λ-CONTINUOUS MARKOV CHAINS[1]

BY

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0. Introduction and summary. We call a Markov operator P which has a representation $P(x,A) = \int_A p(x,y)\lambda(dy)$ with $p(x,y)$ bivariate measurable a λ-continuous Markov operator. It is a special kind of λ-measurable Markov operator of E. Hopf. If the state space is discrete, every Markov operator is λ-continuous where λ assigns measure 1 to every state. In §I various definitions and preliminaries are given. In §II the existence of invariant measures for a λ-continuous conservative P is proved. It is shown that the space is decomposed into at most countably many indecomposable closed sets $C_1, C_2, \cdots$. For each C_i there is a σ-finite invariant measure μ_i which is equivalent to λ on C_i and vanishes outside C_i. Every invariant measure is shown to be of the form $\sum \alpha_i \mu_i$. In §III convergence properties of $\sum_{n=1}^N p^n(z,x) / \sum_{n=1}^N p^n(z,y)$ are studied. It is shown that for a conservative ergodic P the limit of the ratio is $f(x)/f(y)$ where f is the derivative of the invariant measure with respect to λ. All these theorems are well known for a discrete state space (cf. [2, 1.9]).

§IV treats laws of large numbers. The approach used here is similar to that of Harris and Robbins [7]. It contains generalizations of theorems of Chung for discrete state spaces (cf. [2, 1.15]). The theory of λ-measurable Markov operators is extensively used here.

§VI is devoted to some new results on λ-measurable Markov operators which are used in this paper. In §V the theory of Martin boundaries is investigated. The kernel $K(x,y)$ used here is

$$K(x,y) = \lim_{N \to +\infty} \frac{\sum_{n=1}^{N} p^n(x,y)}{\int \pi(dz) \sum_{n=1}^{N} p^n(z,y)},$$

where π is a finite measure equivalent to λ. By using this kernel the space X is embedded in a compact Hausdorff space $\tilde{X}$. Every π-integrable invariant function h is shown to have the representation $h = \int \tilde{y}\tilde{\eta}(d\tilde{y})$ for a Baire measure $\tilde{\eta}$ on $\tilde{X}$. The techniques used here are essentially extensions of that of G. A. Hunt [9]. The space X is assumed to be irreducible, i.e., X is the support of measure $\sum_{n=1}^{\infty} 2^{-n}\pi P^n$.

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I. **Preliminaries.** Let X be a nonempty set, $\mathscr{X}$, a σ-algebra of subsets of X and λ, a σ-finite measure on X. Let $p(x,y)$ be an $\mathscr{X} \times \mathscr{X}$ measurable function defined on $X \times X$ satisfying the following conditions:

1. $p(x,y) \geqq 0$ for $(\lambda \times \lambda)$ almost all (x,y),
2. for (λ) almost all x, $\int p(x,y)\lambda(dy) \leqq 1$.

Let $L_\infty(\lambda)$ be the collection of λ-essentially bounded functions and $\mathscr{A}(\lambda)$, the collection of all finite real valued, countably additive functions on $\mathscr{X}$ which are absolutely continuous with respect to λ. Let $\mathscr{A}^+(\lambda)$ be the collection of all non-negative elements of $\mathscr{A}(\lambda)$. For any $f \in L_\infty(\lambda)$ define Pf by

$$Pf(x) = \int p(x,y)f(y)\lambda(dy). \tag{1.1}$$

For any $\nu \in \mathscr{A}(\lambda)$ define νP by

$$\nu P(A) = \int \nu(dx)\int_A p(x,y)\lambda(dy). \tag{1.2}$$

The operator P here is a special case of λ-measurable Markov operators of E. Hopf (cf. Appendix). We shall call it a *λ-continuous Markov operator* and $(X,\mathscr{X},\lambda)$, the *state space* of P. The iterates of P are then given by

$$P^n f(x) = \int p^{(n)}(x,y)f(y)\lambda(dy),$$

$$\nu P^n(A) = \int \nu(dx)\int_A p^n(x,y)\lambda(dy),$$

where $p^{(n)}(x,y)$ are defined inductively by

$$p^{(n)}(x,y) = \int p^{(n-1)}(x,z)\,p(z,y)\lambda(dz).$$

The function p shall be called the density function of the operator P. (1.1), (1.2) remain meaningful for non-negative f not necessarily λ-essentially bounded and non-negative, σ-finite measure ν. All subsets of X discussed in this paper are elements of $\mathscr{X}$ and all functions on X are $\mathscr{X}$-measurable functions. For two sets A,B, $A \subset B$, $A = B$ mean that $\lambda(A-B)=0$, $\lambda(A \triangle B)=0$ respectively. For two functions f,g on X, $f = g$, $f \leqq g$ mean that the equality and the inequality, respectively, are satisfied except on a λ-null set. For any set A, 1_A represents the function which equals to 1 on A and 0 on the complement A' of A. For any function f and any additive set function ν define

$$I_A f(x) = 1_A(x)f(x),$$

$$\nu I_A(B) = \nu(A \cap B).$$

I_A is a λ-measurable Markov operator. Define

$$P_A^{\#} = \sum_{n=0}^{\infty} (I_{A'}P)^n, \tag{1.3}$$

$$P_A = \sum_{n=0}^{\infty} P(I_{A'}P)^n. \tag{1.4}$$

$P_A^{\#}$, P_A operating on either non-negative functions or measures have well-defined meanings. In our case of λ-continuous P,

$$P_A f(x) = \int p_A(x,y)f(y)\,\lambda(dy),$$

$$\nu P_A(B) = \int \nu(dx) \int_B p_A(x,y)\,\lambda(dy),$$

where

$$p_A(x,y) = \sum_{n=1}^{\infty} p_{A,n}(x,y) \text{ with} \tag{1.5}$$

$$p_{A,n}(x,y) = \int_{A'} \cdots \int_{A'} p(x,z_1)p(z_1,z_2)\cdots p(z_{n-1},y)\,\lambda(dz_1)\cdots\lambda(dz_{n-1}).$$

However $P_A^{\#}I_A$ and P_AI_A are λ-measurable Markov operators and P_AI_A has density function $p_A(x,y)1_A(y)$ [Appendix, Theorem 6.1]. Following E. Hopf and J. Feldman we call a set A a *conservative* set if, for every λ-non-null subset B of A, $P_B1_B = 1$ on B. Let C be the largest conservative set which is then called the conservative part of X. $D = X - C$ is called the dissipative part of X. A set A is said to be *transient* if there is a non-negative number $q < 1$ such that $P_A1_A \leqq q$ on A. Then D is the union of at most countably many transient sets [Appendix, Theorem 6.3]. For any finite measure ν which is equivalent to λ, $\sum_{n=1}^{\infty} \nu P^n(A) = \infty$ if $A \subset C$ and A is λ-non-null and $\sum_{n=1}^{\infty} \nu P^n$ is σ-finite on D. A set A is *closed* if $P1_A = 1$ on A. C is closed. It follows that $\sum_{n=1}^{\infty} p^n(x,y) < \infty$ for $(\lambda \times \lambda)$ almost all $(x,y) \in X \times D$, in particular, $\sum_{n=1}^{\infty} p^n(x,y) = 0$ a.e. $(\lambda \times \lambda)$ on $C \times D$. The collection of all closed subsets of C form a σ-algebra of subsets of C which we shall designate by $\mathscr{C}$. P is *conservative* if $C = X$. P is *dissipative* if $D = X$. An extended real valued non-negative function h is said to be *P-excessive* if $Ph \leqq h$, *P-invariant* if equality holds. h is a *P-potential* if $P^nh \downarrow 0$ a.e. (λ). A σ-finite measure μ is *P-excessive* if $\mu P \leqq \mu$, *P-invariant* if equality holds. μ is a *P-potential* if $d\mu P^n/d\lambda \downarrow 0$ a.e. (λ). Let

$$g(x,y) = \sum_{n=1}^{\infty} p^n(x,y). \tag{1.6}$$

Then $P^n g(\cdot, y) = \sum_{k=n+1}^{\infty} p^k(\cdot, y)$. Hence for (λ) almost all y, $g(\cdot, y)$ is P-excessive, in particular, if $y \in D$, $g(\cdot, y)$ is a P-potential.

For a finite valued P-excessive function h we define

$$(1.7) \qquad p_h(x,y) = \frac{1}{h(x)} p(x,y).$$

p_h is well defined, a.e., $(\lambda_h \times \lambda_h)$ where λ_h is given by

$$\lambda_h(A) = \int_A h(x)\,\lambda(dx).$$

Since $\int p_h(x,y)\,\lambda_h(dy) = (1/h(x)) \int p(x,y) h(y)\,\lambda(dy) \leqq 1$, a λ_h-continuous Markov operator P_h may be defined by:

$$(1.8) \qquad P_h f(x) = \int p_h(x,y) f(y)\,\lambda_h(dy) = \frac{1}{h(x)} \int p(x,y) h(y) f(y)\,\lambda(dy),$$

$$(1.9) \qquad \nu P_h(A) = \int \nu(dx) \int_A p_h(x,y)\,\lambda_h(dy) = \int \nu(dx) \frac{1}{h(x)} \int_A p(x,y) h(y) \lambda(dy).$$

The iterates P_h^n are then given by

$$P_h^n f(x) = \int p_h^{(n)}(x,y) f(y)\,\lambda_h(dy),$$

$$\nu P_h^n(A) = \int \nu(dx) \int_A p_h^{(n)}(x,y) \lambda_h(dy),$$

where

$$p_h^{(n)}(x,y) = \frac{1}{h(x)} p^{(n)}(x,y).$$

For the operator P_h it is easy to see that the space still has the same decomposition $C \cup D$ although sets of λ_h measure 0 have no significance.

