TWO NOTES ON LOCALLY MACAULAY RINGS

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1. **Introduction.** In this paper all rings are assumed to be commutative rings with a unit. The undefined terminology used in this paper (height, altitude, etc.) will be the same as that in [1]. Throughout this paper a number of known properties of locally Macaulay rings are stated, and then are used in the remainder of the paper without explicit mention.

In §2 it is proven that if R is a locally Macaulay ring and if $(a_1,\cdots,a_n)$ is a prime sequence in R, the kernel of the natural homomorphism from $P = R[X_1,\cdots,X_{n-1}]$ onto $R' = R[a_2/a_1,\cdots,a_n/a_1]$ is $(a_1X_1 - a_2, a_1X_2 - a_3,\cdots,a_1X_{n-1} - a_n)P$ (Lemma 2.3). As a consequence, R' is a locally Macaulay ring and $(a_1,a_2/a_1,\cdots,a_n/a_1)$ is a prime sequence in R' (Theorem 2.4). Further, if $R[X_1]$ is a Macaulay ring, then R' is a Macaulay ring (Theorem 2.8). An example is given to show that the converses are not in general true.

In §3 it is proven that, with the same R and a_i, the Rees ring $R^* = R[ta_1,\cdots,ta_n,1/t]$ (t an indeterminant) of R with respect to $A = (a_1,\cdots,a_n)R$ is a locally Macaulay ring (a Macaulay ring if $R[X_1]$ is) and $(1/t,ta_1,\cdots,ta_n)$ is a prime sequence in R^* (Theorems 3.1 and 3.3). A form of the converses of Theorems 3.1 and 3.3 is true (Theorem 3.8). Also, for every $e \geqq 1$, $k \geqq e$, and $i = 1,\cdots,n$, $(a_1^e,a_2^e,\cdots,a_i^e)A^{k-e} = (a_1^e,a_2^e,\cdots,a_i^e) \cap A^k$ (Corollary 3.6). Further, for all $k \geqq 1$, every prime divisor of A^k has height n, and $A^k : a_1R = A^{k-1}$ (Corollary 3.7). It is also proven that if the Rees ring R^* of a Noetherian ring R with respect to an ideal $A = (a_1,\cdots,a_n)R$ is a locally Macaulay ring (a Macaulay ring), then $R' = R[a_1/a,\cdots,a_n/a]$ is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor $a \in A$ (Corollary 3.9).

2. **Transformations of locally Macaulay rings by a prime sequence.**

LEMMA 2.1. *Let R be a ring, let a,b be elements in R such that a is not a zero divisor, and let X be an indeterminant. If $aR : bR = aR$, then the kernel K of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$.*

Proof. Clearly $aX - b \in K$. Let $f(X) = r_nX^n + \cdots + r_0 \in K$. Then $r_nb^n + r_{n-1}ab^{n-1} + \cdots + r_0a^n = 0$, so $r_n \in aR : b^nR = aR$, say $r_n = da$. Since a is not a zero divisor, $g(X) = (db + r_{n-1})X^{n-1} + r_{n-2}X^{n-2} + \cdots + r_0 \in K$, and $f(X) = (aX - b)dX^{n-1} + g(X)$. Hence, by induction on n, $f(X) \in (aX - b)R[X]$, so K is generated by $aX - b$, q.e.d.

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A local (Noetherian) ring R is a *Macaulay local ring* in case there exists a system of parameters $(a_1, \cdots, a_n)$ in R such that a_i is not in any prime divisor of $(a_1, \cdots, a_{i-1})R$ $(i = 1, \cdots, n)$. In particular a_1 is not a zero divisor. A Noetherian ring R is a *locally Macaulay ring* in case R_M is a Macaulay local ring for every maximal ideal M in R. R is a *Macaulay ring* in case R is a locally Macaulay ring such that height $M =$ altitude R for every maximal ideal M in R. It is known that if R is a Macaulay local ring of altitude n and if $(a_1, \cdots, a_k)$ is a subset of a system of parameters in R, then $R/(a_1, \cdots, a_k)R$ is a Macaulay local ring of altitude $n - k$ [3, p. 397]. Also, R is a locally Macaulay ring if and only if the following theorem (the *unmixedness theorem*) holds: If an ideal A in R is generated by k elements and if height $A = k$ $(k \geqq 0)$, then every prime divisor of A has height k [1, p. 85]. These two facts immediately imply that if R is a locally Macaulay ring (a Macaulay ring) and if A is an ideal in R which is generated by k elements and has height k, then R/A is a locally Macualay ring (a Macaulay ring). Finally, it is known that if $X_1, \cdots, X_n$ are algebraically independent over a Noetherian ring R, then $R[X_1, \cdots, X_n]$ is a locally Macaulay ring if and only if R is [1, p. 86].

