SOME PROPERTIES OF PONTRYAGIN CLASSES MOD 3

BY
YASURÔ TOMONAGA

Introduction. In this paper we shall study some properties of Pontryagin classes mod 3 in a way similar to that of Massey in the case of Stiefel-Whitney classes [1; 2]. In the case of Pontryagin classes mod q, where q denotes a prime number larger than 2, Hirzebruch has established a relation similar to that of Wu concerning Stiefel-Whitney classes [3;4]. In the case $q = 3$ the relation takes a simple form and plays a basic role in this paper. After the submission of this paper the author learned that H. Roberts had dealt with a similar problem [6].

1. Let q be a prime number larger than 2 and let X_n be a compact orientable differentiable n-manifold. We denote by $\mathscr{P}^r_q$ the Steenrod power [5]

(1.1) $$\mathscr{P}^r_q : H^i(X_n,Z_q) \to H^{i+2r(q-1)}(X_n,Z_q).$$

For $n = i + 2r(q-1)$ an element

(1.2) $$s^r_q \in H^{2r(q-1)}(X_n,Z_q)$$

is characterized by the Poincaré duality theorem and the relation

(1.3) $$\mathscr{P}^r_q v = s^r_q v \qquad [3;4]$$

for all

(1.4) $$v \in H^{n-2r(q-1)}(X_n,Z_q).$$

Let $\{L_i\}$ be the multiplicative sequence of polynomials corresponding to the power series

(1.5) $$\sum_{i=0}^{\infty} a_i z^i = \frac{\sqrt{z}}{tgh\sqrt{z}}.$$

It is known that

(1.6) $$s^r_q = q^r L_{r(q-1)/2}(p_1, \cdots) \bmod q$$

where $p_i \in H^{4i}(X_n,Z)$ denotes the Pontryagin class and we put

Received by the editors April 23, 1963.

$$p = \sum_{i \geq 0} p_i = \prod_i (1 + \gamma_i). \tag{1.7}$$

It is known that the mod q $4j$-cohomology classes

$$b_{q,j} = \sum \mathscr{P}_q^i s_q^r, \tag{1.8}$$

where the sum is extended over all i, r with

$$2j = i(q-1) + r(q-1), \tag{1.9}$$

satisfy the relation

$$\sum_{j \geq 0} b_{q,j} = \prod_i (1 + \gamma_i^l) \tag{1.10}$$

where

$$l = \tfrac{1}{2}(q-1) \qquad [4]. \tag{1.11}$$

When $q = 3$ (1.10) takes the form

$$\sum_{j \geq 0} b_{3,j} = \prod_i (1 + \gamma_i) = \sum_{i \geq 0} p_i \mod 3. \tag{1.12}$$

Hence we have from (1.8), (1.9) and (1.12)

$$p_j = \sum_{j = i + r} \mathscr{P}_3^i s_3^r \mod 3. \tag{1.13}$$

Hereafter we shall deal exclusively with the case $q = 3$ and use the following notations:

$$\mathscr{P}_3^i = \mathscr{P}^i \quad \text{and} \quad s_3^r = s^r. \tag{1.14}$$

The following arguments will be confined to Z_3 and we shall not use the symbol mod 3 except in important cases. We define a class $\sum_{j \geq 0} \bar{s}^j$ $(\bar{s}^j \in H^{4j}(X_n, Z_3))$ by

$$\sum_{i \geq 0} s^i \sum_{j \geq 0} \bar{s}^j = 1. \tag{1.15}$$

The dual Pontryagin classes $\sum_{j \geq 0} \bar{p}_j$ $(\bar{p}_j \in H^{4j}(X_n, Z))$ are defined by

$$\sum_{i \geq 0} p_i \sum_{j \geq 0} \bar{p}_j = 1. \tag{1.16}$$

As for the Steenrod powers we shall use the following relations:

$$\begin{cases} \text{(i)} \ \mathscr{P}^0 = \text{identity} \quad [5], \\ \text{(ii)} \ \mathscr{P}^r(uv) = \sum_{s=0}^{r} \mathscr{P}^s u \mathscr{P}^{r-s} v, \\ \text{(iii)} \ \mathscr{P}^r \mathscr{P}^s = \sum_i (-1)^{r+i} \mathscr{P}^{r+s-i} \mathscr{P}^i \quad (r < 3s), \\ \text{(iv)} \ \mathscr{P}^j u_k = 0, \quad j > k/2, \quad u_k \in H^k(X_n, Z_3). \end{cases} \tag{1.17}$$

