INVARIANT EIGENDISTRIBUTIONS ON A SEMISIMPLE LIE GROUP

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1. **Introduction.** Let G be a connected semisimple Lie group and $\mathfrak{Z}$ the algebra of all differential operators on G which commute with both left and right translations of G. One of the main objects of this paper is to show that every invariant eigendistribution T of $\mathfrak{Z}$ on G, is actually a locally summable function F which is analytic on the regular set G' of G (Theorem 2). In particular, this implies that the character of an irreducible unitary representation of G is a function.

In the second part we examine the behavior of F around the singular points of G (see §§19, 20). This is done by applying the results of [4(n), §§8, 9]. The third part is devoted to the detailed study of an invariant integral on G, which had been introduced in [4(h), Theorem 2]. Here we have to make use of [4(m), Theorem 1]. The full significance of Theorem 3 for harmonic analysis on G will appear only in later papers. Roughly speaking, it is the group analogue of [4(g), Theorem 3].

Our methods are substantially the same as those introduced in [4(l)] and [4(n)], although they have now to be applied to the group G instead of its Lie algebra $\mathfrak{g}$. Here Theorem 2 of [4(l)] gets replaced by Lemma 22, which is based on Theorem 1 and this, in its turn, depends on Theorem 5 of [4(n)]. The results of §3 enable us to limit ourselves to the semisimple points of G and the reduction procedure, outlined above, can be applied to any such point a provided it does not lie in the center Z of G. However, if $a \in Z$, the translation by a^{-1} reduces the problem to the case $a = 1$. Then we use the results of §14 to transform it, by means of the exponential mapping, into an analogous question on $\mathfrak{g}$ around zero, which has already been discussed in [4(n)]. This general pattern of proof applies to most results of this paper (e.g., Theorems 1 and 2). However, sometimes it is more convenient to reduce the problem around a directly to the corresponding question around zero on the centralizer $\mathfrak{z}$ of a in $\mathfrak{g}$ (see, for example, the proofs of Lemmas 31, 35, 37 and 40).

Theorem 3 is proved by making use of the elementary solution of a certain elliptic "Laplacian" and imitating the argument of [4(g), pp. 208–211]. The Appendix contains a few simple lemmas which are needed in the proof of this theorem.

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2. **The mapping** Γ_x. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra over $\boldsymbol{R}$ and $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}_c$. For any $X \in \mathfrak{g}_c$, let L_X and R_X, respectively, denote the endomorphisms $g \to Xg$ and $g \to gX$ $(g \in \mathfrak{G})$ of $\mathfrak{G}$. Fix $x \in G$ and define(1) (cf. [4(e), p. 114])

$$\sigma_x(X) = L_{x^{-1}X} - R_X \qquad (X \in \mathfrak{g}_c).$$

Note that L_X and R_Y commute and(2)

$$[L_X, L_Y] = L_{[X,Y]}, \quad [R_X, R_Y] = -R_{[X,Y]}$$

for X, $Y \in \mathfrak{g}_c$. Hence it follows immediately that σ_x is a representation of $\mathfrak{g}_c$ on $\mathfrak{G}$. It may, therefore, be extended (uniquely) to a representation of $\mathfrak{G}$ which we shall again denote by σ_x. Let Γ_x denote the linear mapping of $\mathfrak{G} \otimes \mathfrak{G}$ into $\mathfrak{G}$ such that

$$\Gamma_x(g_1 \otimes g_2) = \sigma_x(g_1)g_2 \qquad (g_1, g_2 \in \mathfrak{G}).$$

Let λ denote the canonical mapping (see [4(b), p. 192]) of $S(\mathfrak{g}_c)$ onto $\mathfrak{G}$. We say that an element $g \in \mathfrak{G}$ is homogeneous of degree d if $g \in \mathfrak{G}_d = \lambda(S_d(\mathfrak{g}_c))$ in the notation of [4(k), §6]. Put

$$_d\mathfrak{G} = \sum_{0 \leqq m \leqq d} \mathfrak{G}_m.$$

Then it is obvious that Γ_x defines a linear mapping of $\mathfrak{G}_{d_1} \otimes \mathfrak{G}_{d_2}$ into $_{d_1+d_2}\mathfrak{G}$.

Let $x \to x^a$ $(x \in G)$ be an automorphism of G. Then it defines an automorphism of $\mathfrak{g}$ which can be extended uniquely to an automorphism $g \to g^a$ $(g \in \mathfrak{G})$ of g.

Lemma 1. *For any $x \in G$ and g_1, $g_2 \in \mathfrak{G}$,*

$$\Gamma_{x^a}(g_1^a \otimes g_2{}^a) = (\Gamma_x(g_1 \otimes g_2))^a.$$

Let A denote the automorphism $g \to g^a$ of $\mathfrak{G}$. Then one verifies from the definitions that

$$\sigma_{x^a}(X^a) = A\sigma_x(X)A^{-1}$$

for $X \in \mathfrak{g}$. Our assertion is an immediate consequence of this fact.