II. Invariant measures.

THEOREM 2.1. *Let P be λ-continuous and $\mathscr{C}$ be the σ-algebra of all closed subsets of the conservative part C. Then λ is purely atomic on $\mathscr{C}$. Let $\mathscr{C}$ be generated by distinct atoms $C_1, C_2, \cdots$ and $P_i = PI_{C_i}$; then $P_i P_j = 0$ if $i \neq j$ and $I_C P = I_C(P_1 + P_2 + \cdots)$.*

Proof. Suppose that $\mathscr{C}$ contained a nonatomic set A. We may assume $0 < \lambda(A) < \infty$. For every positive integer n there is a partition $E_1^{(n)}, \cdots, E_{k_n}^{(n)}$ of A such that every $E_i^{(n)}$ belongs to $\mathscr{C}$ and $\lambda(E_i^{(n)}) < n^{-1}$. We may assume that each $E_i^{(n+1)}$ is a subset of $E_j^{(n)}$ for some j. Let $A_n = \bigcup_i [E_i^{(n)} \times E_i^{(n)}]$. Since both $E_i^{(n)}$ and $A - E_i^{(n)}$ are closed sets, $p(x,y) = 0$ a.e. $(\lambda \times \lambda)$ on $A \times A - A_n$. Since A_n is monotonically decreasing and $\lambda \times \lambda(A_n) \leqq n^{-1}\lambda(A) \to 0$ as $n \to \infty$, $p(x,y) = 0$

a.e. $(\lambda \times \lambda)$ on $A \times A$ which contradicts the fact that $\int_A \int_A p(x,y)\,\lambda(dx)\,\lambda(dy) = \lambda(A) > 0$.

For every closed set A, $I_A P = I_A P I_A$, hence $P_i P_j = P I_{C_i} P I_{C_j} = P I_{C_i} I_{C_j} P I_{C_j} = 0$ and $I_C P = I_C P I_C = I_C(P_1 + P_2 + \cdots)$.

A conservative Markov operator P is said to be *ergodic* if the only λ-non-null set in $\mathscr{C}$ is C.

THEOREM 2.2. *If a λ-continuous Markov operator P is conservative and ergodic then P possesses a σ-finite invariant measure μ which is unique up to a constant multiple. Furthermore μ is equivalent to λ.*

Proof. The proof of the existence of invariant measure shall be essentially that of T. E. Harris adapted to the present situation [6]. Since P is conservative $\int p(x,y)\,\lambda(dy) > 0$ implies that $\sum_{n=1}^{\infty} p^n(x,y) = \infty$ for (λ) almost all y on a non-null closed set. Since X is the only non-null closed set, $\sum_{n=1}^{\infty} p^n(x,y) = \infty$ for (λ) almost all y. Hence $\sum_{n=1}^{\infty} p^{(n)}(x,y) = \infty$ a.e. $(\lambda \times \lambda)$. Let M, a be two positive numbers with $2^{-1} < a < 1$. We shall show that there is a set A with $0 < \lambda(A) < \infty$ and a positive integer N such that for (λ) almost all $x \in A$,

$$\lambda\left[y: \sum_{n=1}^{N} p^{(n)}(x,y) > M,\ y \in A\right] > a\lambda(A). \tag{2.1}$$

Let E be an arbitrary set with $0 < \lambda(E) < \infty$. Let

$$f_n(x) = \lambda\left[y: \sum_{i=1}^{n} p^{(i)}(x,y) > M\right] \cap E.$$

Then $f_n \uparrow \lambda(E)$ a.e. (λ). Determine N so that $\lambda[x : f_N(x) > (4/5)\lambda(E), x \in E] > (4/5)\lambda(E)$. Let $A = [x : f_N(x) > (4/5)\lambda(E),\ x \in E]$. Then A satisfies (2.1). Consider P_A and p_A given by (1.4) and (1.5) respectively. $P_A I_A$ is a λ-continuous Markov operator with density function $r(x,y) = p_A(x,y)1_A(y)$. Let $r^n(x,y)$ be the nth iterate of $r(x,y)$,

$$q(x,y) = N^{-1} \sum_{n=1}^{N} r^{(n)}(x,y)$$

and Q be the corresponding λ-continuous Markov operator with density function $q(x,y)$. (2.1) implies that for (λ) almost all $x \in A$,

$$\lambda[y : q(x,y) > M/N,\ y \in A] > a\lambda(A). \tag{2.2}$$

(2.2) implies that there is a probability η such that

$$\operatorname{ess\,sup}_x \left| Q^n 1_E(x) - \eta(E) \right| \to 0 \text{ uniformly in } E$$

(see Appendix of [6]). η is $P_A I_A$-invariant. Let $\mu = \eta P_A$. Then

$$\mu P = \eta P_A P = \eta P_A I_A P + \eta P_A I_{A'} P = \eta P + \eta \sum_{n=1}^{\infty} P(I_{A'} P)^n = \eta P_A = \mu.$$

Hence μ is P-invariant. Since P is ergodic, μ is σ-finite by Theorem 6.4 of Appendix, μ has density with respect to $\lambda : d\mu/d\lambda = \int \eta(dx) p_A(x, \cdot)$. The support of an invariant measure is necessarily a closed set. Hence the support of μ is X and μ is equivalent to λ. Uniqueness of μ follows from Corollary 6.2 of Appendix.

THEOREM 2.3. *Let P be λ-continuous and conservative. Let $C_1, C_2, \cdots$ be distinct atoms which generate $\mathscr{C}$. For each C_i there is a σ-finite P-invariant measure μ_i which is equivalent to λI_{C_i} and every P-invariant measure is of the form $\sum \alpha_i \mu_i$.*

Proof. Let $P_i = P I_{C_i}$. It follows from Theorem 2.1 that there is a P_i-invariant measure which is equivalent to λI_{C_i} which is unique up to a constant multiple. Now $\mu_i P = \mu_i I_{C_i} P = \mu P_i = \mu_i$, hence μ_i is also P-invariant. Conversely, if μ is P-invariant, let $\nu_i = \mu I_{C_i}$; then $\nu_i P_i = \mu I_{C_i} P = \mu P I_{C_i} = \mu I_{C_i} = \nu_i$. ν_i is P_i-invariant, therefore a constant multiple of μ_i. Hence μ is of the form $\sum \alpha_i \mu_i$.

Let

$$B = [P_C 1_C > 0] - C. \tag{2.3}$$

J. Feldmann showed that every excessive measure for a λ-measurable Markov operator P is necessarily absolutely continuous to λI_{X-B}. In our case of λ-continuous Markov operators we have the following corollary.

COROLLARY 2.1. *If P is a λ-continuous Markov operator then there exists an excessive measure μ which is equivalent to λI_{X-B} where B is given by* (2.3).

Proof. Let $\pi \in \mathscr{A}^+(\lambda)$ and be equivalent to $\lambda I_{X-(B\cup C)}$. Let $\eta = \sum_{n=0}^{\infty} \pi P^n$. Then η is equivalent to $\lambda I_{X-(B\cup C)}$ and σ-finite. Let $\mu = \eta + \sum \alpha_i \mu_i$, $\alpha_i > 0$ where μ_i are invariant measures described in Theorem 2.3. μ is the desired excessive measure.

III. **Ratio ergodic theorems.**

For a λ-measurable Markov operator P and $\nu, \eta \in \mathscr{A}^+(\lambda)$ the ratio ergodic theorem states that

$$\frac{\sum_{n=1}^{N} d\nu P^n / d\lambda}{\sum_{n=1}^{N} d\eta P^n / d\lambda}$$

converge a.e. (λ) on the set where the denominator is positive. The theorem was conjectured by E. Hopf and proved by Chacon and Ornstein [1]. The limit function was identified by J. Neveu [10]. For our case of λ-continuous operator P, because of $\mathscr{C}$ being atomic the ratio ergodic theorem may take the form of

Theorem 3.1. But, first, we shall introduce functions u_{C_i}. Let C_i be an atom of C. Let $u_{C_i} = P_{C_i}1_{C_i}$. u_{C_i} is the smallest excessive function which equals to 1 on C_i (Appendix, Corollary 6.1).

THEOREM 3.1. *Let P be λ-continuous and π, a finite measure equivalent to λ; then for $(\lambda \times \lambda)$ almost all (x,z)*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} p^n(x,z)}{\int \pi(dy) \sum_{n=1}^{N} p^n(y,z)} = \frac{u_{C_i}(x)}{\int \pi(dy) u_{C_i}(y)} \quad \text{if } z \in C_i$$

(3.1)

$$= \frac{g(x,z)}{\int \pi(dy) g(y,z)} \quad \text{if } z \in D \cap \left[\bigcup_{n=1}^{\infty} \operatorname{supp} \pi P^n\right].$$

We shall define kernel $K(x,z)$ by

$$K(x,z) = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} p^n(x,z)}{\int \pi(dy) \sum_{n=1}^{N} p^n(y,z)}. \tag{3.2}$$

This kernel shall be used to obtain the exit boundary of P.

Because of the existence of invariant measures for conservative λ-continuous Markov operators we are able to derive the following theorem.

THEOREM 3.2. *Let P be a λ-continuous Markov operator, C_i, an atom of $\mathscr{C}$, μ a P-invariant measure which is equivalent to λI_{C_i} and $f = d\mu/d\lambda$; then for $(\lambda \times \lambda \times \lambda)$ almost all $(x,y,z) \in C_i \times C_i \times C_i$,*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} p^n(z,x)}{\sum_{n=1}^{N} p^n(z,y)} = \frac{f(x)}{f(y)}. \tag{3.3}$$

Proof. It is sufficient to prove the theorem for P being conservative and ergodic and $C_i = X$. We shall define another λ-continuous Markov Q which we shall call the μ-reverse of P (or just the reverse of P for in this case a P-excessive measure is essentially unique). For any $g \in L_\infty(\lambda)$ define Qg to be a function satisfying the following equality for every $h \in L_1(\mu)$:

$$\int h(Qg)d\mu = \int (Ph)g d\mu.$$

In other words, if $d\nu = g d\mu$, Qg is defined by $d\nu P = Qg d\mu$. The P-invariance and λ-equivalence of μ imply that Q is a well-defined λ-measurable Markov operator (cf. Appendix). This Q has been called an "inverse" by S. Kakutani and "μ-adjoint" by J. Feldman. It follows from Theorem 3.1 [5] that Q is conservative. Q is also ergodic (Appendix, Lemma 6.2). Since P is λ-continuous, Q is also λ-continuous with density function $q(x,y)$:

$$q(x,y) = f(y)\,p(y,x)\frac{1}{f(x)}. \tag{3.4}$$

The iterates $q^{(n)}(x,y)$ are given by

$$q^{n}(x,y) = f(y)\,p^{(n)}(y,x)\frac{1}{f(x)}. \tag{3.5}$$

Applying (3.1) to $q(x,y)$ we have for $(\lambda \times \lambda \times \lambda)$ almost all (x,y,z),

$$1 = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} q^{n}(x,z)}{\sum_{n=1}^{N} q^{n}(y,z)}$$

$$= \lim_{N\to\infty} \frac{\sum_{n=1}^{N} p^{(n)}(z,x)}{\sum_{n=1}^{N} p^{(n)}(z,y)} \quad \frac{f(y)}{f(x)}$$

and (3.3) follows immediately.

