RELATIVE HOMOLOGICAL ALGEBRA AND HOMOLOGICAL DIMENSION OF LIE ALGEBRAS

BY

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Introduction. The main problem we shall consider concerns the various kinds of homological dimension that can be attached to a ring or to a pair consisting of a ring and a subring. We begin by investigating the functorial behavior of the relative Tor and Ext functors for pairs of rings $R \supset S$ under a ring epimorphism $R \to R/I$. §1 contains a general result describing this behavior.

In §2, this is applied to the case where R is the ordinary universal enveloping algebra of a restricted Lie algebra and R/I is the restricted universal enveloping algebra of that Lie algebra. We thus obtain an identification of the Tor and Ext functors of the restricted universal enveloping algebra with the relative Tor and Ext functors for the pair (R,S), where S is a subalgebra of R defined from the p-map of the restricted Lie algebra.

The main purpose of §3 is to prove a theorem on the coincidence of the several kinds of homological dimension for a restricted Lie algebra, which is precisely analogous to the well-known result of this type for ordinary Lie algebras. The proof is obtained from an appropriate adaptation of the requisite mapping theorem of Cartan-Eilenberg.

§4 gives an application of the general technique to the partial determination of the global homological dimension of certain factor algebras of the universal enveloping algebra of a Lie algebra, corresponding to a special subclass of the representations of the Lie algebra. These particular factor algebras of the universal enveloping algebra are of special interest, because as was shown by Sridharan in his Columbia thesis, they are precisely those algebras which possess a filtration such that the associated graded algebra is an ordinary polynomial algebra.

In §5, we show that the global homological dimension of the restricted universal enveloping algebra of a restricted Lie algebra is always either 0 or infinite. For a solvable restricted Lie algebra, we show that this global dimension is infinite if and only if the p-map has a nontrivial kernel.

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1. **Isomorphism of the relative Ext and Tor functors induced by a ring epimorphism.** All the rings and subrings we consider are assumed to contain an identity element 1, and all the modules we consider are assumed to be "unitary," in the sense that the identity of the operator ring acts as the identity operator.

Let R be a ring with an identity element 1, and S a subring of R containing 1. An R-module will be regarded also as an S-module, in the natural way. Let I be a two-sided ideal of R, and write $R/I = R'$ and $(S+I)/I = S'$. An R'-module will be regarded also as an R-module, in the natural way. Then any (R',S')-projective resolution of a left R'-module A, or any (R',S')-injective resolution of a left R'-module B, can be regarded as a R-complex annihilated by I. Let

$$\cdots \to X_1 \to X_0 \to A \to 0$$

be an (R',S')-projective resolution of a left R'-module A, and let

$$\cdots \to Y_1 \to Y_0 \to A \to 0$$

be an (R,S)-projective resolution of A. Then the X-sequence is (R,S)-exact. Hence we can successively find R-homomorphisms $Y_i \to X_i$ such that the resulting diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{x_2} & X_1 & \xrightarrow{x_1} & X_0 & \to & A & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \xrightarrow[y_2]{} & Y_1 & \xrightarrow[y_1]{} & Y_0 & \to & A & \to & 0 \end{array}$$

is commutative, where I is the identity map from A to A. Moreover, if (s_i) and (t_i) are any two such systems of R-homomorphisms, there is an R-homotopy connecting them, i.e., a sequence of R-homomorphisms $h_i: Y_i \to X_{i+1}$ such that, if x_i and y_i are the maps of the above sequences, $s_i - t_i = x_{i+1} \cdot h_i + h_{i-1} \cdot y_i$ for all i (where $h_{-1} = 0$).

Hence we can proceed exactly as in the usual theory of Tor^R and get the following fact. Let A be a right R'-module, and let B be a left R-module. Then $B/I \cdot B$ is a left R'-module. The canonical map $B \to B/I \cdot B$ induces the natural homomorphism

$$\mathrm{Tor}^{(R,S)}(A,B) \to \mathrm{Tor}^{(R,S)}(A,B/I \cdot B).$$

The above map of the resolution of A gives the canonical homomorphism

$$\mathrm{Tor}^{(R,S)}(A,B/I \cdot B) \to \mathrm{Tor}^{(R',S')}(A,B/I \cdot B),$$

which is independent of the choice of the resolutions.

Similarly, if A is a right R-module and B is a left R'-module we have a canonical homomorphism

$$\mathrm{Tor}^{(R,S)}(A,B) \to \mathrm{Tor}^{(R',S')}(A/A \cdot I,B).$$

Now let A be a left R'-module, and let B be a left R-module. Let B^I be the submodule of B consisting of all elements annihilated by I. Then we find as above that maps of (R,S)-projective resolutions of A into (R',S')-projective resolutions of A and the injection map $B^I \rightarrow B$ induce a unique canonical homomorphism

$$\mathrm{Ext}_{(R',S')}(A,B^I) \rightarrow \mathrm{Ext}_{(R,S)}(A,B).$$

Dually, we observe that the (R',S')-injective resolutions of A can be mapped into the (R,S)-injective resolutions of A, whence we obtain a unique canonical homomorphism

$$\mathrm{Ext}_{(R',S')}(B/I\cdot B,A) \rightarrow \mathrm{Ext}_{(R,S)}(B,A).$$

We shall see that, under certain conditions, these homomorphisms are actually isomorphisms.

LEMMA 1.1. *Suppose that $I = R(I \cap S)$, and let A be a left R'-module. Then an (R',S')-projective resolution of A is an (R,S)-projective resolution of A, when regarded as an R-complex annihilated by I. If $I = (I \cap S)R$ then an (R',S')-injective resolution of A is an (R,S)-injective resolution of A, when regarded as an R-complex annihilated by I.*

Proof. Let

$$\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$$

be an (R',S')-projective resolution of A. If we regard this as an R-complex, it is clear that an S'-homotopy of the resolution is an S-homotopy of the R-complex. Hence the above sequence is (R,S)-exact.