These facts are used in the proof of

COROLLARY 2.2. *Let R be a locally Macaulay ring, and let a, b be elements in R such that a is not a zero divisor and $aR : bR = aR$. Then $R[b/a]$ is a locally Macaulay ring.*

Proof. $R[X]$ is a locally Macaulay ring, and the kernel of the natural homomorphism from $R[X]$ onto $R[b/a]$ is generated by $aX - b$ (Lemma 2.1). Since $aX - b$ is not a zero divisor in $R[X]$, $R[b/a]$ is a locally Macaulay ring, q.e.d.

Theorem 2.4 below generalizes the above corollary. To obtain the generalization the following definitions and lemma will be used.

An integral domain R satisfies the *altitude formula* in case the following condition holds: If R' is an integral domain which is finitely generated over R, and if p' is a prime ideal in R', then height $p' + \operatorname{trd}(R'/p')/(R/p' \cap R) =$ height $p' \cap R + \operatorname{trd} R'/R$. It is known that if an integral domain R is a homomorphic image of a locally Macaulay ring, then R satisfies the altitude formula [1, p. 130].

If R is a locally Macaulay ring, and if $p \subset q$ are prime ideals in R, then R_q is a Macaulay local ring [1, p. 86], so height $p +$ height $q/p =$ height q [3, p. 399]. This fact will be used in the future without explicit mention.

A sequence $(a_1, \cdots, a_n)$ of nonunits in a Noetherian ring R is a *prime sequence* in case a_1 is not a zero divisor, $(a_1, \cdots, a_i)R : a_{i+1}R = (a_1, \cdots, a_i)R$ $(i = 1, \cdots, n - 1)$, and $(a_1, \cdots, a_n)R \neq R$. It is known that if R is a semi-local ring, and if $(a_1, \cdots, a_n)$ is a prime sequence of elements in the Jacobson radical of R, then $(a_{\pi 1}, \cdots, a_{\pi n})$ is a prime sequence for every permutation π of $\{1, \cdots, n\}$ [3, pp. 394–395].

LEMMA 2.3. *Let R be a locally Macaulay ring, let $(a_1, \cdots, a_n)$ be a prime sequence in R, and let $X_1, \cdots, X_{n-1}$ be algebraically independent over R. Then* *the kernel K of the natural homomorphism ϕ from $P = R[X_1, \cdots, X_{n-1}]$ onto $R' = R[a_2/a_1, \cdots, a_n/a_1]$ is generated by $(a_1X_1 - a_2, a_1X_2 - a_3, \cdots, a_1X_{n-1} - a_n)$.*

Proof(1). The proof is by induction on n. The case $n=1$ is trivial, and Lemma 2.1 proves the case $n = 2$. Let $n > 2$ and assume the conclusion holds for the case $n - 1$. Now $\phi = fg$, where f and g are the natural homomorphisms from $S = R[a_2/a_1, X_2, \cdots, X_{n-1}]$ onto R' and from P onto S respectively. Since the kernel of g is $(a_1X_1 - a_2)P$ (Lemma 2.1), and since $R^* = R[a_2/a_1]$ is a locally Macaulay ring (Corollary 2.2), it is sufficient (by induction) to prove that $(a_1, a_3, a_4, \cdots, a_n)$ is a prime sequence in R^*. Since R and R^* have the same total quotient ring, a_1 is not a zero divisor in R^*, hence height $a_1R^* = 1$. Let $A_i^* = (a_1X_1 - a_2, a_1, a_3, \cdots, a_i)P$ $(i \geqq 3)$. Then $A_i^* = (a_1, a_2, a_3, \cdots, a_i)P$, hence height $A_i^* = i$. Consequently, by the unmixedness theorem $(a_1X_1 - a_2, a_1, a_3, \cdots, a_n)$ is a prime sequence in P, hence $(a_1, a_3, \cdots, a_n)$ is a prime sequence in R^*, q.e.d.