Corollary 3.1. *Let P be λ-continuous, conservative and ergodic. Let μ be P-invariant and $f = d\mu/d\lambda$. If μ is finite and normalized to be a probability measure then for $(\lambda \times \lambda)$ almost all (x,y),*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} p^{(n)}(x,y) = f(y), \tag{3.6}$$

and for (λ) almost all x,

$$\lim_{N\to\infty} \int \left| \frac{1}{N} \sum_{n=1}^{N} p^{n}(x,y) - f(y) \right| \lambda(dy) = 0. \tag{3.7}$$

If μ is not finite then for $(\lambda \times \lambda)$ almost all (x,y)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} p^{(n)}(x,y) = 0. \tag{3.8}$$

Proof. If μ is a probability measure, Theorem 3.1 implies that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} p^{(n)}(x,y)}{N f(y)} = 1,$$

hence (3.6) follows immediately. Since $p^n(x,y)$ are non-negative and

$$\int N^{-1}\left[\sum_{n=1}^{N} p^n(x,y)\right]\lambda(dy) = \int f(y)\lambda(dy),$$

(3.6) implies (3.7). If μ is not finite there is an increasing sequence $\{E_n\}$ of sets such that $\bigcup_n E_n = X$ and $\mu(E_n) < \infty$ for every n. The general ergodic theorem implies that for $(\lambda \times \lambda)$ almost all (x,y),

$$\lim_{N\to\infty} \sum_{n=1}^{N} p^n(x,y) \Big| \sum_{n=1}^{N} \frac{d\mu I_{E_k} P^n}{d\lambda}(y) = \frac{1}{\mu(E_k)}. \tag{3.9}$$

Now for each n,k, $\mu I_{E_k} P^n \leqq \mu$. Hence every term in the summation appearing as the denominator of the left-hand side of (3.9) is $\leqq f(y)$. Hence

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^{N} p^n(x,y)}{Nf(y)} \leqq \frac{1}{\mu(E_k)}.$$

Since $\mu(E_k) \to \infty$, it follows that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} p^{(n)}(x,y)}{Nf(y)} = 0$$

and (3.8) is proved.

The following theorem follows immediately from Theorem 6.5 cf Appendix.

THEOREM 3.3. *Let P be λ-continuous and conservative and $C_1, C_2, \cdots$ are the atoms of $\mathscr{C}$. Let μ be an invariant measure which is equivalent to λ and $\mu_i = \mu I_{C_i}$. Then if $f, g \in \bigcap_i L_1(\mu_i)$, $g > 0$ a.e. (λ) then there exists limit*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} P^n f}{\sum_{n=1}^{N} P^n g} = \sum \frac{a_i}{b_i} 1_{C_i} \quad a.e.\ (\lambda),$$

where $a_i = \int f\,d\mu_i$, $b_i = \int h\,d\mu_i$. The limit is independent of particular μ chosen. Furthermore there exists

$$\lim_{N\to\infty} \frac{1}{N}\left(\sum_{n=1}^{N} P^n f\right) = \sum a_i' 1_{C_i}\ a.e.\ (\lambda),$$

where

$$a_i' = a_i/\mu_i(C_i) \qquad if\ \mu(C_i) < \infty,$$

$$a_i' = 0 \qquad if\ \mu(C_i) = \infty.$$

Again the limit is independent of the particular μ chosen.

IV. **Laws of large numbers.** In this section we shall assume that $\int p(x,y)\lambda(dy) = 1$ for (λ) almost all x.

Let Ω be the infinite product space $\prod_{n=0}^{\infty} X$ and $\mathscr{F}$, the product σ-algebra $\prod_{n=0}^{\infty}\mathscr{X}$ of subsets of Ω. Let X_n be the function on Ω to X defined by $X_n(w) = x_n$ if $w = \{x_0, x_1, x_2, \cdots\}$. Let T be the shift transformation on Ω to Ω defined by $Tw = \{x_1, x_2, \cdots\}$ if $w = \{x_0, x_1, x_2, \cdots\}$. For any function f defined on Ω we define function Tf by $Tf(w) = f(Tw)$. If f is a function on X then $Tf(X_n) = f(X_{n+1})$. For any measure ϕ on $\mathscr{F}$ define measure ϕT by $\langle \phi T, f\rangle = \langle \phi, Tf\rangle$ for every non-negative $\mathscr{F}$-measurable function f ($\langle \phi, f\rangle = \int f d\phi$ if the integral is well defined).

A function $\mathscr{P}(x, E)$, $x \in X$, $E \in \mathscr{F}$ is defined in the following manner. If $E = [X_0 \in A_0, \cdots, X_n \in A_n]$ where $A_i \in \mathscr{X}$,

$$(4.1)\quad \mathscr{P}(x,E) = \int_{A_0}\lambda(dx_0)\int_{A_1}\lambda(dx_1)\cdots\int_{A_n}\lambda(dx_n)\,p(x,x_0)\,p(x_0,x_1)\cdots p(x_{n-1},x_n).$$

Then the definition of $\mathscr{P}(x,E)$ is extended to arbitrary $E \in \mathscr{F}$ by a well-known measure extension argument. We have that for (λ) almost all x $\mathscr{P}(x,\cdot)$ is a probability measure on $\mathscr{F}$ and for every $E \in \mathscr{F}$ $\mathscr{P}(\cdot, E)$ is $\mathscr{X}$-measurable. For a real valued, bounded or non-negative $\mathscr{F}$-measurable function f we define

$$(4.2)\qquad \mathscr{P}f = \int \mathscr{P}(\cdot, dw')f(\cdot, w').$$

In the above we write $w = \{x_0, x_1, \cdots\}$ as a pair (x_0, w') where $w' = Tw$. $\mathscr{P}f$ is $\mathscr{X}$-measurable and

$$(4.3)\qquad \mathscr{P}T^n f = P^n \mathscr{P}f.$$

(4.3) may be first proved for f being of the form 1_E where $E = [X_0 \in A_0, \cdots, X_n \in A_n]$ and then extended to arbitrary f. If f is of the form $f(X_n)$ with f being $\mathscr{X}$-measurable then $\mathscr{P}f(X_n) = Pf$. For any measure η absolutely continuous with respect to λ, a measure $\boldsymbol{\eta}$ is defined on $\mathscr{F}$ by letting

$$(4.4)\qquad \langle \boldsymbol{\eta}, f\rangle = \langle \eta, \mathscr{P}f\rangle = \int \eta(dx)\int \mathscr{P}(x, dw')f(x, w')$$

for every non-negative $\mathscr{F}$-measurable function f. $\boldsymbol{\eta}$ is σ-finite. Since $\langle \boldsymbol{\eta}, f\rangle = 0$ if and only if $\mathscr{P}f = 0$ a.e. (η), $\langle \boldsymbol{\lambda}, f\rangle = 0$ implies that $\langle \boldsymbol{\eta}, f\rangle = 0$ so that $\boldsymbol{\eta}$ is absolutely continuous with respect to $\boldsymbol{\lambda}$. Since

$$\langle \boldsymbol{\lambda} T, f\rangle = \langle \boldsymbol{\lambda}, Tf\rangle = \langle \lambda, \mathscr{P}Tf\rangle = \langle \lambda, P\mathscr{P}f\rangle = \langle \lambda P, \mathscr{P}f\rangle$$

and since λP is absolutely continuous with respect to λ, $\boldsymbol{\lambda} T$ is absolutely continuous with respect to $\boldsymbol{\lambda}$. Hence T is a $\boldsymbol{\lambda}$-measurable Markov operator acting on $(\Omega, \mathscr{F}, \boldsymbol{\lambda})$ and the theory of $\boldsymbol{\lambda}$-measurable Markov operator is applicable.

We shall begin with the ergodic decomposition of Ω for $\boldsymbol{T}$. In the following all subsets of Ω are understood to be $\mathscr{F}$-measurable sets and two sets are equal if they are equal modulo $\boldsymbol{\lambda}$-null sets.