Since X_i is (R',S')-projective, X_i is R'-isomorphic with a direct R'-module summand of $R' \otimes_{S'} X_i$. Hence X_i, as an R-module, is isomorphic with a direct R-module summand of $R' \otimes_{S'} X_i$. We have the following exact sequence of R-module homomorphism:

$$I \otimes_S X_i \rightarrow R \otimes_{S'} X_i \rightarrow R' \otimes_{S'} X_i \rightarrow 0.$$

Since $I = R(I \cap S)$, the first map is the 0-map. Hence $R' \otimes_{S'} X_i$ is isomorphic, as an R-module, with $R \otimes_S X_i$. Hence it is (R,S)-projective, whence also X_i is (R,S)-projective. Thus X is also an (R,S)-projective resolution of A.

Now let

$$0 \rightarrow A \rightarrow Y_0 \rightarrow Y_1 \rightarrow \cdots$$

be an (R',S')-injective resolution of A. As before, we see that this is an (R,S)-exact sequence of R-module homomorphisms. Since Y_i is (R',S')-injective, it may be identified with a direct R'-module summand of $\mathrm{Hom}_{S'}(R',Y_i)$. We have the following exact sequence of R-module homomorphisms.

$$0 \to \mathrm{Hom}_{S'}(R', Y_i) \to \mathrm{Hom}_S(R, Y_i) \to \mathrm{Hom}_S(I, Y_i).$$

Since $I = (I \cap S) \cdot R$, the last map is the 0-map. Hence $\mathrm{Hom}_{S'}(R', Y_i)$ is isomorphic as an R-module with $\mathrm{Hom}_S(R, Y_i)$, and is therefore (R,S)-injective. Hence Y_i is also (R,S)-injective, so that the sequence Y is an (R,S)-injective resolution of A.

PROPOSITION 1.1. *Let A and B be left R-modules. Suppose that $IA = 0$, and let B^I be the submodule of B consisting of all elements annihilated by I. Then, if $I = R(I \cap S)$, the canonical homomorphism of $\mathrm{Ext}_{(R',S')}(A, B^I)$ into $\mathrm{Ext}_{(R,S)}(A, B)$ is an isomorphism. Similarly, if A is arbitrary, $IB = 0$ and $I = (I \cap S)R$, the canonical homomorphism of $\mathrm{Ext}_{(R',S')}(A/I \cdot A, B)$ into $\mathrm{Ext}_{(R,S)}(A,B)$ is an isomorphism.*

If A is a left R-module with $I \cdot A = (0)$, B is a right R-module and $I = R(I \cap S)$, then the canonical homomorphism of $\mathrm{Tor}^{(R,S)}(B,A)$ into $\mathrm{Tor}^{(R',S')}(B/B \cdot I, A)$ is an isomorphism. If $B \cdot I = (0)$, A is an arbitrary left R-module and $I = (I \cap S)R$, then the canonical homomorphism of $\mathrm{Tor}^{(R,S)}(B, A)$ into $\mathrm{Tor}^{(R',S')}(B, A/I \cdot A)$ is an isomorphism.

Proof. If $I \cdot A = 0$ and $I = R \cdot (I \cap S)$, then an (R',S')-projective resolution of A is also an (R,S)-projective resolution by Lemma 1.1.

Let $\cdots \to X_1 \to X_0 \to A \to 0$ be an (R',S')-projective resolution of A. Then

$$\mathrm{Ext}_{(R',S')}(A, B^I) = H(Hom_{R'}(X, B^I)).$$

Now

$$\mathrm{Hom}_R(X_i, B) = \mathrm{Hom}_R(X_i, B^I) = \mathrm{Hom}_{R'}(X_i, B^I).$$

Hence

$$\mathrm{Ext}_{(R',S')}(A, B^I) = H(\mathrm{Hom}_R(X, B)).$$

Since X is an (R,S)-projective resolution of A, this identifies $\mathrm{Ext}_{(R',S')}(A, B^I)$ with $\mathrm{Ext}_{(R,S)}(A, B)$.

If $IB = 0$ and $I = (I \cap S) \cdot R$, then an (R',S')-injective resolution of B is also an (R,S)-injective resolution, by Lemma 1.1.

Let

$$0 \to B \to Y_0 \to Y_1 \to \cdots$$

be an (R',S')-injective resolution of B. Then

$$\mathrm{Ext}_{(R,S)}(A, B) = H(\mathrm{Hom}_R(A, Y))$$

and

$$\mathrm{Ext}_{(R',S')}(A/IA, B) = H(\mathrm{Hom}_{R'}(A/IA, Y)).$$

Now

$$\mathrm{Hom}_{R'}(A/IA, Y_i) = \mathrm{Hom}_R(A/IA, Y_i) = \mathrm{Hom}_R(A, Y_i).$$

Hence $\mathrm{Ext}_{(R',S')}(A/IA, B)$ is identified with $\mathrm{Ext}_{(R,S)}(A,B)$.

If $I \cdot A = (0)$ and $I = R \cdot (I \cap S)$, then we have

$$\mathrm{Tor}^{(R,S)}(B,A) = H(B \otimes_R X_i)$$

and

$$\mathrm{Tor}^{(R',S')}(B/B \cdot I, A) = H(B/B \cdot I \otimes_{R'} X_i).$$

Now

$$B \otimes_R X_i \approx B/B \cdot I \otimes_R X_i \approx B/B \cdot \otimes_{R'} X_i.$$

Hence $\mathrm{Tor}^{(R,S)}(B,A)$ is identified with $\mathrm{Tor}^{(R',S')}(B/B \cdot I, A)$.

If B is a right R-module such that $B \cdot I = (0)$ and $I = (I \cap S) \cdot R$, then an (R',S')-projective resolution X' of B is an (R,S)-projective resolution of B, by Lemma 1.1. Hence

$$\mathrm{Tor}^{(R,S)}(B,A) = H(X_i' \otimes_R A)$$

and

$$\mathrm{Tor}^{(R',S')}(B, A/I \cdot A) = H(X_i' \otimes_{R'} A/I \cdot A).$$

Now

$$X_i' \otimes_{R'} A/I \cdot A \approx X_i' \otimes_R A/I \cdot A \approx X_i \otimes_R A.$$

Hence $\mathrm{Tor}^{(R,S)}(B,A)$ is identified with $\mathrm{Tor}^{(R',S')}(B, A/IA)$. This completes the proof of Proposition 1.1.