Lemma 2. *Suppose X_i and Y_j $(1 \leqq i \leqq r,\ 1 \leqq j \leqq s)$ are elements in $\mathfrak{g}_c$. Fix $x \in G$ and put $X_i' = x^{-1}X_i - X_i$ $(1 \leqq i \leqq r)$. Then*

$$\Gamma_x(\lambda(X_1X_2 \cdots X_r) \otimes \lambda(Y_1Y_2 \cdots Y_s)) \equiv \lambda(X_1'X_2' \cdots X_r'\ Y_1Y_2 \cdots Y_s) \mod {}_{(r+s-1)}\mathfrak{G}.$$

It follows by an easy induction on r that

(1) As usual $xX = X^x = \mathrm{Ad}(x)X$ for $x \in G$ and $X \in \mathfrak{g}_c$.

(2) $[A, B] = AB - BA$ for two endomorphisms A and B of a vector space.

$$\Gamma_x(X_1X_2\cdots X_r\otimes Y_1Y_2\cdots Y_s)\equiv X_1'X_2'\cdots X_r'Y_1Y_2\cdots Y_s \bmod_{(r+s-1)}\mathfrak{G},$$

where all the products are in $\mathfrak{G}$. Hence our assertion is an immediate consequence of the following well-known fact (see [4(a), p. 902]).

Lemma 3. *Let $Z_1,\cdots,Z_d$ be elements of $\mathfrak{g}_c$ and $(i_1,i_2,\cdots,i_d)$ a permutation of $(1,2,\cdots,d)$. Then*

$$Z_1Z_2\cdots Z_d - Z_{i_1}Z_{i_2}\cdots Z_{i_d}\in {}_{(d-1)}\mathfrak{G}.$$

As usual we regard elements of $\mathfrak{G}$ as left-invariant differential operators on G. Moreover, for every $X\in\mathfrak{g}$, let $\rho(X)$ denote the right-invariant vector-field on G given by(3)

$$f(x;\rho(X))=(df(\exp tX\cdot x)/dt)_{t=0}\qquad (x\in G, f\in C^\infty(G)).$$

A simple argument shows that $[\rho(X),\rho(Y)]=-\rho([X,Y])$ and therefore ρ can be extended uniquely to an anti-homomorphism of $\mathfrak{G}$ into the algebra of all differential operators on G. We define

$$f(g;x)=f(x;\rho(g))$$

for $x\in G$, $g\in\mathfrak{G}$ and $f\in C^\infty(G)$. If $X_1,\cdots,X_r\in\mathfrak{g}$, then(4)

$$\begin{aligned} f(X_1X_2\cdots X_r;x) &= f(x;\rho(X_1X_2\cdots X_r)) = f(x;\rho(X_r)\cdots\rho(X_2)\rho(X_1))\\ &= \{\partial^r f(\exp t_1X_1\cdots\exp t_rX_r\cdot x)/\partial t_1\cdots\partial t_r\}_{t_1=\cdots=t_r=0}\\ &= f(x;(X_1X_2\cdots X_r)^{x^{-1}}) \end{aligned}$$

since $\exp t_1X_1\cdots\exp t_rX_r\cdot x = x(\exp t_1X_1\cdots\exp t_rX_r)^{x^{-1}}$. Therefore

$$f(g;x)=f(x;g^{x^{-1}})\qquad (g\in\mathfrak{G}).$$

It is obvious that X and $\rho(Y)$ $(X,Y\in\mathfrak{g})$ commute (in the algebra of differential operators on G) and therefore g_1 and $\rho(g_2)$ $(g_1,g_2\in\mathfrak{G})$ also commute.

Now G operates on itself by means of inner automorphisms so that $y^x=xyx^{-1}$ $(x,y\in G$, see [4(k), §5]). Let Ω_0 and G_0 be two open sets in G and f a C^∞-function on $\Omega=\Omega_0^{G_0}$. Put $f(x:y)=f(y^x)$ $(x\in G_0,\ y\in\Omega_0)$. The significance of the mapping Γ_y arises from the following lemma.

Lemma 4. *Let $g_1,g_2\in\mathfrak{G}$. Then*

$$f(x;g_1:y;g_2)=f(x:y;\Gamma_y(g_1\otimes g_2))$$

for $x\in G_0$ and $y\in\Omega_0$.

(3) We use here the notation of [4(k), §2].

(4) For any $x\in G$, we extend $\mathrm{Ad}(x)$ to an automorphism $g\to g^x$ of $\mathfrak{G}$ and define $y^x=xyx^{-1}$ $(y\in G)$.