Lemma 4.1. *Let $X = C \cup D$ be the ergodic decomposition of X for P; then $\boldsymbol{C} = \prod_{n=0}^{\infty} C$ is the conservative part of Ω for $\boldsymbol{T}$ and $\Omega - \boldsymbol{C}$ is the dissipative part of Ω.*

Proof. To show that $\boldsymbol{C}$ is $\boldsymbol{T}$-conservative we shall show that, for every $\boldsymbol{T}$-transient set $\boldsymbol{E}$, $\boldsymbol{\lambda}(\boldsymbol{E} \cap \boldsymbol{C}) = 0$. If $\boldsymbol{E}$ is $\boldsymbol{T}$-transient then there is a number c for which $\sum_{n=0}^{\infty} \boldsymbol{T}^n 1_{\boldsymbol{E}} \leqq c$ a.e. ($\boldsymbol{\lambda}$). Let $f = \mathscr{P} 1_{\boldsymbol{E}}$. Then $P^n f = \mathscr{P} \boldsymbol{T} 1_{\boldsymbol{E}}$ so that $\sum_{n=0}^{\infty} P^n f = \mathscr{P}(\sum_{n=0}^{\infty} \boldsymbol{T}^n 1_{\boldsymbol{E}}) \leqq c$ a.e. (λ). Hence $f = 0$ a.e. (λ) on C. Now

$$\boldsymbol{\lambda}(\boldsymbol{E} \cap \boldsymbol{C}) = \int_C \mathscr{P} 1_{\boldsymbol{E} \cap \boldsymbol{C}} d\lambda \leqq \int_C \mathscr{P} 1_{\boldsymbol{E}} d\lambda = 0.$$

To show that $\Omega - \boldsymbol{C}$ is $\boldsymbol{T}$-dissipative it is sufficient to show that the set $[X_n \in D]$ is dissipative for $n = 0, 1, 2, \cdots$. Let A be P-transient (therefore $A \subset D$) then $\sum_{n=0}^{\infty} P^n 1_A \leqq d$ a.e. (λ) for some number d. Let η be a finite measure equivalent to λ. Then $\boldsymbol{\eta}$ is equivalent to $\boldsymbol{\lambda}$. Now

$$\left\langle \boldsymbol{\eta}, \sum_{k=0}^{\infty} \boldsymbol{T}^k 1_{[X_n \in A]} \right\rangle = \left\langle \eta, \sum_{k=0}^{\infty} P^{k+n} 1_A \right\rangle < \infty.$$

Hence $\sum_{k=0}^{\infty} \boldsymbol{T}^k 1_{[X_n \in A]} < \infty$ a.e. ($\boldsymbol{\lambda}$). It follows from Theorem 2.1 [5] that $[X_n \in A]$ is $\boldsymbol{T}$-dissipative. Since D is a countable union of P-transient sets, $[X_n \in D]$ is $\boldsymbol{T}$-dissipative.

We shall designate by Γ the collection of $\boldsymbol{T}$-closed subsets of $\boldsymbol{C}$.

Lemma 4.2. *Let P be conservative. Then $\boldsymbol{T}$ is also conservative. If a non-negative function f is P-invariant ($f = Pf$ a.e. (λ)) then $f(X_n)$ is $\boldsymbol{T}$-invariant ($f(X_n) = f(X_{n+1})$ a.e. ($\boldsymbol{\lambda}$)). Conversely if a non-negative function $\boldsymbol{g}$ on Ω is $\boldsymbol{T}$-invariant then there is a P-invariant function f such that $\boldsymbol{g} = f(X_n)$ a.e. ($\boldsymbol{\lambda}$).*

Proof. If a P-invariant function f is of the form 1_A with $A \in \mathscr{C}$ then

$$\mathscr{P} 1_A(X_n) \cdot 1_A(X_{n+1}) = P^n I_A P 1_A = P^n 1_A = 1_A = P^{n+1} 1_A.$$

Since $\mathscr{P} 1_A(X_n) = P^n 1_A$ and $\mathscr{P} 1_A(X_{n+1}) = P^{n+1} 1_A$ we have

$$\boldsymbol{\lambda}[1_A(X_n) \neq 1_A(X_n) \cdot 1_A(X_{n+1})] = \boldsymbol{\lambda}[1_A(X_{n+1}) \neq 1_A(X_n) \cdot 1_A(X_{n+1})] = 0.$$

Hence $1_A(X_n) = 1_A(X_{n+1})$ a.e. ($\boldsymbol{\lambda}$). In general f may be approximated from below by linear combinations of functions of the form 1_A. $f(X_n) = f(X_{n+1})$ a.e. (λ) is obtained by the usual limiting process.

Conversely if a bounded function $\boldsymbol{g}$ is $\boldsymbol{T}$-invariant and $f = \mathscr{P} \boldsymbol{g}$ then f is P-invariant. By the Martingale convergence theorem $\{f(X_n)\}$ converges a.e. ($\boldsymbol{\lambda}$) to $\boldsymbol{g}$

(cf. the proof of Theorem 1.1 on p. 460, [3]). Since $f(X_n) = f(X_0)$ a.e. (λ), $g = f(X_n)$ a.e. (λ). It follows that every set E in Γ is of the form $[X_n \in A]$ where $A \in \mathscr{C}$. Since every non-negative T-invariant function g is Γ-measurable, g is of the form $f(X_n)$ where f is $\mathscr{C}$-measurable.

COROLLARY 4.1. *A set E belongs to Γ if and only if E is of the form $[X_0 \in A]$ for some $A \in \mathscr{C}$.*

LEMMA 4.3. *If a measure μ on $\mathscr{X}$ is P-invariant (P-excessive) then $\boldsymbol{\mu}$ on $\mathscr{F}$ is T-invariant (T-excessive).*

Proof. The lemma follows from the following equalities:

$$\begin{aligned}\boldsymbol{\mu}(E) &= \langle \mu, \mathscr{P}1_E \rangle \leqq \langle \mu P, \mathscr{P}1_E \rangle = \langle \mu, P\mathscr{P}1_E \rangle \\ &= \langle \mu, \mathscr{P}T1_E \rangle = \langle \boldsymbol{\mu}, T1_E \rangle = \langle \boldsymbol{\mu} T, 1_E \rangle \\ &= \boldsymbol{\mu} T(E).\end{aligned}$$

THEOREM 4.1. *Let P be λ-continuous, $C_1, C_2, \cdots$, atoms of the σ-algebra $\mathscr{C}$ of P-closed subsets of C and $\mu_1, \mu_2, \cdots$, the P-invariant measures equivalent to λI_{C_1}, $\lambda I_{C_2}, \cdots$ respectively. Then the σ-algebra Γ of T-closed subsets of $\boldsymbol{C}$ is also purely atomic with atoms $\boldsymbol{C}_1, \boldsymbol{C}_2, \cdots$ where $\boldsymbol{C}_i = [X_0 \in C_i]$. Each $\boldsymbol{\mu}_i$ is T-invariant and equivalent to $\lambda I_{\boldsymbol{C}_i}$.*

Proof. By Corollary 4.1, $\mathscr{C}$ and Γ are isomorphic. Since $\mathscr{C}$ is purely atomic, Γ is purely atomic with atoms $[X_0 \in C_i]$, $i = 1, 2, \cdots$. By Lemma 4.3, $\boldsymbol{\mu}_i$ is T-invariant. $\boldsymbol{\mu}_i$ is equivalent to $\boldsymbol{\lambda} I_{\boldsymbol{C}_i}$ since μ_i is equivalent to λI_{C_i}.

THEOREM 4.2. *Let P be λ-continuous and $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \cdots$ be T-invariant measures of Theorem 4.1. Let $E_i = \bigcup_{n=0}^{\infty} [X_n \in C_i]$. If f, g are $\boldsymbol{\mu}_i$-integrable functions with $g \geqq 0$, $\int g d\boldsymbol{\mu}_i > 0$ then*

$$\lim_{N \to \infty} \frac{\sum_{n=0}^{N} T^n f}{\sum_{n=0}^{N} T^n g} = \frac{\int f d\boldsymbol{\mu}_i}{\int g d\boldsymbol{\mu}_i} \quad a.e.\ (\boldsymbol{\lambda})\ on\ E_i. \tag{4.4}$$

The above limit is independent of the particular T-invariant, $\boldsymbol{\lambda} I_{\boldsymbol{C}_i}$ equivalent measure $\boldsymbol{\mu}_i$ chosen. Furthermore

$$\lim_{N \to \infty} N^{-1}\left(\sum_{n=0}^{N} T^n f\right) = \int f d\boldsymbol{\mu}_i / \mu_i(C_i) \quad a.e.\ (\boldsymbol{\lambda})\ on\ E_i \tag{4.5}$$

(the right-hand side of (4.5) is interpreted to be 0 if $\mu_i(C_i) = \infty$).

Proof. Since $\boldsymbol{C}_i$ is an atom of Γ and $\boldsymbol{\mu}_i$ is a T-invariant measure with support $\boldsymbol{C}_i$, (4.4) is true a.e. (λ) on $\boldsymbol{C}_i$ by Theorem 6.5 of Appendix. Let

$$F = \left[w: \lim_{N\to\infty} \frac{\sum_{n=0}^{N} T^n f(w)}{\sum_{n=0}^{N} T^n g(w)} = \frac{\int f d\mu_i}{\int g d\mu_i} \right].$$

Then $\mathscr{P}1_{F\cap C_i} = 1$ a.e. (λ) on C_i. Let $E^k = [X_0 \notin C_i, \cdots, X_{k-1} \notin C_i, X_k \in C_i]$. To show that (4.4) is also true a.e. (λ) on $E_i - C_i$ it is sufficient to show $\lambda(E^k - E^k \cap F) = 0$ for $k = 1, 2, \cdots$. We shall show this by showing $\mathscr{P}1_{E^k} = \mathscr{P}1_{E^k\cap F}$ a.e. (λ). It is clear that $1_{E^k\cap F}(w) = 1_{E^k}(w)\cdot 1_{E\cap C_i}(T^{k+1}w)$. Hence for (λ) almost all x,

$$\mathscr{P}1_{E^k\cap F}(x) = 1_{X-C_i}(x) \int_{X-C_i}\cdots\int_{C_i} p(x,y_1)\cdots p(y_{k-1},y_k)\lambda(dy_1)\cdots\lambda(dy_k)\mathscr{P}1_{F\cap C_i}(y_k)$$

$$= 1_{X-C_i}) \int_{X-C_i}\cdots\int_{C_i} p(x,y_1)\cdots p(y_{k-1},y_k)\lambda(dy_1)\cdots\lambda(dy_k)$$

$$= \mathscr{P}1_{E_k}(x).$$

Any T-invariant λI_{Ci} equivalent measure is a constant multiple of μ_i. Hence the limit is independent of the particular μ_i chosen. If $\mu_i(C_i) < \infty$ and g is chosen to equal to 1 on C_i then (4.4) becomes (4.5). If $\mu(C_i) = \infty$ we may choose a monotone nondecreasing sequence $\{g_k\}$ such that $0 \leqq g_k \leqq 1$, $\int g_k d\mu_i < \infty$, $g_k \uparrow 1$ a.e. (λ); then

$$\limsup_{N\to\infty} \frac{\sum_{n=0}^{N} T^n f}{N} \leqq \lim_{N\to\infty} \frac{\sum_{n=0}^{N} T^n f}{\sum_{n=0}^{N} T^n g_k} = \frac{\int f d\mu_i}{\int g_k d\mu_i} \quad \text{a.e. } (\lambda) \text{ on } E_i.$$

Since $\int g_k d\mu_i \uparrow \infty$ as $k\to\infty$, $\lim_{N\to\infty} N^{-1}\sum_{n=0}^{N} T^n f = 0$ a.e. (λ) on E_i.