2. **Restricted Lie algebras.** Let L be a restricted Lie algebra over a field F of characteristic $p \neq 0$. Let R be the ordinary universal enveloping algebra of L, and let R' be the restricted universal enveloping algebra of L. Let $x^{[p]}$ be the image under the p-map of an element x of L, and let S be the subalgebra of R that is generated by the elements $x^p - x^{[p]}$, with $x \in L$, and the elements of F. Let I be the two-sided ideal of R generated by the elements $x^p - x^{[p]}$, with $x \in L$. Then $R' = R/I$, $S' = (S + I)/I = F$ and $R \cdot (S \cap I) = I = (S \cap I) \cdot R$. Hence we can apply Proposition 1.1. We obtain the following result.

THEOREM 2.1. *Let L be a restricted Lie algebra over a field F of characteristic $p \neq 0$, and let A and B be restricted L-modules. Then*

$$\mathrm{Ext}_{(R,S)}(A,B) \approx \mathrm{Ext}_{R'}(A,B)$$

and

$$\mathrm{Tor}^{R'}(A,B) \approx \mathrm{Tor}^{(R,S)}(A,B).$$

Proof. From Proposition 1.1, we get

$$\mathrm{Ext}_{(R',F)}(A,B) \approx \mathrm{Ext}_{(R,S)}(A,B)$$

and

$$\mathrm{Tor}^{(R',F)}(A,B) \approx \mathrm{Tor}^{(R,S)}(A,B).$$

Since F is a field, every (R',F)-projective (injective) resolution is an R'-projective (injective) resolution. Hence

$$\mathrm{Ext}_{(R',F)}(A,B) = \mathrm{Ext}_{R'}(A,B)$$

and

$$\mathrm{Tor}^{(R',F)}(A,B) = \mathrm{Tor}^{R'}(A,B).$$

THEOREM 2.2. *Let L be a restricted Lie algebra over a field F of characteristic $p \neq 0$. Let R^*, S^* and R'^* be the opposite rings of R, S and R' respectively. Let A and B be $R' \otimes_F R'^*$-modules. Then*

$$\mathrm{Ext}_{(R' \otimes R'^*)}(A,B) \approx \mathrm{Ext}_{(R \otimes R^*, S \otimes S^*)}(A,B),$$

and

$$\mathrm{Tor}^{(R \otimes R^*, S \otimes S^*)}(A,B) \approx \mathrm{Tor}^{(R' \otimes R'^*)}(A,B).$$

Proof. $I \otimes_F R^* + R \otimes_F I^* = J$ is a two-sided ideal of $R \otimes_F R^*$, where I^* is the opposite ring of I.

$$R \otimes_F R^*/J \approx R' \otimes_F R'^*,$$

$$(S \otimes_F S^* + J)/J \approx F,$$

and

$$(S \otimes_F S^* \cap J)\cdot(R \otimes_F R^*) = (R \otimes_F R^*)\cdot(S \otimes_F S^* \cap J) = J.$$

Hence Proposition 1.1 applies, and Theorem 2.2 is proved in the same way as Theorem 2.1.

3. **The enveloping algebra in relative homology.** If R and R' are rings with a homomorphism E of R into R' then any R'-module can be regarded as an R-module via the homomorphism E. We denote this R-module by A_E or ${}_EA$ according to whether A is a right R'-module or a left R'-module.

LEMMA 3.1. *Let R and R' be rings with a homomorphism E of R into R', and let S and S' be subrings of R and R' respectively such that $E(S) \subset S'$. Then, for any left (R,S)-projective module A, $(R')_E \otimes_R A$ is an (R',S')-projective module.*

Proof. Since A is (R,S)-projective, it is a direct R-module summand of $R \otimes_S A$. Hence $(R')_E \otimes_R A$ is a direct R'-module summand of $(R')_E \otimes_R (R \otimes_S A)$.

But this is isomorphic with $(R')_E \otimes_S A = R' \otimes_{S'} (S')_E \otimes_S A$. Hence it is (R', S')-projective. Hence $(R')_E \otimes_R A$ is (R', S')-projective.

Now let R be an algebra over a commutative ring F containing 1, and let S be a subalgebra of R containing 1. Assume that R and S are F-projective. Let R^* be the opposite ring of R and write $R^e = R \otimes_F R^*$. Then R^e is the enveloping algebra of R in the sense of [1, p. 167]. Since R and S^* (or R^* and S) are F-projective, we may identify S^e with its canonical image in R^e.

Let E be an algebra homomorphism of R into R^e such that $E(S) \subset S \otimes_F S^*$, and let A be a left R-module. Let

$$\cdots \to X_1 \to X_0 \to A \to 0$$

be an (R, S)-projective resolution of A, and let

$$\cdots \to X_1' \to X_0' \to (R^e)_E \otimes_R A \to 0$$

be an (R^e, S^e)-projective resolution of $(R^e)_E \otimes_R A$. Lemma 3.1, $(R^e)_E \otimes_R X_i$ is (R^e, S^e)-projective, for all i. Hence there is a map of complexes

$$G: (R^e)_E \otimes_R X \to X'$$

over the identity map of $(E^e)_E \otimes_R A$, and G is unique up to a homotopy. This yields a homomorphism

$$H(B \otimes_{R^e} (R^e)_E \otimes_R X) \to \mathrm{Tor}^{(R^e, S^e)}(B, (R^e)_E \otimes_R A),$$

for an R^e-right module B. But

$$B \otimes_{R^e} (R^e)_E \otimes_R X \approx B_E \otimes_R X.$$

Thus we have a natural homomorphism

$$\mathrm{Tor}^{(R,S)}(B_E, A) \to \mathrm{Tor}^{(R^e, S^e)}(B, (R^e)_E \otimes_R A).$$

We call this the canonical homomorphism induced by E.

Dually, if C is a left R^e-module, E induces a homomorphism

$$\mathrm{Hom}_{R^e}(X', C) \to \mathrm{Hom}_{R^e}((R^e)_E \otimes_R X, C),$$

and hence a homomorphism of $\mathrm{Ext}_{(R^e, S^e)}((R^e)_E \otimes_R A, C)$ into

$$H(\mathrm{Hom}_{R^e}((R^e)_E \otimes_R X, C)).$$

But $\mathrm{Hom}_{R^e}((R_e)_E \otimes_R X, C)$ may be identified with $\mathrm{Hom}_R(X, {}_E C)$. Thus we have a natural homomorphism

$$\mathrm{Ext}_{(R^e, S^e)}((R^e)_E \otimes_R A, C) \to \mathrm{Ext}_{(R,S)}(A, {}_E C),$$

which we call the canonical homomorphism.

