (contrary to previous usage) the maximum term and central index of the series $\sum a_p z^p$. If we choose $n = \nu(r)$, the inequality of Lemma 5.1b can be written as

$$\log\{(1+\varepsilon)\alpha_n|a_n|^{1/n}\} > \frac{\log\mu(r)}{\nu(r)} - \frac{8}{\varepsilon}\frac{\log M(r)}{\nu(r)}.$$

Since $\sum a_p z^p$ is of infinite order, it follows that

$$\liminf\frac{\log M(r)}{\nu(r)} = 0$$

from a theorem of S. M. Shah [7]. Therefore

$$\limsup(1+\varepsilon)\alpha_n|a_n|^{1/n} \geqq 1,$$

which establishes (5.2) and completes the proof.

Theorem 5.1 adds an interesting footnote to certain more general results on zeros of sections of power series. If $\sum a_p z^p$ is an entire function of infinite order and $\varepsilon > 0$, it is known [1], [6] that, for infinitely many integers n, all but $o(n)$ zeros of $s_n(z)$ lie in the annulus

$$(1-\varepsilon)|a_n|^{-1/n} < |z| < (1+\varepsilon)|a_n|^{-1/n}.$$

If $\sum a_p z^p$ omits the value zero, we have shown that, for infinitely many integers n, no zero of $s_n(z)$ lies in the interior of the inner circle of the annulus.

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