THEOREM 4.3. *Let $D = \bigcap_{n=0}^{\infty}[X_n \in D]$ and $\eta \in \mathscr{A}^+(\lambda)$. Let η be defined by (4.4) and $\mu = \sum_{n=0}^{\infty} \eta I_D T^n$. Then μ is T-excessive and $\sum_{n=0}^{\infty} T^n f$ converges a.e. (λ) on D for every μ-integrable function f.*

Proof. It is clear that D is T-closed and dissipative. μ is σ-finite with D as its support. The convergence of $\sum_{n=0}^{\infty} T^n f$ then follows from Theorem 6.5 of Appendix.

The following corollaries are special cases of Theorem 4.2 and Theorem 4.3.

COROLLARY 4.2. *Let C_i be an atom of $\mathscr{C}$ and μ_i a P-invariant measure equivalent to λI_{C_i}. If f, g are μ_i-integrable functions with $g \geqq 0$, $\int g d\mu_i > 0$ then*

$$\lim_{N\to\infty} \frac{f(X_1)+\cdots+f(X_N)}{g(X_1)+\cdots+g(X_n)} = \frac{\int f d\mu_i}{\int g d\mu_i} \quad a.e.\ (\lambda)\ on\ \mathbf{E}_i,$$

where $\mathbf{E}_i = \bigcup_{n=0}^{\infty}[X_n \in C_i]$ *and*

$$\lim_{N\to\infty} \frac{f(X_1)+\cdots+f(X_n)}{N} = \frac{\int f d\mu_i}{\mu_i(C_i)} \quad a.e.\ (\lambda)\ on\ \mathbf{E}_i,$$

where $1/\infty$ *is* 0.

COROLLARY 4.3. *Under the same assumption as in Theorem* 4.3 *if* f *is a function on* X *and* $g = \mathscr{P}1_{\mathbf{D}}$ *and if* fg *is integrable with respect to* $\sum_{n=0}^{\infty}\eta I_D(PI_D)^n$ *then* $\sum_{n=0}^{\infty} f(X_n)$ *converges a.e.* (λ) *on* $\mathbf{D}$.

Proof. We only need to point out that $\int f(X_0)d\boldsymbol{\mu} = \int fgd(\sum_{n=0}^{\infty}\eta I_D(PI_D)^n)$. Then the corollary follows immediately from Theorem 4.3.

V. **Martin boundaries.** Let π be a finite measure equivalent to λ which is fixed all through this section and h, a P-excessive π-integrable function. Let measure π_h, λ_h be defined by $\pi_h(A) = \int_A h d\pi$, $\lambda_h(A) = \int_A h d\lambda$. π_h is finite and λ_h is σ-finite. Let P_h be given by (1.8), (1.9). P_h is a λ_h-continuous Markov operator of which p_h of (1.7) is the density with respect to λ_h. Let the kernel $K(x,y)$ be given by (3.2). We shall construct measures on the product σ-algebra $\mathscr{F}$ of subsets of the product space Ω as in §IV. However in this section the component spaces of Ω will be $X \cup [\rho]$. ρ is usually called an "absorbing state." An element w of Ω is a sequence $\{x_0, x_1, x_2, \cdots\}$ with x_n being elements of X or equal to ρ. X_n shall be the function on Ω to $X \cup [\rho]$ defined by $X_n(w) = x_n$ if $w = \{x_0, x_1, x_2, \cdots\}$. For a π-integrable P-excessive function h a function $\mathscr{P}_h(x, \mathbf{E})$ $(x \in X,\ \mathbf{E} \in \mathscr{F})$ is defined as follows. If $\mathbf{E} = [X_0 \in A_0, \cdots, X_n \in A_n]$ where $A_i \in \mathscr{X}$, let

$$\mathscr{P}_h(x,\mathbf{E}) = \int_{A_0} \lambda_h(dx_0) \int_{A_1} \lambda_h(dx_1) \cdots \int_{A_n} \lambda_h(dx_n) p_h(x,x_0)\, p_h(x_0,x_1) \cdots p_h(x_{n-1},x_n).$$

If $\mathbf{E} \in \mathscr{F}$ and is of the form $\Omega - \mathbf{E}$ we let $\mathscr{P}_h(x,\mathbf{F}) = 1 - \mathscr{P}_h(x,\mathbf{E})$. Then the definition of $\mathscr{P}_h(x,\mathbf{E})$ is extended to arbitrary $\mathbf{E} \in \mathscr{F}$ by the usual measure extension argument. Measure $\boldsymbol{\pi}_h$ on $\mathscr{F}$ is defined by

$$\boldsymbol{\pi}_h(\mathbf{E}) = \int_{A \cap X} \pi_h(dx_0) \mathscr{P}(x_0, \mathbf{E}_1) \tag{5.1}$$

if $\mathbf{E}$ is of the form $[X_0 \in A,\ \{X_1, X_2, \cdots\} \in \mathbf{E}_1]$. Then the definition is again extended to arbitrary $\mathbf{E}$. $\boldsymbol{\pi}_h$ is a finite measure on $\mathscr{F}$.

THEOREM 5.1. *Let* h *be* P-*excessive and* π-*integrable. Let* $\boldsymbol{\pi}_h$ *be defined by* (5.1) *and*

$$\begin{aligned} K(x,X_n(w)) &= K(x,y) \text{ if } X_n(w) = y \in X, \\ &= 0 \qquad \text{if } X_n(w) = \rho. \end{aligned}$$

Then $\{K(x,X_n(w))\}$ *converges for* $(\lambda \times \pi_h)$ *almost all* (x,w). (*Note that* $\pi_h\{X_n \in D \cap [\bigcup_{n=1}^{\infty} \operatorname{supp} \pi P^n]\} = 0$ *for* $n \geqq 1$. *Hence* $K(x,X_n(w))$ *are well defined a.e.* $(\lambda \times \pi_h)$ *for* $n \geqq 1$.)

Proof. If $X_n(w)$ are in C for $n \geqq 1$ the convergence of $K(x,X_n(w))$ is obvious since $K(x,\cdot)$ is constant on every C_i and C, C_i are also P_h-closed. Since $K(x,y)1_D(y)$ is the kernel for operator PI_D it is sufficient to prove the theorem for a dissipative P.

Let $\mu = \sum_{n=1}^{\infty} \pi P^n$. μ is a σ-finite, P-excessive measure. Let $f = d\mu/d\lambda$; then $f(y) = \int \pi(dx) g(x,y)$ where $g(x,y)$ is given by (1.6). Let

$$q^n(y,x) = f(x)p^n(x,y)/f(y).$$

Then we may define a μ-continuous Markov operator Q with $q(x,y)$ as its density function with respect to μ. Q is actually the μ-reverse of P (cf. Appendix). Since P is dissipative so is Q. Hence for any finite measure v absolutely continuous with respect to μ, $\int v(dy) \sum_{n=1}^{\infty} q^n(y,x)$ is finite for (μ) almost all x. Since

$$\sum_{n=1}^{\infty} q^n(y,x) = f(x) g(x,y)/f(y) = f(x) K(x,y),$$

hence $\int v(dy) K(x,y) < \infty$ for (μ) almost all x.

Let a, b be two real numbers with $0 \leqq a < b$. Let $\beta(w)$ be the number of down crossings of $[a,b]$ by the sequence $K(x,X_1(w))$, $K(x,X_2(w)), \cdots$. Now we shall proceed to prove the theorem by showing that for (μ) almost all x, $\int \beta(w) \pi_h(dw) < \infty$.

Since P_h is also dissipative there is a monotone nondecreasing sequence of transient sets E of which the union is X. Let

$$\begin{aligned} \Omega_E &= [w : X_n(w) \in E \text{ for some } n > 0], \\ \tau_E(w) &= \sup[n : X_n(w) \in E] \text{ if } w \in \Omega_E \qquad (5.2) \\ &= 0 \text{ otherwise.} \end{aligned}$$

τ_E is finite valued a.e. (π_h). Let

$$\begin{aligned} L_E(x) &= \mathscr{P}_h\left(x, \bigcap_{n=1}^{\infty} X_n \notin E\right) \text{ if } x \in E \\ &= 0 \text{ otherwise.} \end{aligned}$$

On Ω_E define functions $Y_0, Y_1, Y_2, \cdots$ with values in $X \cup [\rho]$ as follows:

$$\begin{aligned} Y_0(w) &= X_{\tau_E(w)}(w), \\ Y_1(w) &= X_{\tau_E(w)-1}(w) \text{ if } \tau_E(w) \geqq 2 \\ &= \rho \text{ otherwise,} \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