PROPOSITION 3.1. *Let I_1 be the map $R \to R^e$ such that $I_1(r) = r \otimes 1$, for all $r \in R$. Suppose that S is contained in the center of R, and that the F-linear maps $I_1 \otimes_{R^e} E$; $R \otimes_F R \to R^e$ and $I_1 \otimes_{S^e} E : S \otimes_F S \to S^e$ are isomorphisms of F-modules. Then the canonical maps*

$$\mathrm{Tor}^{(R,S)}(B_E, A) \to \mathrm{Tor}^{(R^e,S^e)}(B, (R^e)_E \otimes_R A)$$

and

$$\mathrm{Ext}_{(R^e,S^e)}((R^e)_E \otimes_R A, C) \to \mathrm{Ext}_{(R,S)}(A, {}_E C)$$

are isomorphisms.

Proof. Let $\cdots \to X_1 \to X_0 \to A \to 0$ be an (R,S)-projective resolution of A. We have

$$(R^e)_E \otimes_R X_i = (R \otimes_F E(R)) \otimes_R X_i \approx R \otimes_F X_i .$$

Let π be an S-homotopy of the resolution X. We define an endomorphism π' of $R \otimes_F X$ such that, with $r \in R$ and $x \in X$, $\pi'(r \otimes x) = r \otimes \pi(x)$. We show that π' is an S^e-homomorphism. Note that when $(R^e)^E \otimes_R X$ is identified with $R \otimes_F X$, and S^e with $S \otimes_F E(S)$, the S^e-module structure takes the following form:

$$(s_1 \otimes 1)(r \otimes x) = (s_1 \otimes 1)(r \otimes 1) \otimes x = s_1 r \otimes x .$$

Also if $E(s_2) = \sum s_i \otimes s_j^*$ then

$$\begin{aligned} E(s_2)(r \otimes x) &= (\textstyle\sum s_i \otimes s_j^*)(r \otimes 1) \otimes x \\ &= (\textstyle\sum s_i r \otimes s_j^*) \otimes x = (\sum r s_i \otimes s_j^*) \otimes x \\ &= (r \otimes 1) E(s_2) \otimes x = r \otimes s_2 x \end{aligned}$$

because S was assumed to lie in the center of R. Hence

$$(s_1 \otimes E(s_2))(r \otimes x) = s_1 r \otimes s_2 x .$$

Hence it is clear that π' is an S^e homomorphism and thus an S_e-homotopy, of the complex $(R^e)_E \otimes_R X$. Thus we conclude that $(R^e)_E \otimes_R X$ is an (R^e, S^e)-projective resolution of A. Hence it is clear from the definition of the canonical homomorphisms in question that under the present assumptions, they are isomorphisms.

As a particular case where Proposition 3.1 can be applied, consider a restricted Lie algebra L of finite dimension over a field F of characteristic $0 \neq p$. Let R be the ordinary universal enveloping algebra of L. Denote the p-map in L by $x \to x^{[p]}$. Let S be the subalgebra of R that is generated by the elements $x^p - x^{[p]}$ with $x \in L$, and the elements of F. Let $u \to u^*$ be the anti-isomorphism of R onto its opposite algebra R^*. Let E be the algebra homomorphism from R to R^e that is characterized by: $E(x) = x \otimes 1 - 1 \otimes x^*$, for all $x \in L$. Then all the conditions of Proposition 3.1 are satisfied. In order to see this, let L^* be the canonical image of L in R^*, and identify L^* and L with their canonical images in R^e.

Every element of L commutes with every element of L^*. Hence $L+L^*$ is a Lie algebra, and $E(L)=\{x-x^*, x\in L\}$ is a subalgebra of $L+L^*$, because

$$\begin{aligned}[x-x^*, y-y^*] &= xy-yx+x^*y^*-y^*x^*\\ &= [x,y]+[y,x]^* = [x,y]-[x\,y]^*\in E(L).\end{aligned}$$

Moreover

$$\begin{aligned}E(x^p-x^{[p]}) &= E(x^p)-E(x^{[p]}) = E(x)^p-E(x^{[p]})\\ &= x^p-x^{*p}-x^{[p]}+x^{[p]*} = x^p-x^{[p]}-(x^{*p}-x^{[p]*})\\ &= x^p-x^{[p]}-(x^p-x^{[p]})^*\in S\otimes S^*,\end{aligned}$$

where we take L^* as a restricted Lie algebra with the p-map given by $x^{*[p]}=x^{[p]*}$. Hence $E(S)\subset S\otimes S^*$.

Consider the natural F-linear map

$$I_1\otimes_{R^e}E:\ R\otimes_F R\ \to\ R^e.$$

Note that R^* is the ordinary universal enveloping algebra of L^*. Hence R^e is the ordinary universal enveloping algebra of $L+L^*$, and $E(R)$ may be identified with the ordinary universal enveloping algebra of $E(L)$. Let $(x_1,\cdots,x_n)$ be an F-basis of L. Then $(x_1,\cdots,x_n,E(x_1),\cdots,E(x_n))$ is a basis for $L+E(L)=L+L^*$. Writing the elements of R^e as F-linear combinations of ordered monomials in these basis elements, we see (using the Poincaré-Birkhoff-Witt Theorem) that our homomorphism $I_1\otimes_{R^e}E$ is an F-linear isomorphism. Since S lies in the center of R, it is now clear that the conditions of Proposition 3.1 are satisfied.

Hence we obtain the following result immediately from Proposition 3.1.