THEOREM 2.4. *With the same notation as Lemma 2.3, $R_i' = R[a_2/a_1, \cdots, a_i/a_1]$ $(2 \leqq i \leqq n)$ is a locally Macaulay ring, and $(a_1, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)$ is a prime sequence in R_i', where $\{b_1, \cdots, b_k\}$ is a subset of $\{a_2/a_1, \cdots, a_i/a_1\}$, and $0 \leqq j \leqq n - i$. (For $j = 0$ the sequence is $(a_1, b_1, \cdots, b_k)$.)*

Proof. That R_i' is a locally Macaulay ring follows immediately from Lemma 2.3 and the remarks preceding the proof of Corollary 2.2. Let $A^* = (a_1, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)R_i'$. Since $(a_1X_1 - a_2, \cdots, a_1X_{i-1} - a_i, a_1)R[X_1, \cdots, X_{i-1}]$ is generated by $(a_1, \cdots, a_i)$, A^* is a proper ideal. Hence by the unmixedness theorem, since j and k are arbitrary, it is sufficient to prove height $A^* = j + k + 1$. Let p' be a minimal prime divisor of A^*, let q' be a (minimal) prime divisor of zero in R_i' such that $q' \subset p'$ and let $p = p' \cap R$, $q = q' \cap R$. By the altitude formula (for R'/q' over R/q), height $p'/q' + \operatorname{trd} R'/p'/R/p =$ height p/q (since $a_1 \notin q$). Also, height $p'/q' \leqq j + k + 1$, $\operatorname{trd} R'/p'/R/p \leqq i - 1 - k$, and height $p/q =$ height $p \geqq i + j$. Hence, height $p' =$ height $p'/q' = j + k + 1$. Therefore height $A^* = j + k + 1$, q.e.d.

REMARK 2.5. The last step in the proof of Theorem 2.4 shows the following results. For every (minimal) prime divisor p' of A^* and for every prime divisor q' of zero contained in p', p'/q' is a minimal prime divisor of $(A^* + q')/q'$. Since height $p'/q' = j + k + 1$, none of the elements $a_1, \cdots, a_{i+j}, b_1, \cdots, b_k$ are in q'. Also the elements $a_2/a_1, \cdots, a_i/a_1$ which are not in p' are such that their p' residues are algebraically independent over $R/(p' \cap R)$.

REMARK 2.6. In Theorem 2.4, if every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in R (for example, if R is a semi-local locally Macaulay ring and $a_1, \cdots, a_n$ are in the Jacobson radical of R), then every permutation of $(a_1, a_{i+1}, \cdots, a_n, a_2/a_1, \cdots, a_i/a_1)$ is a prime sequence in R_i'.

(1) The author is indebted to the referee for the following proof which is considerably simpler than the author's original proof, and which leads to a more direct proof of Theorem 2.4.