Let $A_0,\cdots,A_n \in X$. We shall compute the following measure:

$$\pi_h[Y_0 \in A_0,\ Y_1 \in A_1,\cdots,Y_n \in A_n]$$

$$= \sum_{m=n+1}^{\infty} \pi_h[X_{m-n} \in A_n,\ X_{m-n-1} \in A_{n-1},\cdots,X_0 \in A_0] \cap [\tau_E = m]$$

$$= \sum_{m=n+1}^{\infty} \int_{A_n} \pi P^{m-n}(dy_n) \int_{A_{n-1}} \lambda(dy_{n-1}) \cdots \int_{A_0} \lambda(dy_0)\, p(y_n, y_{n-1}) \cdots$$
$$\cdots p(y_1, y_0)\, h(y_0)\, L_E(y_0)$$

$$= \int_{A_n} \mu(dy_n) \int_{A_{n-1}} \lambda(dy_{n-1}) \cdots \int_{A_0} \lambda(dy_0) p(y_n,y_{n-1}) \cdots p(y_1,y_0) h(y_0) L_E(y_0) \tag{5.3}$$

$$= \int_{A_n} \mu(dy_n) \int_{A_{n-1}} \lambda(dy_{n-1}) \cdots \int_{A_0} \lambda(dy_0)\, q(y_{n-1}, y_n) \cdots q(y_0, y_1) f(y_0) h(y_0) L_E(y_0)$$

$$= \int_{A_0} f(y_0) h(y_0) L_E(y_0) \lambda(dy_0) \int_{A_1} \lambda(dy_1) \cdots \int_{A_n} \lambda(dy_n) q(y_0, y_1) \cdots q(y_{n-1}, y_n).$$

Let $\eta_E(A) = \pi_h[Y_0 \in A]$. η_E is the distribution of Y_0. η_E has density function $f(y_0)\, h(y_0)\, L_E(y_0)$ with respect to λ. The collection of x for which $\int \eta_E(dy) K(x,y) = \infty$ for some E in the sequence is a μ-null set. Let x belong to the complement of this μ-null set. Then $\int K(x, Y_0) d\pi_h < \infty$. Let

$\beta_E(w) =$ the number of down crossings of $[a,b]$
by $K(x, X_1(w)),\cdots, K(x, X_{\tau_E}(w))$ if $W \in \Omega_E$,
$= 0$ otherwise.

Then $\beta_E(w)$ is also the number of up crossings of $[a,b]$ by $K(x, Y_0(w))$, $K(x, Y_1(w)),\cdots$. Since $K(x,\cdot)$ is a Q-potential, (5.3) implies that $K(x, Y_n)$ is a supermartingale and $\int K(x, Y_n) d\pi_h \downarrow 0$. It follows from a well-known inequality of Doob [3, p. 316, Theorem 3.3] that $E[\beta_E] \leqq b/(b-a)\pi_h(\Omega_E)$. $\int \beta_E d\pi_h \uparrow \int \beta d\pi_h$ as $E \uparrow X$. Hence $\int \beta d\pi_h \leqq b/(b-a)$ and the theorem is proved.

The kernel $K(x,y)$ has the property that for (λ) almost all $y \in \bigcup_{n=1}^{\infty} \operatorname{supp} \pi P^n$, $K(\cdot, y)$ is an element of $L_1(\pi)$ with L_1 norm $= 1$. For an element ϕ of $L_\infty(\pi)$ define

$$l_\phi(y) = \int \phi(x) K(x,y)\, \pi\, d(x). \tag{5.4}$$

COROLLARY 5.1. *Let h be P-excessive and π-integrable. Let $\phi \in L_\infty(\pi)$ and $l_\phi(y)$ be defined by* (5.4) *and $l_\phi(\rho) = 0$. Then $\{l_\phi(X_n)^k\}$ converges a.e. (π_h) and also in $L_1(\pi_h)$ for every positive integer k. Furthermore, if P is dissipative, E is transient and $E \uparrow X$ and τ_E be defined by* (5.2), *then $\{l_\phi(X_{\tau_E})^k\}$ converges a.e. (π_h) and also in $L_1(\pi_h)$ as $E \uparrow X$ where the value of $l_\phi(X_{\tau_E})$ is taken to be* 0 *if*

$w \notin \Omega_E$. The limit is independent of the particular sequence of transient sets E chosen.

Proof. It is sufficient to prove the corollary for a non-negative ϕ. For the a.e. (π_h)-convergence of $\{l_\phi(X_n)\}$ the same proof of Theorem 5.1 goes through since l_ϕ is a Q-potential if P is dissipative. Let $\tau(w) = \lim \tau_E(w)$ as $E \uparrow X$. If $\tau(w)$ is finite the convergence of $l_\phi(X_{\tau_E}(w))^k$ is trivial and the limit is $l_\phi(X_{\tau(w)}(w))^k$. If $\tau(w) = \infty$, $l_\phi(X_{\tau_{E(w)}}(w))^k$ is a subsequence of $l_\phi(X_n(w))^k$, therefore, its convergence follows from the convergence of $l_\phi(X_n(w))^k$ and both sequences have the same limit. The $L_1(\pi_h)$-convergence follows from the fact that $|l_\phi(y)| \leqq \|\phi\|_\infty$.

For the rest of this section we shall assume that $X = \bigcup_{n=1}^{\infty} \operatorname{supp} \pi P^n$. We shall also assume that for every $y \in X$, $K(\cdot, y)$ is a non-negative element of $L_1(\pi)$ with L_1 norm 1. This may be accomplished by discarding a λ-null set. Consider the map T:

$$T(y) = K(\cdot, y). \tag{5.5}$$

This is a map of X into $L_1(\pi)$. We shall consider $L_1(\pi)$ as a subset of its second dual. Let $\tilde{X}$ be the weak closure of the image of X under T. $\tilde{X}$ is a compact Hausdorff space under the weak topology.

Lemma 5.1. *Let $\tilde{\mathscr{X}}$ be the σ-algebra of all Baire subsets of $\tilde{X}$. Then T given by (5.5) is a measurable transformation of $(X, \mathscr{X})$ to $(\tilde{X}, \tilde{\mathscr{X}})$.*

Proof. We shall use the symbol $\langle \tilde{y}, \phi \rangle$ for $\phi \in L_\infty(\pi), \tilde{y} \in \tilde{X}$ to denote the value of $\tilde{y}$ at ϕ. To prove Lemma 5.1 we shall show that for every real valued continuous function f on $\tilde{X}$, fT is $\mathscr{X}$-measurable. We shall show this by two steps. First, we shall show that every real continuous function on $\tilde{X}$ may be uniformly approximated by polynomials of functions of the type $\langle \cdot, \phi \rangle$. Second, we shall show that $\langle T(\cdot), \phi \rangle$ is $\mathscr{X}$-measurable for every $\phi \in L_\infty(\pi)$.

Let Λ be the collection of all functions on the unit ball S of $L_\infty(\pi)$ into the closed interval $[-1, +1]$. The topology on Λ shall be the Tychonoff topology. Then $\tilde{X}$ may be considered as a closed subset of Λ. For any $\eta \in \Lambda$, $\phi \in S$ we shall denote the value of η at ϕ by $\langle \eta, \phi \rangle$. The polynomials of functions of the type $\langle \cdot, \phi \rangle$ form a subalgebra of the algebra of all real valued continuous functions on Λ. This subalgebra separates points of Λ and contains the constant functions. By the Stone-Weierstrass theorem every continuous function on Λ may be uniformly approximated by elements of this subalgebra. Now every continuous real valued function on $\tilde{X}$ may be extended to be real continuous function on Λ. Hence the first step is finished. The $\mathscr{X}$-measurability of $\langle T(\cdot), \phi \rangle$ follows from the equality:

$$\langle T(\cdot), \phi \rangle = \int \pi(dx) \phi(x) K(x, \cdot).$$

Theorem 5.2. *For every non-negative P-invariant π-integrable function h there is a Baire measure $\tilde{\eta}$ on $\tilde{\mathscr{X}}$ for which the following formula holds,*

$$h = \int \tilde{y}\tilde{\eta}(d\tilde{y}). \tag{5.6}$$

(5.6) *is interpreted to be that for every* $\phi \in L_\infty(\pi)$

$$\langle h, \phi \rangle = \int \langle \tilde{y}, \phi \rangle \tilde{\eta}(d\tilde{y}).$$

Proof. Let

$$\tilde{y}_{C_i} = \frac{u_{C_i}}{\int \pi(dy) u_{C_i}(y)}.$$

$\tilde{y}_{C_i}$ is then the image of all points y in C_i under the transformation T. If h is of the form $\sum a_i u_{C_i}$, then $\tilde{\eta}$ in (5.6) is atomic which assigns measure $a_i \int \pi(dy) u_{C_i}(y)$ to $\tilde{y}_{C_i}$ and the formula is valid. In general if $h = a_i$ on C_i then $h - \sum a_i u_{C_i} = 0$ on C. $\sum a_i u_{C_i}$ is P-invariant and, by Theorem 6.2 of Appendix, is the smallest P-excessive function which is equal to h on C. Hence $h - \sum a_i u_{C_i} \geqq 0$ and is P-invariant. Hence we only need to establish (4.4) for h which equals to 0 on C.

Since $h = 0$ on C, we may replace P by PI_D. In other words we may assume P to be dissipative. Then P_h is also dissipative. Let E be P_h-transient and $E \uparrow X$. Let τ_E be defined by (5.2). Let us consider X_{τ_E} to be defined on Ω_E and η_E to be its distribution. Let $\tilde{X}_E = TX_{\tau_E}$ and $\tilde{\eta}_E$ be its distribution. We shall show that $\tilde{\eta}_E$ converges weakly, i.e., for every continuous function f on $\tilde{X}$, $\int f d\tilde{\eta}_E$ converges as $E \uparrow X$. Since $\{\tilde{\eta}_E(\tilde{X})\}$ is a bounded sequence, it is sufficient to show the convergence for f being a polynomial of functions of the form $\langle \cdot , \phi \rangle$ where $\phi \in L_\infty(\pi)$. It is clear that

$$\int (\langle \tilde{y}, \phi \rangle)^k \tilde{\eta}_E(d\tilde{y}) = \int_{\Omega_E} l_\phi(X_{\tau_E})^k d\pi_h$$

where l_ϕ is given by (5.4). The convergence of $\{\int (\langle \tilde{y}, \phi \rangle)^k \tilde{\eta}_E(d\tilde{y})\}$ is then a consequence of Corollary 5.1. Let $\tilde{\eta}$ be the limit measure of $\{\tilde{\eta}_E\}$.