THEOREM 3.1. *Let R be the ordinary universal enveloping algebra of a restricted Lie algebra L of finite dimension over a field F of characteristic $p\neq 0$. Let $x\to x^{[p]}$ denote the p-map in L, and let S be the subalgebra of R generated by the elements of the form $x^p-x^{[p]}$, with $x\in L$, and the elements of F. Let E be the homomorphism $R\to R^e$ defined above. Let B be a right R^e-module, A a left R-module and C a left R^e-module. Then E induces isomorphisms*:

$$E^*\colon \operatorname{Tor}^{(R,S)}(B_E,A)\to \operatorname{Tor}^{(R^e,S^e)}(B,(R^e)E\otimes_R A),$$

$$E^*\colon \operatorname{Ext}_{(R^e,S^e)}((R^e)_E\otimes_R A,C)\to \operatorname{Ext}_{(R,S)}(A,{}_EC).$$

Now let I be the ideal of R generated by the elements $x^p-x^{[p]}$, with $x\in L$. Let $R'=R/I$ be the restricted universal enveloping algebra of L. Since $E(I)\subset I\otimes R^*+R\otimes I^*$, E induces a homomorphism E' from R' to $(R')^e$. We have $S'=S+I/I=F$ and $E'(F)=F$. Let I_1 be the homomorphism of R'

into $(R')^e$ such that $I_1(r)=r\otimes 1$. As before, using Poincaré-Birkhoff-Witt Theorem for R', we see that the map

$$I_1\otimes_{(R')^e}E' : R'\otimes_F R' \to (R')^e$$

is an isomorphism. Using Proposition 3.1, we obtain the following result.

THEOREM 3.2. *Let L be a restricted Lie algebra of finite dimension over a field F of characteristic $p\neq 0$. Let R' be the restricted universal enveloping algebra of L, and let E' be the homomorphism $R'\to(R')^e$ defined above. Let B be a right $(R')^e$-module, A a left R'-module and C a left $(R')^e$-module. Then E' induces isomorphism:*

$$(E')_*\colon \operatorname{Tor}^{R'}(B_{E'},A) \approx \operatorname{Tor}^{(R')^e}(B,(R')^e_{E'}\otimes_{R'}A),$$

$$(E')^*\colon \operatorname{Ext}_{(R')^e}((R')^e_{E'}\otimes_{R'}A,C) \approx \operatorname{Ext}_{R'}(A,{}_{E'}C).$$

COROLLARY 3.1.

$$\dim_{(R')^e}(R') = \dim_{R'}(F) = \operatorname{gl.dim}(R').$$

Proof. By Theorem 3.2,

$$\operatorname{Ext}_{(R')^e}((R')^e_{E'}\otimes_{R'}F,C) \approx \operatorname{Ext}_{R'}(F,{}_{E'}C),$$

where F is regarded as a left R'-module via the supplementation $E\colon R'\to F$. We have

$$(R')^e_{E'}\otimes_{R'}F = (R'\otimes_F E'(R'))_{E'}\otimes_{(R')}F = R'.$$

We wish to determine the $(R')^e$-module structure induced on R' by this identification. Identify L with its canonical image in R', and let $r\to r^*$ be the anti-isomorphism $R'\to(R')^*$. Let $x\in L$ and $r\in R'$. Then r is the natural image of $(r\otimes 1)\otimes 1$ in $(R')^e_{E'}\otimes_{R'}F$.

$$(x\otimes 1)(r\otimes 1)\otimes 1 = (xr\otimes 1)\otimes 1,$$

and the natural image of this in R' is xr. On the other hand,

$$\begin{aligned}(1\otimes x^*)\cdot((r\otimes 1)\otimes 1) &= (r\otimes x^*)\otimes 1\\ &= (r\otimes 1)(x\otimes 1-E'(x))\otimes 1\\ &= (rx\otimes 1)\otimes 1-((r\otimes 1)E'(x))\otimes 1\\ &= (rx\otimes 1)\otimes 1.\end{aligned}$$

The natural image of this is in R' is rx. Hence it is clear that the $(R')^e$-module structure induced on R' is the usual one, where

$$(r_1\otimes r_2^*)\cdot r = r_1 r r_2.$$

Thus we have

$$\operatorname{Ext}_{(R')^e}(R',C) \approx \operatorname{Ext}_{R'}(F,{}_EC).$$

Hence $\dim_{(R')^e}(R') \leqq \dim_{R'}(F) \leqq \mathrm{gl.dim}(R')$. On the other hand, by Corollary 4.4, p. 170, of [1],

$$\mathrm{Ext}_{(R')^e}(R', \mathrm{Hom}_F(B,C)) \approx \mathrm{Ext}_{R'}(B,C),$$

for all left R'-modules B and C. Hence

$$\mathrm{gl.dim}\, R' \leqq \dim_{(R')^e}(R').$$

With the above, this gives the conclusion of the corollary.

4. **An application of relative homological algebra to Lie algebras.** Let R be an algebra over a commutative ring F. Let R^* be the opposite algebra of R. Let $r \to r^*$ be the anti-isomorphism of R onto R^*. Suppose we are given a homomorphism

$$D: R \to R \otimes_F R$$

and an F-algebra anti-endomorphism ψ of R. Let ω be the homomorphism of R into R^* defined by $\omega(r) = \psi(r)^*$. Let U and V be left R-modules. Then $U \otimes_F V$ is a left $R \otimes_F R$-module, and $\mathrm{Hom}_F(U, V)$ is a left $R \otimes_F R^*$-module, in the natural fashion.

We define left R-module structures on $U \otimes_F V$ and $\mathrm{Hom}_F(U, V)$ from these structures and the homomorphisms

$$D: R \to R \otimes_F R,$$
$$(R \otimes \omega)D: R \to R \otimes_F R^*,$$

respectively. Now let A, B and C be left R-modules. Then, by double applications of the above definitions, we obtain left R-module structures on $\mathrm{Hom}_F(A, \mathrm{Hom}_F(B,C))$ and $\mathrm{Hom}_F(B \otimes_F A, C)$. There is a canonical F-module isomorphism $h \to h^*$ of $\mathrm{Hom}_F(A, \mathrm{Hom}_F(B,C))$ onto $\mathrm{Hom}_F(B \otimes_F A, C)$ such that $h^*(b \otimes a) = h(a)(b)$.