Proof. Let $(c_1, \cdots, c_n)$ be a permutation of $(a_1, a_{i+1}, \cdots, a_n, a_2/a_1, \cdots, a_i/a_1)$. Since no a_i is a zero divisor in R, c_1 is not a zero divisor in R'_i. Also $(c_1, \cdots, c_n)R'_i \neq R'_i$. Therefore, by the unmixedness theorem, it remains to prove height $(c_1, \cdots, c_h) R'_i = h$ $(h = 2, \cdots, n-1)$. Let p' be a minimal prime divisor of $(c_1, \cdots, c_h)R'_i$, let q' be a prime divisor of zero in R'_i which is contained in p', and let $p = p' \cap R$, $q = q' \cap R$. If $a_1 \notin p'$, then $\operatorname{trd} R'/p'/R/p = 0$. Hence by the altitude formula (for R'/q' over R/q), height $p'/q' =$ height p/q. Now height $p' \leqq h$ and height $p \geqq h$ (by the assumption on $(a_1, \cdots, a_n)$), so height $p' =$ height $p = h$. If $a_1 \in p'$, let k of the elements $c_1, \cdots, c_h$ be in $\{a_2/a_1, \cdots, a_i/a_1\}$. Then height $p \geqq i + (h - 1 - k)$ (by the assumption on $(a_1, \cdots, a_n)$), and $\operatorname{trd} R'/p'/R/p \leqq i - 1 - k$. By the altitude formula for R'/q' over R/q, height $p' =$ height $p'/q' = h$, q.e.d.

Remark 2.6 is of some interest because of the following

LEMMA 2.7. *Let R be a locally Macaulay ring, and let $(a_1, \cdots, a_n)$ be a prime sequence in R such that every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in R. Let $A = (a_1, \cdots, a_n)R$. Then, for all $k \geqq 1$,* (1) *every prime divisor of A^k has height n, and* (2) $A^k \colon a_iR = A^{k-1}$ $(i = 1, \cdots, n)$.

Proof. This can be proved in the same way as Lemmas 5 and 6 in [3, pp. 401–402]. Without assuming that every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in R, Corollary 3.7 below proves (1) is still true and (2)$'$ $A^k \colon a_1R = A^{k-1}$ (for all $k \geqq 1$), q.e.d.

It is known that if R is a Macaulay ring and if $X_1, \cdots, X_n$ are algebraically independent over R, then $R[X_1, \cdots, X_n]$ is a Macaulay ring if and only if there does not exist an ideal p in R such that R/p is a semi-local integral domain of altitude one [1, p. 87]. Hence if $R[X_1]$ is a Macaulay ring, then $R[X_1, \cdots, X_n]$ is a Macaulay ring. This fact is used in the proof of the next theorem.

THEOREM 2.8. *If R and $R[X]$ are Macaulay rings (X transcendental over R), and if $(a_1, \cdots, a_n)$ is a prime sequence in R, then $R' = R[a_2/a_1, \cdots, a_n/a_1]$ is a Macaulay ring.*

Proof. The kernel K of the natural homomorphisms from $P = R[X_1, \cdots, X_{n-1}]$ onto R' has height $n-1$. Since P is a Macaulay ring, if M is a maximal ideal in P which contains K, then altitude $R + n - 1 =$ altitude $P =$ height $M =$ height $M/K +$ height K. Hence, if M' is a maximal ideal in R', then height $M' =$ altitude $P -$ height $K =$ altitude R. Since R' is a locally Macaulay ring by Theorem 2.4, R' is a Macaulay ring, q.e.d.

REMARK 2.9. If R is a locally Macaulay ring (a Macaulay ring such that $R[X]$ is a Macaulay ring), and if $(a_1, \cdots, a_n)$ is a prime sequence in R, then $R[a_1/a, \cdots, a_n/a]$ is a locally Macaulay ring (a Macaulay ring) for every non-zero-divisor $a \in (a_1, \cdots, a_n)R$. This follows from Theorems 3.1 and 3.3 and Corollary 3.9 below.