Now we shall compute $\mathscr{P}_h(x, [X_{\tau_E} \in A, \tau_E > 0])$. For (π_h) almost all x

$$\begin{aligned} \mathscr{P}_h(x, [X_{\tau_E} \in A, \tau_E > 0]) &= \frac{1}{h(x)} \sum_{m=1}^{\infty} \int_A p^{(m)}(x,y) h(y) L_E(y) \lambda(dy) \\ &= \frac{1}{h(x)} \int_A g(x,y) h(y) L_E(y) \lambda(dy) \\ &= \frac{1}{h(x)} \int_A K(x,y) \eta_E(dy). \end{aligned}$$

Hence

$$\begin{aligned} \int \phi(x) h(x) \mathscr{P}_h(x, [\tau_E > 0]) \pi(dx) &= \iint \phi(x) K(x,y) \pi(dx) \eta_E(dy) \\ &= \int \langle \tilde{y}, \phi \rangle \tilde{\eta}_E(d\tilde{y}). \end{aligned} \tag{5.7}$$

Now $\mathscr{P}_h(x,[\tau_E > 0]) \geqq \mathscr{P}_h(x,[X_1 \in E]) \uparrow 1$ a.e. (π_h) since h is P-invariant. (5.6) is then obtained by letting $E \uparrow X$ in (5.7).

VI. **Appendix on λ-measurable Markov operators.** Let λ be a σ-finite measure and $\mathscr{A}(\lambda)$ be the Banach space of the collection of all finite countably additive set functions which are absolutely continuous with respect to λ. Let $L_\infty(\lambda)$ be the dual of $\mathscr{A}(\lambda)$. Let P: $\nu \to \nu P$ be a positive linear operator and a contraction on $\mathscr{A}(\lambda)$ to $\mathscr{A}(\lambda)$. Let $P: f \to Pf$ acting on $L_\infty(\lambda)$ be its dual. Operator P on $L_\infty(\lambda)$ is characterized by (1) P is positive, (2) $P1 \leqq 1$, (3) $f_k \downarrow 0$ implies that $Pf_k \downarrow 0$. For an extended real valued non-negative function f which is not λ-essentially bounded Pf shall be the limit of Pf_n where $f_n \in L_\infty(\lambda), f_n \uparrow f$. Similarly for any measure ν absolutely continuous with respect to λ, νP shall be the limit of $\nu_n P$ where $\nu_n \in \mathscr{A}(\lambda)$, $\nu_n \uparrow \nu$. $\langle \nu, f \rangle$ is to designate the integral $\int f d\nu$ whenever the latter is well defined. We have $\langle \nu P, f\rangle = \langle \nu, Pf\rangle$. We shall call P a *λ-measurable Markov operator*. An extended real valued non-negative function h is *excessive* if $Pf \leqq f$, a *potential* if $P^n f \downarrow 0$. A σ-finite measure μ is *excessive* if $\mu P \leqq P$, a *potential* if $d\mu P^n/d\lambda \downarrow 0$. It should be emphasized that for a λ-measurable Markov operator λ-null sets are irrelevant and $=$, $\leqq$ mean $=$ a.e. (λ), $\leqq$ a.e. (λ).

For any set A, A' designates its complement. Let

$$P_A^{\#} = \sum_{n=0}^{\infty} (I_{A'}P)^n, \qquad P_A = \sum_{n=0}^{\infty} P(I_{A'}P)^n,$$

then $P_A = PP_A^{\#}$ and

$$P_A^{\#} I_A = I_A + I_{A'} P_A I_A. \tag{6.1}$$

LEMMA 6.1. *If h is an excessive function and l is an excessive function which is $\geqq h$ on A then $P_A^{\#} I_A h \leqq l$. It follows that $P_A^{\#} I_A h \leqq h$.*

Proof. We need to show

$$\sum_{n=0}^{N} (I_{A'}P)^n I_A h \leqq l \tag{6.2}$$

for $N = 0,1,2,\cdots$. For $N=0$ (6.2) becomes $I_A h \leqq l$. Assume (6.2) to be true for N; then

$$P\sum_{n=0}^{N} (I_{A'}P)^n I_A h \leqq Pl \leqq l.$$

Hence

$$I_A h + I_{A'} P \sum_{n=0}^{N} (I_{A'}P)^n I_A h \leqq l, \text{ i.e., } \sum_{n=0}^{N+1} (I_{A'}P)^n I_A h \leqq l.$$

THEOREM 6.1. *$P_A^{\#} I_A$ and $P_A I_A$ are λ-measurable Markov operators.*

Proof. We need to prove the theorem for $P_A^{\#} I_A$ only. By Lemma 6.1, $P_A^{\#} I_A 1 \leqq 1$. It follows that $f \in L_\infty(\lambda)$ implies that $P_A^{\#} I_A f \in L_\infty(\lambda)$. Positivity of $P_A I_A$ is obvious.

Let $f_k \in L_\infty(\lambda), f_k \geqq 0, f_k \downarrow 0$. The theorem is proved if we can show that $P_A^\# I_A f_k \downarrow 0$. We may assume $f_k \leqq 1$ for all k. Then

$$P_A^\# I_A f_k(x) \leqq \sum_{n=0}^{N} (I_{A'}P)^n I_A f_k(x) + \sum_{n=N+1}^{\infty} (I_{A'}P)^n 1_A(x). \tag{6.3}$$

Let x be fixed. For any $\varepsilon > 0$ choose N so large that the second term at the right-hand side of (6.3) is $\leqq \varepsilon/2$, then choose K so large that the first term at the right-hand side of (6.3) is $\leqq \varepsilon/2$ whenever $k \geqq K$. Then $P_A^\# I_A f_k(x) \leqq \varepsilon$ whenever $k \geqq K$ and the assertion that $P_A^\# I_A f_k \downarrow 0$ is proved.

THEOREM 6.2. *If h is excessive then $P_A^\# I_A h$ is the smallest excessive function which is $\geqq h$ on A.*

Proof. We need only to prove that $P_A^\# I_A h$ is excessive. Then the rest of the theorem follows from Lemma 6.1. By Lemma 6.1, $P_A^\# I_A h \leqq h$. Hence $P_A I_A h \leqq h$. By (6.1)

$$P_A^\# I_A h = I_A h + I_{A'} P_A I_A h \geqq I_A P_A I_A h + I_{A'} P_A I_A h = P_A I_A h = P P_A^\# I_A h.$$

COROLLARY 6.1. *If A is conservative then $P_A 1_A$ is an invariant function and is the smallest excessive function which is $= 1$ on A.*

Proof. $P_A 1_A = P P_A I_A 1$. Hence $P_A 1_A$ is excessive and is $\leqq P_A I_A 1$. However A being conservative implies $P_A 1_A = 1$ on A. Hence, by Theorem 6.2, $P_A 1_A 1 = P_A 1_A$. Hence $P_A 1_A$ is invariant and is the smallest excessive function $= 1$ on A.

A set A is said to be recurrent if $P_A 1_A = 1$ on A. A set B is *conservative* if every subset of B is *recurrent*. A set E is *dissipative* if every λ-non-null subset of E is not conservative. A set T is *transient* if there is a non-negative number $q < 1$ such that $P_T 1_T \leqq q$ on T. A subset of a transient set is transient.

THEOREM 6.3. *Any dissipative set E is a union of at most countably many transient sets.*

Proof. First we shall show that every λ-non-null nonrecurrent set A contains a λ-non-null transient set. The set $(P_A 1_A < 1) \cap A$ is λ-non-null, hence, there is a non-negative number $q < 1$ such that $T = (P_A 1_A \leqq q) \cap A$ is non-null. By Theorem 6.2, $P_T 1_T \leqq P_A 1_A$. Hence $P_T 1_T \leqq P_A 1_A \leqq q$ on T and T is transient.

If E is transient then there is nothing to prove. We assume that E is not transient. For every countable ordinal number $\alpha > 0$ we shall define a transient subset T_α of E by transfinite induction. Since E is λ-non-null (because E is not transient) and dissipative, E contains a λ-non-null subset which is not recurrent. Hence E contains a λ-non-null transient set T_1. Suppose T_β are defined for all $\beta < \alpha$. If $E - U_{\beta<\alpha} T_\beta$ is λ-null define T_α to be the null set. Otherwise $E - U_{\beta<\alpha} T_\beta$ contains a λ-non-null transient set T_α. There must be an α for which T_α is null. For, if not, E would contain uncountably many disjoint λ-non-null sets which is impossible. Let α_0 be the first ordinal number such that T_{α_0} is null. Then $E = U_{\alpha<\alpha_0}T_\alpha$ and the theorem is proved.

For any $\nu \in \mathscr{A}^+(\lambda)$ the *support* of ν, supp ν, is the set for which $A \subset \text{supp}\,\nu$ and A being λ-non-null imply that $\nu(A) > 0$, and $B \subset X - \text{supp}\,\nu$ implies that $\nu(B) = 0$. For any set A, we define the *consequent* of A, $F(A)$ by

$$F(A) = \bigcup_{n=0}^{\infty} \text{supp}\,\nu P^n$$

where $\nu \in \mathscr{A}^+(\lambda)$ has A as its support. The particular ν chosen clearly does not matter. If A is recurrent then $F(A)$ is the smallest closed set containing A.

THEOREM 6.4. *If A is recurrent then for every $\nu \in \mathscr{A}^+(\lambda)$, νP_A is σ-finite on $F(A)$. In particular, if P is conservative and ergodic, then νP_A is σ-finite for every non-null set A, and every $\nu \in \mathscr{A}^+(\lambda)$.*

Proof. It is sufficient to prove that $\nu P_A^{\#}$ is σ-finite on $F(A)$. Since $I_{A'}P$ is a λ-measurable Markov operator and $P_A^{\#} = \sum_{n=0}^{\infty}(I_{A'}P)^n$ it is sufficient to show that $F(A)$ is dissipative under the operator $I_{A'}P$.