Now assume that our "diagonal map" D is associative and ψ-symmetric, in the sense that

$$(R \otimes D)D = (D \otimes R)D$$

and

$$D\psi = (\psi \otimes \psi)D.$$

We claim that then the above F-module isomorphism is actually an R-module isomorphism. Write $D(r) = \sum_{\alpha,\beta} r_\alpha \otimes r_\beta$. Let $h \in \mathrm{Hom}_F(A, \mathrm{Hom}_F(B,C))$, $a \in A$, $b \in B$. Then we have

$$\begin{aligned}(r \cdot h^*)(b \otimes a) &= ((R \otimes \omega)D(r)h^*)(b \otimes a) \\ &= \sum_{\alpha,\beta} [(r_\alpha \otimes \omega(r_\beta))h^*](b \otimes a) \\ &= \sum_{\alpha,\beta} r_\alpha \cdot h^*(D(\psi(r_\beta)) \cdot (b \otimes a)) \\ &= \sum_{\alpha,\beta;\alpha',\beta'} r_\alpha h(\psi(r_\beta)_{\beta'} \cdot a)(\psi(r_\beta)_{\alpha'} \cdot b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(r\cdot h)^*(b\otimes a) &= (r\cdot h)(a)(b)\\ &= ((R\otimes\omega)D(r)\cdot h)(a)(b)\\ &= \sum_{\alpha,\beta}((r_\alpha\otimes\omega(r_\beta))\cdot h)(a)(b)\\ &= \sum_{\alpha,\beta}(r_\alpha\cdot h(\psi(r_\beta)\cdot a))(b)\\ &= \sum_{\alpha,\beta}[(R\otimes\omega)D(r_\alpha)h(\psi(r_\beta)\cdot a)](b)\\ &= \sum_{\alpha,\beta;\alpha',\beta'}(r_\alpha)_{\alpha'}[h(\psi(r_\beta)\cdot a)(\psi(r_\alpha)_{\beta'}\cdot b)].\end{aligned}$$

Now we have

$$(R\otimes D)D(r) = (D\otimes R)D(r),$$

i.e.,

$$(R\otimes D)\sum_{\alpha,\beta} r_\alpha\otimes r_\beta = (D\otimes R)\sum_{\alpha,\beta} r_\alpha\otimes r_\beta.$$

Hence, applying $R\otimes\psi\otimes\psi$,

$$\sum_{\alpha,\beta;\alpha',\beta'} r_\alpha\otimes\psi((r_\beta)_{\alpha'})\otimes\psi((r_\beta)_{\beta'}) = \sum_{\alpha,\beta;\alpha',\beta'} (r_\alpha)_{\alpha'}\otimes\psi((r_\alpha)_{\beta'})\otimes\psi(r_\beta).$$

Since $(\psi\otimes\psi)\,D(r) = D\psi(r)$, this gives

$$\sum_{\alpha,\beta;\alpha',\beta'} r_\alpha\otimes\psi(r_\beta)_{\alpha'}\otimes\psi(r_\beta)_{\beta'} = \sum_{\alpha,\beta;\alpha',\beta'} (r_\alpha)_{\alpha'}\otimes\psi((r_\alpha)_{\beta'})\otimes\psi(r_\beta).$$

Hence $(r\cdot h)^*(b\otimes a) = (r\cdot h^*)(b\otimes a)$, whence $(r\cdot h)^* = r\cdot h^*$, showing that our isomorphism is an R-module isomorphism.

Now assume that the F-algebra R is supplemented by an F-algebra epimorphism $\varepsilon: R\to F$ with kernel I. Let J be the kernel of the canonical map $R\otimes_F R^*\to R$, sending $r\otimes s^*$ onto rs. Assume that $J = (R\otimes_F R^*)(R\otimes\omega)D(I)$. Let U and V be left R-modules, and consider the left R-module $\mathrm{Hom}_F(U,V)$ as defined above, using

$$(R\otimes\omega)D: R\to R\otimes_F R^*.$$

We claim that $\mathrm{Hom}_R(U,V)$ is the F-submodule of $\mathrm{Hom}_F(U,V)$ consisting of all elements annihilated by I. Clearly, $\mathrm{Hom}_R(U,V)$ consists precisely of all the elements of $\mathrm{Hom}_F(U,V)$ that are annihilated by J, under the natural $R\otimes R^*$-module structure of $\mathrm{Hom}_F(U,V)$ (note that J is the left ideal of $R\otimes_F R^*$ that is generated by the elements of the form $r\otimes 1 - 1\otimes r^*$). Since $(R\otimes\omega)D(I)\subset J$, it follows that every element of $\mathrm{Hom}_R(U,V)$ is annihilated by I, under our R-module structure of $\mathrm{Hom}_F(U,V)$. Conversely, suppose that $h\in\mathrm{Hom}_F(U,V)$ and is annihilated by I. Then, under the natural $R \otimes R^*$-module structure of $\mathrm{Hom}_F(U, V)$, h is annihilated by $(R \otimes \omega) D(I)$ and hence by $(R \otimes_F R^*)(R \otimes \omega) D(I) = J$. Thus $h \in \mathrm{Hom}_R(U, V)$. It follows that under the present assumptions our above isomorphism $\mathrm{Hom}_F(A, \mathrm{Hom}_F(B, C)) \to \mathrm{Hom}_F(B \otimes_F A, C)$ sends $\mathrm{Hom}_R(A, \mathrm{Hom}_F(BC))$ isomorphically onto $\mathrm{Hom}_R(B \otimes_F A, C)$. The essential part of above is contained in Chapter XI, Proposition 8.1 of [1].

LEMMA 4.1. *Let R be a supplemented algebra over a commutative ring F, with the supplementation $\varepsilon: R \to F$, and let S be a subalgebra of R. Let D be a homomorphism $R \to R \otimes_F R$ and ψ an anti-endomorphism of R satisfying all the above conditions. Assume also that D maps S into the canonical image of $S \otimes_F S$ in $R \otimes_F R$ and that $\psi(S) \subset S$. Let C and B be left R-modules. If the sequence*

$$0 \to C \to Y_0 \to Y_1 \to \cdots$$

is an (R,S)-injective resolution of C, then the sequence

$$0 \to \mathrm{Hom}_F(B, C) \to \mathrm{Hom}_F(B, Y_0) \to \cdots$$

is an (R,S)-injective resolution of $\mathrm{Hom}_F(B,C)$, *where* $\mathrm{Hom}_F(B,C)$ *and the* $\mathrm{Hom}_F(B, Y_i)$ *are regarded as left R-modules via $(R \otimes \omega)D$, with $\omega(r) = \psi(r)^*$.*

Proof. First, we show that $\mathrm{Hom}_F(B, Y_i)$ is (R,S)-injective. Let

$$0 \to U_1 \to U_2 \to U_3 \to 0$$

be an (R,S)-exact sequence of left R-modules. We have seen above that

$$\mathrm{Hom}_R(U_i, \mathrm{Hom}_F(B, Y_j)) \approx \mathrm{Hom}_R(B \otimes_F U_i, Y_j).$$