It will now be shown that the converses of Theorems 2.4 and 2.8 are not in general true. Let $S = k[X, Y]$, where k is a field and X and Y are algebraically independent over k. Let $P = (X - 1, Y)S$, $R_1 = S_P$, and $N_1 = PR_1$. Let $Q = (X, Y)S$, $R_2 = S_Q$, and $N_2 = QR_2$. Let $R' = R_1 \cap R_2$, $M_1 = N_1 \cap R'$, and $M_2 = N_2 \cap R'$. Further let $R = k + (M_1 \cap M_2)$, and let $M = (M_1 \cap M_2)R$. Then R' is the intersection of two regular local rings, hence R' is normal. The following statements are easily verified: (1) M_1 and M_2 are the maximal ideals in R', and $R'_{M_i} = R_i$ is Noetherian ($i = 1,2$). Therefore R' is Noetherian [1, p. 203], so R' is a normal semi-local Macaulay domain. (2) Since $R'/M_i = k$ ($i = 1,2$), R is a local domain and R' is its derived normal ring [1, p. 204]. (3) $XY, Y \in R$, $X \notin R$, $R' = R[XY/Y]$, and $(Y, X = XY/Y)$ is a prime sequence in R'. (4) If p is a height one prime ideal in R, then R_p is a regular local ring. Since $R \neq R'$, M is an imbedded prime divisor of every nonzero element in M [1, p. 41], hence R is not a Macaulay domain.

3. **The Rees ring of a locally Macaulay ring.** Let R be a Noetherian ring, let $A = (a_1, \cdots, a_n)R$ be an ideal in R, let t be an indeterminant, and set $u = t^{-1}$. The graded Noetherian ring $R^* = R[ta_1, \cdots, ta_n, u]$ is called the *Rees ring* of R with respect to A.

THEOREM 3.1. *Let R be a locally Macaulay ring, and let $a_1, \cdots, a_n$ be a prime sequence in R. Then the Rees ring R_i^* of R with respect to $(a_1, \cdots, a_i)R$ $(1 \leqq i \leqq n)$ is a locally Macaulay ring, and $(u, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)$ is a prime sequence in R_i^*, where $\{b_1, \cdots, b_k\}$ is a subset of $\{ta_1, \cdots, ta_i\}$ and $0 \leqq j \leqq n - i$. (For $j = 0$ the sequence is $(u, b_1, \cdots, b_k)$.)*

Proof. Since u is transcendental over R, $R[u]$ is a locally Macaulay ring, hence $(u, a_1, \cdots, a_n)$ is a prime sequence in $R[u]$. Since $ta_j = a_j/u$, R_i^* is a locally Macaulay ring and $(u, a_{i+1}, \cdots, a_{i+j}, b_1, \cdots, b_k)$ is a prime sequence in $R_i^*$by Theorem 2.4, q.e.d.

REMARK 3.2. In Theorem 3.1, if every permutation of $(a_1, \cdots, a_n)$ is a prime sequence in R, then every permutation of $(u, a_1, \cdots, a_n)$ is a prime sequence in $R[u]$ (since $R[u]$ is a locally Macaulay ring and u is transcendental over R), hence by Remark 2.6 every permutation of $(u, a_{i+1}, \cdots, a_n, ta_1, \cdots, ta_i)$ is a prime sequence in R_i^*.

THEOREM 3.3. *If R and $R[X]$ are Macaulay rings (X transcendental over R), and if $a_1, \cdots, a_n$ is a prime sequence in R, then the Rees ring R^* of R with respect to $(a_1, \cdots, a_n)$ R is a Macaulay ring.*

Proof. Considering the natural homomorphism from $R[u, X_1, \cdots, X_n]$ onto R^* and the ideal $(u, a_1, \cdots, a_n)$ of R^*, the proof is the same as the proof of Theorem 2.8, q.e.d.

LEMMA 3.4. *Let R^* be the Rees ring of a locally Macaulay ring R with respect to a prime sequence $(a_1, \cdots, a_n)$ in R. Then $(ta_1, \cdots, ta_i, u)$ is a prime sequence in R^* $(i = 1, \cdots, n)$.*

Proof. Since R^* is a locally Macaulay ring and height $(u, ta_1, \cdots, ta_i)R^* = i+1$ (Theorem 3.1), it is sufficient to prove height $(ta_1, \cdots, ta_i)R^* = i$. Let p be a minimal prime divisor of $A_i^* = (ta_1, \cdots, ta_i)R^*$. Then height $p \leqq i$, hence $u \notin p$. Let $T = R[u, t]$, so T is a quotient ring of R^*. Since pT is a minimal prime divisor of $A_i^*T = (a_1, \cdots, a_i)T$, and since height $(a_1, \cdots, a_i)R[u] = i$, height $A_i^*T = i$. Therefore height $p = i$, so height $A_i^* = i$, q.e.d.