From (1.13), (1.15), (1.16) and (1.17)(i),(ii),(iii) we have

$$\bar{p}_j = \sum_{i+r=j;\, i,r \geqq 0} \mathscr{P}^i \bar{s}^r \mod 3 \tag{1.18}$$

because

$$\begin{aligned} 1 &= \Big(\sum_{i\geqq 0} \mathscr{P}^i\Big)\Big\{\Big(\sum_{j\geqq 0} s^j\Big)\Big(\sum_{k\geqq 0} \bar{s}^k\Big)\Big\} = \Big(\sum_{i\geqq 0}\mathscr{P}^i\Big)\sum_{j,k\geqq 0} s^j\bar{s}^k \\ &= \sum_{i\geqq 0}\ \sum_{j,k\geqq 0}\ \sum_{q=0}^{i} \mathscr{P}^{i-q}s^j\mathscr{P}^q\bar{s}^k = \sum_{t,j\geqq 0}\mathscr{P}^t s^j \sum_{q,k\geqq 0}\mathscr{P}^q\bar{s}^k = p\sum_{q,k\geqq 0}\mathscr{P}^q\bar{s}^k. \end{aligned} \tag{1.19}$$

2. Let us prove

Lemma 1. *For any $x \in H^{n-4k}(X_n, Z_3)$ $(0 < k < n/4)$, it holds that*

$$x\bar{p}_k = \sum_{r=1}^{k} (-\mathscr{P}^r x)\bar{p}_{k-r} \mod 3. \tag{2.1}$$

Proof. We have from (1.18)

$$\bar{p}_k = \sum_{i+r=k} \mathscr{P}^i\bar{s}^r = \bar{s}^k + \sum_{i=1}^{k}\mathscr{P}^i\bar{s}^{k-i}. \tag{2.2}$$

On the other hand we have from (1.15)

$$0 = \bar{s}^k + \sum_{i=1}^{k} s^i\bar{s}^{k-i}. \tag{2.3}$$

Hence we have from (2.2) and (2.3)

$$\bar{p}_k = \sum_{i=1}^{k}(\mathscr{P}^i\bar{s}^{k-i} - s^i\bar{s}^{k-i}). \tag{2.4}$$

We have from (2.4)

$$x\bar{p}_k = \sum_{i=1}^{k}(x\mathscr{P}^i\bar{s}^{k-i} - xs^i\bar{s}^{k-i}). \tag{2.5}$$

Meanwhile we have from (1.1) and (1.17)(ii)

$$xs^i\bar{s}^{k-i} = s^i(x\bar{s}^{k-i}) = \mathscr{P}^i(x\bar{s}^{k-i}) = \sum_{r=0}^{i}\mathscr{P}^r x\mathscr{P}^{i-r}\bar{s}^{k-i}. \tag{2.6}$$

Hence we have from (2.5), (2.6) and (1.18)

$$\begin{aligned} x\bar{p}_k &= \sum_{i=1}^{k}\sum_{r=1}^{i}(-\mathscr{P}^r x\mathscr{P}^{i-r}\bar{s}^{k-i}) = \sum_{0<r\leqq i\leqq k}(-\mathscr{P}^r x)(\mathscr{P}^{i-r}\bar{s}^{k-i}) \\ &= \sum_{r=1}^{k}\Big[(-\mathscr{P}^r x)\sum_{j\geqq 0}^{k-r}\mathscr{P}^j\bar{s}^{k-j-r}\Big] = \sum_{r=1}^{k}(-\mathscr{P}^r x)\bar{p}_{k-r}. \end{aligned} \tag{2.7}$$

Q.E.D.