Since the sequence

$$0 \to U_1 \to U_2 \to U_3 \to 0$$

is (R,S)-exact, so is the sequence

$$0 \to B \otimes_F U_1 \to B \otimes_F U_2 \to B \otimes_F U_3 \to 0,$$

because F operates on U_i via S. Hence

$$0 \to \mathrm{Hom}_R(U_3, \mathrm{Hom}_F(B, Y_j)) \to \mathrm{Hom}_R(U_2, \mathrm{Hom}_F(B, Y_j))$$
$$\to \mathrm{Hom}_R(U_1, \mathrm{Hom}_F(B, Y_j)) \to 0$$

is exact and we conclude that $\mathrm{Hom}_F(B, Y_j)$ is (R,S)-injective. Also, it is clear that the sequence of the $\mathrm{Hom}_F(B, Y_j)$ is (R,S)-exact. This completes the proof of Lemma 4.1.

LEMMA 4.2. *Let L be a Lie algebra of finite dimension n over a field F,* *and let z be an element in the center of L. Let M be the ideal of the universal enveloping algebra R of L that is generated by $z-1$. Then*

$$\mathrm{gl.dim}(R/M) \leqq n-1.$$

Proof. There is a homomorphism D of R into $R \otimes_F R$ that is characterized by $x \to x \otimes 1 + 1 \otimes x$, for $x \in L$. There is also an anti-automorphism ψ of R characterized by $\psi(x) = -x$, for $x \in L$. These maps satisfy all the assumptions we made above. Also, if S is the subalgebra of R generated by the elements of F and the element $z-1$, we have $D(S) \subset S \otimes_F S$. The only assumption needing verification is that

$$J = (R \otimes R^*)\cdot(R \otimes \omega)\cdot D(I),$$

where I is the ideal of R generated by L. If $x \in L$, $(R \otimes \omega)D(x) = x \otimes 1 - 1 \otimes x^*$. Hence it is clear that $(R \otimes \omega)D(I) \subset J$. To prove that

$$J \subset (R \otimes R^*)\cdot(R \otimes \omega)\cdot D(I),$$

it suffices to show that the elements of the form $r \otimes 1 - 1 \otimes r^*$ with $r \in R$ lie in the left ideal generated by the elements $x \otimes 1 - 1 \otimes x^*$, with $x \in L$. Suppose this has already been shown for some $r \in R$, and let $x \in L$. Then we have

$$\begin{aligned} rx \otimes 1 - 1 \otimes (rx)^* &= (r \otimes 1)(x \otimes 1 - 1 \otimes x^*) + r \otimes x^* - 1 \otimes x^* r^* \\ &= (r \otimes 1)(x \otimes 1 - 1 \otimes x^*) + (1 \otimes x^*)(r \otimes 1 - 1 \otimes r^*). \end{aligned}$$

Hence the same is true for rx in the place of r. Hence an evident induction shows that this holds for all $r \in R$.

We have $(S \cap M)\cdot R = R(S \cap M) = M$ and $(S + M)/M = F$. Hence, if A and B are left R/M-modules, Proposition 1.1 gives

$$\mathrm{Ext}_{R/M}(A,B) \approx \mathrm{Ext}_{(R,S)}(A,B).$$

Hence

$$\mathrm{gl.dim}\, R/M \leqq \mathrm{gl.dim}(R,S).$$

Let Y be an (R,S)-injective resolution of B. Then $\mathrm{Hom}_R(A,Y) \approx \mathrm{Hom}_R(F, \mathrm{Hom}_F(A,Y))$, where the R-module structure of $\mathrm{Hom}_F(A,Y)$ is via the homomorphism $(R \otimes \omega)D$ of R into $R \otimes R^*$. By Lemma 4.1, $\mathrm{Hom}_F(A,Y)$ is an (R,S)-injective resolution of $\mathrm{Hom}_F(A,B)$. Hence,

$$\mathrm{Ext}_{(R,S)}(A,B) \approx \mathrm{Ext}_{(R,S)}(F, \mathrm{Hom}_F(A,B)).$$

Hence

$$\mathrm{gl.dim}(R,S) \leqq \dim_{(R,S)}(F).$$

Let C be any left R-module, and let N be the ideal of R generated by z. Then

$$R\cdot(N\cap S) = (S\cap N)\cdot R = N,$$

and $(S+N)/N \approx F$. Hence, by Lemma 1.1,

$$\mathrm{Ext}_{(R,S)}(F,C) \approx \mathrm{Ext}_{R/N}(F,C^N).$$

Hence

$$\dim_{(R,S)}(F) \leqq \dim_{R/N}(F).$$

Since R/N is isomorphic with the universal enveloping algebra of the Lie algebra L/N,

$$\dim_{R/M}(F) = n-1.$$

Therefore

$$\mathrm{gl.dim}\, R/M \leqq n-1.$$

This result also can be seen from Theorem 5.4 of *Filtered algebras and representation of Lie algebras*, Trans. Amer. Math. Soc. **100** (1961), p. 542.

THEOREM 4.1. *Let L be a Lie algebra of finite dimension over a field F, and let P be an ideal of L. Let ζ be a nontrivial 1-dimensional representation of P (i.e., a nonzero Lie algebra homomorphism $P\to F$) such that $\zeta([L,P]) = (0)$, and let M be the two-sided ideal of the universal enveloping algebra R of L that is generated by the elements $x-\zeta(x)$ with $x\in P$. Then*

$$\mathrm{gl.dim}(R/M) \leqq [L/P:F].$$

Proof. Let T be the kernel of the representation ζ. Then P/T is one-dimensional. Let $x_1,\cdots,x_r$ be a basis of P such that $x_2,\cdots,x_r$ is a basis of T. By assumption, $[T,L]\subset[P,L]\subset T$, so that T is an ideal of L. Hence the universal enveloping algebra $\bar{R}$ of L/T is isomorphic with R/Q, where Q is the ideal of R generated by the elements of T. Clearly, M is the ideal of R generated by the elements $x_1-\zeta(x_1),x_2,\cdots,x_r$. Hence M/Q can be identified with the ideal $\bar{M}$ of $\bar{R}$ generated by $x_1-\zeta(x_1)$, regarding x_1 as an element of L/T. Therefore $R/M\approx R/Q/M/Q\approx\bar{R}/\bar{M}$. By Lemma 4.2, this gives $\mathrm{gl.dim}\, R/M = \mathrm{gl.dim}\, \bar{R}/\bar{M} \leqq n\approx(r-1)-1 = n-r$. This proves Theorem 4.1.