REMARK 3.5. Let $(a_1, \cdots, a_n)$ be a prime sequence in a locally Macaulay ring R. Then the radical of $(a_1, \cdots, a_n)R$ is the radical of $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})R$ $(e_i \geqq 1, i = 1, \cdots, n)$. Hence, by the unmixedness theorem, $(a_1^{e_1}, a_2^{e_2}, \cdots, a_n^{e_n})$ is a prime sequence in R. Therefore $R[a_2^{e_2}/a_1^{e_1}, \cdots, a_n^{e_n}/a_1^{e_1}]$ and $R[ta_1^{e_1}, \cdots, ta_n^{e_n}, u]$ are locally Macaulay rings.

Let R be a Noetherian ring and let R^* be the Rees ring of R with respect to an ideal A in R. Let $T = R[t, u]$, so T is a quotient ring of R^*. For any ideal B in R let $B' = BT \cap R^*$. For any homogeneous ideal B^* in R^* let $[B^*]_k$ be the set of elements $r \in R$ such that $rt^k \in B^*$. It is immediately seen that $[B^*]_k$ is an ideal in R and $A^k \supseteq [B^*]_k \supseteq [B^*]_{k+1} \supseteq A[B^*]_k$ for all integers k (with the convention that $A^k = R$ if $k \leqq 0$). Also, since R^* is Noetherian, if k is greater than or equal to the maximum degree of the generators of B^*, then $[B^*]_{k+1} = A[B^*]_k$, and if k is less than or equal to the degree of the generators of B^*, then $[B^*]_{k-1} = [B^*]_k$ [2]. Let $B = (b_1, \cdots, b_i)R \subseteq A^e$. Clearly $B' = BT \cap R^* \supseteq (b_1t^e, b_2t^e, \cdots, b_it^e)R^* = B^*$, and for $k \leqq e$, $[B']_k = B \cap A^k = B \supseteq [B^*]_k = [B]_e \supseteq B$. Hence for $k > e$, $[B']_k = B \cap A^k \supseteq [B^*]_k = BA^{k-e}$. Since $B'T = B^*T = BT$, $B^* = B'$ if and only if u is not in any prime divisor of B^*. Hence if $(b_1t^e, b_2t^e, \cdots, b_it^e, u)$ is a prime sequence in R^*, then $B' = B^*$. In particular, by Lemma 3.4 and Remark 3.5 we have proved the following

COROLLARY 3.6. *Let R be a locally Macaulay ring, let $(a_1, \cdots, a_n)$ be a prime sequence, and let $A = (a_1, \cdots, a_n)R$. Then, for every $e \geqq 1$, $k \geqq e$, and $i = 1, \cdots, n$, $(a_1^e, a_2^e, \cdots, a_i^e)A^{k-e} = (a_1^e, a_2^e, \cdots, a_i^e)R \cap A^k$.*

COROLLARY 3.7. *Let $(a_1, \cdots, a_n)$ be a prime sequence in a locally Macaulay ring R. Set $A = (a_1, \cdots, a_n)R$. Then, for all $k \geqq 1$, (1) every prime divisor of A^k has height n, and (2) $A^k \colon a_1R = A^{k-1}$.*

Proof. By Corollary 3.6, $a_1A^{k-1} = a_1R \cap A^k$. Since a_1 is not a zero divisor in R, $A^{k-1} = a_1A^{k-1} \colon a_1R = (a_1R \cap A^k) \colon a_1R = A^k \colon a_1R$, hence (2) holds. For (1), $u^kR^* \cap R = A^k$, where $R^* = R[ta_1, \cdots, ta_n, u]$, and $k \geqq 1$. Since R^* is a locally Macaulay ring, every prime divisor of uR^* has height one, and the prime divisors of u^kR^* are the prime divisors of uR^* (Remark 3.5). Let p' be a prime divisor of uR^*, let q' be a minimal prime divisor of zero in R^* which is contained in p', and let $p = p' \cap R$, $q = q' \cap R$. Applying Remark 2.5 (with $A^* = uR^*$) and the altitude formula for R^*/q' over R/q, height $p = n$ (since $\mathrm{trd}R^*/q'/R/q = 1$), so p is a prime divisor of A^k. Since $u^kR^* \cap R = A^k$, (1) holds, q.e.d.