Applying the formula (2.1) many times we can express $x\bar{p}_k$ as a product of x and a certain sum of iterated Steenrod powers:

$$(2.8) \qquad x\bar{p}_k = \sum_{i_1+\dots+i_r=k} \mathscr{P}^{i_1}\cdots\mathscr{P}^{i_r}x.$$

3. Suppose that

$$(3.1) \qquad \bar{p}_k \neq 0 \mod 3,\ 0 < k < \frac{n}{4}.$$

Then the homomorphism

$$(3.2) \qquad x \to x\bar{p}_k, \quad H^{n-4k}(X_n,Z_3) \to H^n(X_n,Z_3)$$

is not zero and takes the form (2.8). On the other hand any iterated Steenrod power $\mathscr{P}^{i_1}\cdots\mathscr{P}^{i_r}$ can be expressed as a sum of the admissible ones by virtue of (1.17)(iii):

$$(3.3) \qquad \mathscr{P}^I = \mathscr{P}^{i_1}\cdots\mathscr{P}^{i_r},$$

$$(3.4) \qquad n(I) = i_1 + \cdots + i_r = k,$$

$$(3.5) \qquad i_1 \geqq 3i_2,\ i_2 \geqq 3i_3, \cdots, i_{r-1} \geqq 3i_r > 0.$$

Hence we have

THEOREM 1. *Let X_n be a compact orientable differentiable n-manifold. If $\bar{p}_k \neq 0$ mod 3 $(0 < k < n/4)$, then it holds for some admissible iterated Steenrod power $\mathscr{P}^I$ $(n(I)=k)$ and some nonzero $x \in H^{n-4k}(X_n, Z_3)$ that $\mathscr{P}^I x \neq 0$ mod 3.*

We put as follows:

$$(3.6) \qquad i_1 = 3i_2 + \alpha_1,\ i_2 = 3i_3 + \alpha_2, \cdots i_{r-1} = 3i_r + \alpha_{r-1}, i_r = \alpha_r,$$

$$(3.7) \qquad e(I) = \alpha_1 + \cdots + \alpha_r.$$

Let us prove

LEMMA 2. *For any admissible iterated Steenrod power $\mathscr{P}^I$ we have $\mathscr{P}^I x = 0$ mod 3 provided that degree $x < 2e(I)$.*

Proof. We have from (3.4), (3.6) and (3.7)

$$(3.8) \qquad n(I) = e(I) + 3(i_2 + \cdots + i_r).$$

Hence we have

$$(3.9) \qquad i_1 - 2i_2 - \cdots - 2i_r = e(I) > \tfrac{1}{2}(\text{degree}\, x),$$

i.e.,

$$(3.10) \qquad 2i_1 - 4i_2 - \cdots - 4i_r > \text{degree}\, x.$$

Therefore we have

$$(3.11) \qquad 2i_1 > \text{degree}\,(\mathscr{P}^{i_2}\cdots\mathscr{P}^{i_r}x).$$

Hence we have from (1.17)(iv)

$$\mathscr{P}^{i_1} \cdots \mathscr{P}^{i_r} x = 0. \tag{3.12}$$ Q.E.D.

We put $q = n - 4k$ and assume that $\mathscr{P}^I$ is admissible and

$$e(I) \leqq \frac{q}{2}, \quad n(I) = k, \quad 1 < q < n. \tag{3.13}$$

First we consider the case where $e(I) = q/2$. For any $x \in H^q(X_n, Z_3)$ we have

$$\begin{aligned} \text{degree } \mathscr{P}^I x &= 4n(I) + q = 6i_1 - 2e(I) + q \\ &= 2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r) - 2e(I) + q \\ &= 2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r). \end{aligned} \tag{3.14}$$

The last bracket of (3.14) consists of powers of 3 whose number is equal to

$$\alpha_1 + \cdots + \alpha_r = e(I) = q/2. \tag{3.15}$$

Hence we have from (3.14)

$$n = \text{degree } \mathscr{P}^I x = 2(3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q/2}}) \quad (h_1 \geqq h_2 \geqq \cdots \geqq h_{q/2} \geqq 1) \tag{3.16}$$

provided that $\mathscr{P}^I x$ is not zero mod 3. Next we consider the case where $2e(I) < q$. We put

$$\alpha_0 = q - 1 - 2e(I). \tag{3.17}$$

In the same way we have

$$\begin{aligned} n = \text{degree } \mathscr{P}^I x &= 2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r) + \alpha_0 + 1 \\ &= \{2(3\alpha_1 + 3^2\alpha_2 + \cdots + 3^r\alpha_r) + 3\alpha_0\} - 2\alpha_0 + 1. \end{aligned} \tag{3.18}$$