5. **Projective dimension of restricted universal enveloping algebras.**

LEMMA 5.1 (N. JACOBSON). *Let L be a finite-dimensional restricted Lie algebra over a field F of characteristic $p\neq 0$, and let R be the ordinary universal enveloping algebra of L. Let $x_1,\cdots,x_n$ be an F-basis for L. Let $F[t_1,\cdots,t_n]$ be the polynomial algebra in n variables $t_1,\cdots,t_n$ over F. Then the homomorphism ψ of $F[t_1,\cdots,t_n]$ into R that sends each t_i onto $x_i^p-x_i^{[p]}$ is an algebra monomorphism of $F[t_1,\cdots,t_n]$ into the center of R.*

Moreover R is a free $\psi(F[t_1,\cdots,t_n])$-module with the ordered monomials $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$, $0 \leqq e_i \leqq p$ as a free basis. A proof of this is contained in [5]; the result follows easily from the Poincaré-Birkhoff-Witt Theorem.

COROLLARY 5.1. *For $i = 1,\cdots,n$, let P_i be the ideal of R generated by the elements $x_j^p - x_j^{[p]}$, $1 \leqq j \leqq i$ and let $P_0 = (0)$. Then, for each i, the canonical image of $x_i^p - x_i^{[p]}$ in R/P_{i-1} is not a zero-divisor.*

Proof. Immediate from Lemma 5.1.

LEMMA 5.2 (KAPLANSKY). *Let R be a ring with identity element, and let x be a central element of R that is not a unit nor a zero-divisor. Let I be the two-sided ideal generated by x. Then, for any R/I-module A with $\dim_{R/I}(A) < \infty$*

$$\dim_R(A) = \dim_{R/I}(A) + 1.$$

For a proof, see Theorem 1.3, p. 6, in [6].

THEOREM 5.1. *Let L be a restricted Lie algebra of finite dimension n over a field F of characteristic $p \neq 0$. Let R' be the restricted universal enveloping algebra of L, and let R be the ordinary universal enveloping algebra of L. Then, for any R'-module A, $\dim_{R'}(A)$ is either 0 or ∞. If $\dim_{R'}(A) = 0$, then $\dim_R(A) = n$.*

Proof. Let P_i, $1 \leqq i \leqq n$, be the ideal of R generated by the elements $x_j^p - x_j^{[p]}$ $1 \leqq j \leqq i$, and let $P_0 = (0)$. Let u_i be the canonical image of $x_i^p - x_i^{[p]}$ in R/P_{i-1}. By Corollary 5.1, u_i is the center of R/P_{i-1} and not a zero-divisor. If $u_i r = 1 \in R/P_{i-1}$, for some $r \in R$, then $1 \in P_i$, so that $P_i = R$, which is impossible by Lemma 5.1. Hence u_i is not a unit of R/P_{i-1}. From Lemma 5.2, if $\dim_{R/P_i}(A) < \infty$, then

$$\dim_{R/P_{i-1}}(A) = \dim_{R/P_i} A + 1.$$

Hence $\dim_{R/P_n}(A) < \infty$ implies $\dim_{R/P_0}(A) = \dim_{R/P_n}(A) + n$. Since $\dim_{R/P_0}(A) = \dim_R(A) \leqq \mathrm{gl.dim}(R) = n$, we conclude that $\dim_{R'}(A) < \infty$ implies $\dim_{R'}(A) = 0$. Moreover, if $\dim_{R'}(A) = 0$, we must have $\dim_R(A) = n$. This completes the proof of Theorem 5.1.

It is known from [3] that $\mathrm{gl.dim}(R') = 0$ if and only if L is abelian and the elements $x^{[p]}$ with $x \in L$ span L over F.

THEOREM 5.2. *Let L be a restricted Lie algebra of finite dimension over a field F of characteristic $p \neq 0$. Suppose that, as an ordinary Lie algebra, L is solvable and that, for every $0 \neq x \in L$, $x^{[p]} \neq 0$; then the global dimension of the restricted universal enveloping algebra of L is 0, so that L is abelian.*

Proof. Since L is solvable, there is a basis $(x_1,\cdots,x_n)$ of L such that, for all i and j with $i < j, [x_i,x_j]$ is an F-linear combination of the x_k's with $k < j$. For each i, let L_i denote the smallest restricted Lie subalgebra of L containing x_i. Let R_i' denote the restricted universal enveloping algebra of L_i. We identify R_i' with its canonical image in the restricted universal enveloping algebra R' of L. Let $I_i = L_i R_i'$. Consider the left R'-module $V_i = R'I_1 + \cdots + R'I_i$ of R'. We claim that $V_i R_{i+1}' \subset V_i$. In order to prove this, it suffices to show that, for all $k \leqq i$, $x_k x_{i+1} \in V_i$. But this is clear, because $x_k x_{i+1} = [x_k, x_{i+1}] + x_{i+1} x_k$, and $[x_k, x_{i+1}]$ is a linear combination of x_e's with $e \leqq i$. We shall show that for each i, R'/V_i is R'-projective. Note that L_i is abelian and that the p-map annihilates no nonzero element of L_i. Hence L_i is spanned over F by $L_i^{[p]}$. From this, it is easily seen that R_i' is semisimple as an F-algebra (cf. [3]). In particular, F is projective as an R_i'-module. Now $R'/V_1 \approx R' \otimes_{R_1'} F$, and, since F is R_1'-projective, this shows that R'/V_1 is R'-projective. Generally $R'/V_{i+1} \approx (R'/V_i) \otimes_{R_{i+1}'} F$. Hence we see successively, by repeating the argument just made, that each R'/V_i is R'-projective. Since $R'/V_n \approx F$, we conclude that F is R'-projective. Now, by Corollary 3.1,

$$\mathrm{gl.dim}(R') = \dim_{R'}(F) = 0.$$

This completes the proof of Theorem 5.2.

It is shown by the author that the solvability in Theorem 5.2 can be removed by assuming F is a perfect field (to appear in Proceedings of the American Mathematical Society).

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