If $(a_1,\cdots,a_n)$ is a prime sequence in a locally Macaulay ring R, then $(ta_1,\cdots,ta_n,u)$ is a prime sequence in the locally Macaulay ring $R[ta_1,\cdots,ta_n,u]$ (Theorem 3.1 and Lemma 3.4). Theorem 3.8 contains the converse of this.

THEOREM 3.8. *Let R be a Noetherian ring and let A be an ideal in R. If the Rees ring R^* of R with respect to A is a locally Macaulay ring (a Macaulay ring), then R is a locally Macaulay ring (R and $R[X]$ are Macaulay rings). If also there are elements $b_1,\cdots,b_n$ in A such that $(b_1t^{e_1},\cdots,b_nt^{e_n},u)$ is a prime sequence in R^*, then $(b_1,\cdots,b_n)$ is a prime sequence in R.*

Proof. Let R^* be a locally Macaulay ring. Then, since $T=R^*[t]$ is a quotient ring of R^*, T is a locally Macaulay ring. Let M be a maximal ideal in R. Since T is a quotient ring of R^* and of $R[u]$, $T_{MT}=R[u]_{MR[u]}$ is a Macaulay local ring. Since u is transcendental over R, a system of parameters in R_M is a system of parameters in $R[u]_{MR[u]}$. It is known that if a local ring has one system of parameters which form a prime sequence, then each system of parameters forms a prime sequence [3, p. 399]. Hence R is a locally Macaulay ring. Therefore, if $(b_1t^{e_1},\cdots,b_nt^{e_n},u)$ is a prime sequence in R^*, then, for $i=1,\cdots,n$, every prime divisor of $(b_1t^{e_1},\cdots,b_it^{e_i})T=(b_1,\cdots,b_i)T$ has height i. Hence height $(b_1,\cdots,b_i)R=i$, and so $(b_1,\cdots,b_n)$ is a prime sequence in R. Let R^* be a Macaulay ring. By what has already been proved, R and $R[X]$ are locally Macaulay rings. To prove that R is a Macaulay ring, let M be a maximal ideal in R. Then $N^*=(M,u-1)T\cap R^*$ is a maximal ideal in R^*. Therefore, altitude $R+1=$ altitude $R^*=$ height $N^*=$ height $N^*T=$ height $M+1$, hence R is a Macaulay ring. Finally, let N be a maximal ideal in $R[u]$. If there is a maximal ideal N^* in R^* such that $N^*\cap R[u]=N$, then altitude $R[u]=$ altitude $R^*=$ height $N^*=$ (since R^*/N^* is a field) height N^*+ trd$R^*/N^*/R[u]/N=$ (altitude formula) height $N\leqq$ altitude $R[u]$. If there does not exist such N^*, then $NT=T$, hence $u\in N$. Therefore $R/N\cap R=R[u]/N$ is a field, so altitude $R[u]=$ altitude $R+1=$ height $N\cap R+1=$ height N^*. Hence $R[X]\cong R[u]$ is a Macaulay ring, q.e.d.

COROLLARY 3.9. *Let R be a Noetherian ring. If there exists an ideal $A=(a_1,\cdots,a_n)R$ in R such that the Rees ring R^* of R with respect to A is a locally Macaulay ring (a Macaulay ring), then for every non-zero-divisor $a\in A$, $R'=R[a_1/a,\cdots,a_n/a]$ is a locally Macaulay ring (a Macaulay ring).*

Proof. Since $(a-u)R[t,u]=(at-1)R[t,u]$ is the kernel of the mapping from $R[t,u]$ onto $R[1/a,a]$ (Lemma 2.1), and since $R[t,u]$ is a quotient ring of R^*, to prove the two statements about R' it is sufficient to prove that u is not in any prime divisor of $(ta-1)R^*$. If u is in some (minimal) prime divisor p of $(ta-1)R^*$, then p is a prime divisor of uR^*. But uR^* is a graded ideal, hence p is a graded deal. This implies the contradiction $1\in p$. Therefore u is not in any prime divisor of $(ta-1)R^*$, q.e.d.