The number of powers of 3 in the last bracket of (3.18) is equal to $\alpha_0 + 2e(I) = q - 1$. Hence we have

$$n = (3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q-1}}) - 2\alpha_0 + 1 \tag{3.19}$$

where $h_1 \geqq h_2 \geqq \cdots \geqq h_{q-1} \geqq 1$ and $0 \leqq \alpha_0 \leqq q - 3$.

In the case $q = 1$ it follows from (1.17)(iv) that

$$\mathscr{P}^I x = 0 \mod 3$$

for any nonidentity $\mathscr{P}^I$. Hence we have from Theorem 1

$$\bar{p}_k = 0 \mod 3.$$

The same thing holds for the case $n = 4k$ by (1.3) and (2.4). Thus we have

THEOREM 2. *Let X_n be a compact orientable differentiable n-manifold. I $\bar{p}_k \neq 0 \bmod 3$ and $n > n - 4k > 1$ we have either*

$$n = 2(3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q/2}}) \quad (h_1 \geqq h_2 \geqq \cdots \geqq h_{q/2} \geqq 1, \ q = n - 4k)$$

or

$$n = 3^{h_1} + 3^{h_2} + \cdots + 3^{h_{q-1}} - 2\alpha_0 + 1 \quad (h_1 \geqq h_2 \geqq \cdots \geqq h_{q-1} \geqq 1, \;\; q = n-4k, \;\; 0 \leqq \alpha_0 \leqq q-3).$$

If $0 \leqq n - 4k \leqq 1$, *then* $\bar{p}_k = 0 \mod 3$.

4. We can derive various corollaries from Theorems 1 and 2. For example we consider the case where $n = 14$. If $\bar{p}_3 \neq 0 \mod 3$ we have from Theorem 2

$$14 = 2 \cdot 3^h, \qquad h \geqq 1, \tag{4.1}$$

and this is impossible. The second case of Theorem 2 is also impossible because $q = 2$. Hence we have

$$\bar{p}_3 = 0 \mod 3. \tag{4.2}$$

Thus we have

COROLLARY 1. *For any compact orientable differentiable* 14-*manifold* X_{14}

$$\bar{p}_3 = 0 \mod 3.$$

We can prove the above corollary otherwise. If $\bar{p}_3 \neq 0 \mod 3$ we have from Theorem 1

$$\mathscr{P}^I x \neq 0 \mod 3 \quad (n(I) = 3)$$

for some nonzero $x \in H^2(X_{14}, Z_3)$ and some admissible $\mathscr{P}^I$. However, the only possible $\mathscr{P}_I$ is $\mathscr{P}^3$ and we have

$$\mathscr{P}^3 x = 0 \mod 3$$

by (1.17)(iii). Hence we have $\bar{p}_3 = 0 \mod 3$. In such a way we see that in most cases $\bar{p}_k = 0 \mod 3$ provided that $4k$ is close enough to n.

REFERENCES

1. W. S. Massey, *On the Stiefel-Whitney classes of a manifold*, Amer. J. Math. **82** (1960), 92–102.

2. ———, *On the Stiefel-Whitney classes of a manifold*. II, Proc. Amer. Math. Soc. **13** (1962), 938–942.

3. F. Hirzebruch, *On Steenrod's reduced powers, the index of inertia and the Todd genus*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 951–956.

4. ———, *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) **60** (1954), 213–236.

5. J. Adem, *The relations on Steenrod powers of cohomology classes*, Algebraic geometry and topology, pp. 191–238, Princeton Univ. Press, Princeton, N. J., 1957.

6. H. Roberts, *On the Pontryagin classes of a manifold*, Abstract 63T-23, Notices Amer. Math. Soc. **10** (1963), 129.

UTSUNOMIYA UNIVERSITY,
UTSUNOMIYA, JAPAN