We shall first show that $P_A 1_A = 1$ on $F(A)$. It follows that $P_A^{\#}1_A = 1$ on $F(A)$ since $P_A^{\#}1_A$ is P-excessive (Theorem 6.2) and $P_A 1_A = PP_A^{\#}1_A$. Let $F_n(A) = \text{supp}\,\eta P^n$ where η has A as its support. We shall show that $P_A 1_A = 1$ on $F_n(A)$ for $n = 0,1,2\cdots$. By the definition of recurrency of A, $P_A 1_A = 1$ on $F_0(A) = A$. Assuming that $P_A 1_A = 1$ on $F_n(A)$ we proceed to show that $P_A 1_A = 1$ on $F_{n+1}(A)$.

$$\begin{aligned} \eta P^n(x) &= \eta P^n(F_n(A)) = \langle \eta P^n, P_A 1_A\rangle \\ &= \langle \eta P^n, PP_A^{\#}1_A\rangle = \langle \eta P^{n+1}, P_A^{\#}1_A\rangle \\ &= \langle \eta P^{n+1}, 1_A\rangle + \langle P^{n+1}, I_{A'}P_A 1_A\rangle \qquad (6.5)\\ &\leqq \langle \eta P^{n+1}, 1_A\rangle + \langle \eta P^{n+1}, 1_{A'}\rangle = \langle \eta P^{n+1}, 1\rangle \\ &= \eta P^{n+1}(X). \end{aligned}$$

The fifth equality in (6.5) is owing to (6.1) and the inequality following it is owing to the fact $P_A 1_A \leqq 1$ (Lemma 6.1). Since $\eta P^n(X) \geqq \eta P^{n+1}(X)$, equality holds all through (6.5). Hence

$$\langle \eta P^{n+1}, I_{A'}P_A 1_A\rangle = \langle \eta P^{n+1}, 1_{A'}\rangle$$

and $P_A 1_A = 1$ on $\text{supp}\,\eta P^{n+1} - A = F_{n+1}(A) - A$. Hence $P_A 1_A = 1$ on $F_{n+1}(A)$.

Now we have the following equality:

$$P^n = \sum_{k=0}^{n-1} (I_{A'}P)^k I_A P^{n-k} + (I_{A'}P)^n.$$

Since $F(A)$ is closed, for $\nu \in \mathscr{A}^+(\lambda)$ with $\text{supp}\,\nu \subset F(A)$, we have

$$\begin{aligned}\nu(F(A)) &= \nu P^n(F(A)) = \langle \nu P^n, 1\rangle \\ &= \left\langle \nu \sum_{k=0}^{n-1} (I_{A'}P)^k I_A P^{n-k}, 1\right\rangle + \langle \nu(I_{A'}P)^n, 1\rangle \\ &= \left\langle \nu \sum_{k=0}^{n-1} (I_{A'}P)^k I_A, 1\right\rangle + \nu(I_{A'}P)^n(X) \\ &= \left\langle \nu, \sum_{k=0}^{n-1} (I_{A'}P)^k 1_A\right\rangle + \nu(I_{A'}P)^n(X).\end{aligned}$$

Now

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (I_{A'}P)^k 1_A = P_A^{\#}\, 1_A = 1 \text{ on } F(S).$$

Hence

$$\lim_{n\to\infty} \nu(I_{A'}P)^n(X) = 0. \tag{6.6}$$

If $F(A)$ were not dissipative under the operator $I_{A'}P$ then $F(A)$ would contain an $I_{A'}P$-recurrent set B. Let ν have support B. Then $\nu(X) = \nu(I_{A'}P)^n(X)$ for all positive integers n, which contradicts (5.6).

Let μ be a σ-finite excessive measure. Then, if ν is absolutely continuous to μ, so is νP. P acting on $\mathscr{A}(\mu)$ is a μ-measurable Markov operator which we shall designate by P_μ. Let $g \in L_\infty(\mu)$ with $L_\infty(\mu)$ norm $\| g \|$ and μ_g be defined by $\mu_g(A) = \int_A g d\mu$. $\mu_g P$ is σ-finite and absolutely continuous with respect to μ. Define

$$Qg = d\mu_g P/d\mu.$$

Since $\pm \mu_g \leqq \| g \| \mu$, $\pm \mu_g P \leqq \| g \| \mu P \leqq \| g \| \mu$, hence $Qg \in L_\infty(\mu)$ with $\| Qg \| \leqq \| g \|$. Clearly $g_k \in L_\infty(\mu)$, $g_k \downarrow 0$ imply that $Qg_k \downarrow 0$, so that Q is a μ-measurable Markov operator. Q is characterized by the following equality:

$$\int Pf \cdot g d\mu = \int f \cdot Qg d\mu \tag{6.7}$$

whenever one side of (6.7) is well defined. J. Feldman showed that the space X has the same decomposition $X = C \cup D$ for Q as that of P although μ-null sets are irrelevant to Q [5, Theorem 3.1]. We shall call Q the *μ-reverse* of P. Since for $g \geqq 0$

$$\langle \mu, Qg\rangle = \int 1 \cdot Qg d\mu = \int P1 \cdot g d\mu \leqq \int g d\mu = \langle \mu, g\rangle,$$

μ is also Q-excessive. The symmetric roles played by Q, P in (6.7) imply that P_μ is the μ-reverse of Q.

LEMMA 6.2. *Let P be a conservative λ-measurable Markov operator which possesses an invariant measure μ. Let Q be its μ-reverse. If set A is P-closed then it is Q-closed. Conversely if A is a subset of the support of μ and A is Q-closed then it is P-closed.*

Proof. We shall prove the lemma for the case that μ is equivalent to λ. Then. applying it to P_μ and noting that a subset of the support of μ is P-closed if and only if it is P_μ-closed, we obtain the assertion for the general case.

When μ is equivalent to λ the roles played by P, Q are symmetric. Hence we only need to prove the sufficiency part. Let S be Q-closed. Let f be non-negative, bounded, μ-integrable and $[f > 0] = S$. Let μ_f be defined by $\mu_f(A) = \int_A f d\mu$. Then $S = \bigcup_{n=0}^{\infty} \operatorname{supp} \mu_f Q^n$. Now (6.7) implies that $d\mu_f Q^n/d\mu = P^n f$. Hence $S = \bigcup_{n=0}^{\infty} [P^n f > 0]$. However $[P^n f > 0] = [P^n 1_S > 0]$. Hence

$$S = \bigcup_{n=0}^{\infty} [P^n 1_S > 0] = [P_S 1_S > 0].$$

Now by Corollary 6.1, $P_S 1_S$ is P-invariant, therefore $\mathscr{C}$-measurable [7, Theorem 9.1]. Hence the set $[P_S 1_S > 0] = S$ is P-closed.

LEMMA 6.3. *Let P possess an excessive measure μ and Q be its μ-reverse.*

1. *If h is a finite valued P-excessive function then the measure μ_h defined by $\mu_h(A) = \int_A h d\mu$ is Q-excessive. If, in addition, h is a P-potential then μ_h is a Q-potential.*
2. *If η is a P-excessive measure absolutely continuous with respect to μ then $d\eta/d\mu$ is Q-excessive. If, in addition, η is a P-potential then $d\eta/d\mu$ is a Q-potential.*

Proof. For any non-negative function f, μ_f shall always denote the measure defined by $\mu_f(A) = \int_A f d\mu$. Now if $0 \leqq Ph \leqq h$, $g \geqq 0$ then by (6.7)

$$\langle \mu_h Q^n, g\rangle = \langle \mu_h, Q^n g\rangle = \langle \mu_g, P^n h\rangle \leqq \langle \mu_h, g\rangle.$$

Hence μ_h is Q-excessive. It is clear that $P^n h = d\mu_h Q^n/d\mu$ so that h being a P-potential implies that μ_h is a Q-potential. If η is a P-excessive measure absolutely continuous to μ, then, by (6.7), for $g \geqq 0$

$$\langle \eta, P^n g\rangle = \langle \mu_g, Q^n(d\eta/d\mu)\rangle.$$

However

$$\langle \mu_g, d\eta/d\mu\rangle = \langle \eta, g\rangle \geqq \langle \eta P^n, g\rangle = \langle \mu_g, d\eta P^n/d\mu\rangle.$$

Hence

$$\langle \mu_g, d\eta/d\mu\rangle \geqq \langle \mu_g, Q^n(d\eta/d\mu)\rangle,$$

$$Q^n(d\eta/d\mu) = d\eta P^n/d\mu$$

and the conclusion 2 follows.

COROLLARY 6.2. *If P is conservative and ergodic and if P possesses a nontrivial invariant measure μ then it is unique up to a constant multiple.*

Proof. By Lemma 6.2, Q, the μ-reverse of P, is also ergodic. Hence the only Q-excessive functions are constants. If η is another P-invariant measure then $d\eta/d\mu$ is Q-excessive and is a constant by Lemma 6.3. Hence η is a constant multiple of μ.

THEOREM 6.5. *If μ is a P-excessive measure and if f, $g \in L_1(\mu)$, $g > 0$ a.e. (μ) then $\sum_{n=1}^{\infty} P^n f$ converges a.e. (μ) on D and, on C, there exists*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} P^n f}{\sum_{n=1}^{N} P^n g} = \frac{\bar{f}}{\bar{g}} \quad a.e.\ (\mu)$$

where $\bar{f}$, $\bar{g}$ are $\mathscr{C}$-measurable and satisfy

$$\int_A \bar{f} d\mu_1 = \int_A f d\mu, \qquad \int_A \bar{g} d\mu_1 = \int_A g d\mu$$

for every $A \in \mathscr{C}$ where μ_1 is a finite measure equivalent to μ.

Proof. Let Q be the μ-reverse of P. It follows from (6.7) that $d\mu_f Q^n/d\mu = P^n f$, $d\mu_g Q^n/d\mu = P^n g$. For Q, $\mathscr{C}$ is still the σ-algebra of conservative Q-closed sets by Lemma 6.2. The theorem is then obtained immediately by applying the general ratio ergodic theorem.

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