Theorem 3.8 is of some interest, since the Rees ring R^* of a locally Macaulay ring R with respect to an ideal A which cannot be generated by a prime sequence may be a locally Macaulay ring. For example, let R be a semi-local Macaulay ring of altitude $n \geqq 2$, and let $(a_1, \cdots, a_n)$ be a prime sequence in the Jacobson radical of R. Let $A = (a_1, \cdots, a_n)R$ and fix an integer $e \geqq 2$. Then A^e cannot be generated by n elements, but *the Rees ring of R with respect to A^e is a locally Macaulay ring.* For convenience of notation this will be proved for the case $n = 2$ (the general case being exactly the same). Let $a = a_1$ and $b = a_2$, and let N be a maximal ideal in $R^* = R[ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e, u]$. If $u \notin N$, then R^*_N contains $T = R[t, u]$. Since T is a locally Macaulay ring, R^*_N is a Macaulay local ring. If $(ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e)R^*$ is not contained in N, say $ta^f b^{e-f} \notin N$. Then $ta^{f+1}b^{e-f-1}/ta^f b^{e-f} = a/b \in R^*_N$ (if $f < e$), and/or $b/a \in R^*_N$ (if $f > 0$). Since (a, b) and (b, a) are prime sequences in R, $R_e = R[a/b]$, $R_0 = R[b/a]$, and $R_f = R[a/b, b/a]$ are locally Macaulay rings, and at least one of these rings (call it R') is contained in R^*_N. Hence $S = R'[ta^f b^{e-f}]$ is a locally Macaulay ring contained in R^*_N, and S contains $R[ta^e, \cdots, ta^g b^{e-g}, \cdots, tb^e]$. Since $ta^f b^{e-f} \notin N' = NR^*_N \cap S$, $u = a^f b^{e-f}/ta^f b^{e-f} \in S_{N'}$. Hence $R^*_N = S_{N'}$ is a locally Macaulay ring. Clearly the only maximal ideals in R^* which contain $(ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e, u)R^*$ are the ideals $N_i = (M_i, ta^e, \cdots, ta^f b^{e-f}, \cdots, tb^e, u)R^*$, where M_i is a maximal ideal in R. Therefore it remains to prove that the semi-local ring $R^*_{R^* - \cup N_i}$ is a Macaulay ring. For this, it will be shown that (ta^e, tb^e, u) is a prime sequence in R^* (since the N_i contain this sequence). Since (a^e, b^e) is a prime sequence in the locally Macaulay ring $R^*[t]$, to prove (ta^e, tb^e, u) is a prime sequence, it is sufficient to prove that u is not in any prime divisor of either of the ideals $ta^e R^*$ or $(ta^e, tb^e)R^*$. This is equivalent to proving $ta^e R^* = a^e T \cap R^*$ and $(ta^e, tb^e)R^* = (a^e, b^e)T \cap R^*$, where $T = R[t, u]$. With the notation used in the proof of Corollary 3.6, these latter equalities are equivalent to $[ta^e R^*]_k = [a^e T \cap R^*]_k$ and $[(ta^e, tb^e)R^*]_k = [(a^e, b^e)T \cap R^*]_k$ for all k. Since the degrees of the generators of the four ideals are all non-negative, and since $[ta^e R^*]_0 = [a^e T \cap R^*]_0 = a^e R$ and $[(ta^e, tb^e)R^*]_0 = [(a^e, b^e)T \cap R^*]_0 = (a^e, b^e)R$, it must be shown that $a^e(A^e)^{k-1} = a^e R \cap (A^e)^k$ and $(a^e, b^e)(A^e)^{k-1} = (a^e, b^e)R \cap (A^e)^k$ for all $k \geqq 1$. These equalities hold by Corollary 3.6.

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