If $X \in \mathfrak{g}$, it is clear that

$$y^{\exp tX} = y \exp(tX^{y^{-1}})\exp(-tX) \qquad (y \in G,\ t \in \boldsymbol{R}).$$

On the other hand if $x \in G_0$ and $y \in \Omega_0$, it is clear that

$$f(x \exp tX : y) = f(x : y^{\exp tX})$$

provided $|t|$ is small. Therefore

$$f(x; X : y) = f(x : y; X^{y^{-1}} - X) = f(x : y; \rho(X) - X).$$

Since $\rho(X)$ commutes with g_2, it follows by differentiation with respect to y that

$$\begin{aligned} f(x; X : y; g_2) &= f(x : y; \rho(X) \circ g_2 - g_2 X) \\ &= f(x : y; X^{y^{-1}} g_2 - g_2 X) = f(x : y; \sigma_y(X) g_2). \end{aligned}$$

Hence if $X_1, \cdots, X_r \in \mathfrak{g}$ and $g_1 = X_1 X_2 \cdots X_r$, it follows by induction on r that

$$f(x; g_1 : y; g_2) = f(x : y; \sigma_y(g_1) g_2).$$

The statement of the lemma is now obvious.

3. **Completely invariant sets.** Assume that G is connected. Consider the polynomial

$$\det(t + 1 - \mathrm{Ad}(x)) = \sum_{0 \leq j \leq n} t^j D_j(x) \qquad (x \in G),$$

where t is an indeterminate and $n = \dim G$. Then D_j are analytic functions on G and $D_n = 1$. Let l be the least integer such that $D_l \neq 0$. Then $l = \operatorname{rank} G = \operatorname{rank} \mathfrak{g}$ and an element $x \in G$ is called singular or regular according as $D_l(x) = 0$ or not. Let G' be the set of all regular elements. Then it is obvious that G' is open and dense in G and the set of singular elements is of measure zero with respect to the left-invariant Haar measure of G.

We say that G is reductive if $\mathfrak{g}$ is reductive. An element $x \in G$ is called semisimple if the endomorphism $\mathrm{Ad}(x)$ of $\mathfrak{g}$ is semisimple. Let $\mathfrak{z}_x$ denote the centralizer of x in $\mathfrak{g}$. We assume, from now on, that G is reductive.

LEMMA 5. *Let x be an element of G. Then $x \in G'$ if and only if $\mathfrak{z}_x$ is a Cartan subalgebra of* $\mathfrak{g}$. *Moreover, if x is semisimple then $\mathfrak{z}_x$ is reductive in* $\mathfrak{g}$ *and* $\operatorname{rank} \mathfrak{z}_x = \operatorname{rank} \mathfrak{g}$. *Finally, a regular element is always semisimple.*

It is clearly enough to consider the case when $\mathfrak{g}$ is semisimple. The first and last statements follow from [4(e), Lemma 5]. Put $B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$ $(X, Y \in \mathfrak{g})$ and let B_x denote the restriction of the bilinear form B on $\mathfrak{z}_x$. Now assume that x is semisimple. An elementary argument (see [2, p. 391]) shows that B_x is nondegenerate. Hence it follows from [2, Proposition 3.4] and [3] that $\mathfrak{z}_x$ is reductive in $\mathfrak{g}$. Finally rank $\mathfrak{z}_x = \operatorname{rank} \mathfrak{g}$ from [2, Proposition 4.6].

COROLLARY. *If x is semisimple, it is contained in a Cartan subgroup of G.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}_x$ and A the centralizer of $\mathfrak{h}$ in G. Since rank $\mathfrak{z}_x = \operatorname{rank} \mathfrak{g}$, A is a Cartan subgroup of G and $x \in A$.

Let $\mathcal{N}$ denote the set of all nilpotent elements (see [4(n), §3]) of $\mathfrak{g}$. Put $\mathcal{N}_G = \exp \mathcal{N} \subset G$. The mapping $X \to \exp \operatorname{ad} X$ $(X \in \mathcal{N})$ is known (see [4(h), §3]) to be univalent on $\mathcal{N}$.

LEMMA 6. *Every $x \in G$ can be written uniquely in the form $x = hn$, where h is a semisimple element of G, $n \in \mathcal{N}_G$ and h, n commute with each other. Let Z_x denote the centralizer of x in G. Then h and n lie in the center of Z_x.*

It is obviously enough to consider the case when $\mathfrak{g}$ is semisimple and G is the connected component of 1 in the adjoint group G_0 of $\mathfrak{g}$. Then G_0 is the set of all real points of a linear algebraic group defined over $\boldsymbol{R}$. Therefore the lemma follows from well-known results on algebraic groups (see, for example, [1, §8]). h and n, respectively, are called the semisimple and unipotent components of x.

COROLLARY(5). $h \in \mathrm{Cl}(x^G)$.

Choose $X \in \mathcal{N}$ such that $n = \exp X$ and let $\mathfrak{z}$ denote the centralizer of h in $\mathfrak{g}$. Then $X \in \mathfrak{z}$. We may obviously assume that $X \neq 0$. Then by the Jacobson-Morosow theorem, we can choose $H \in \mathfrak{z}$ such that $[H, X] = 2X$ (see [4(n), §3]). Put $y_t = \exp(-tH)$. Then

$$x^{y_t} = (h \exp X)^{y_t} = h \exp(e^{-2t} X) \to h$$

as $t \to +\infty$. This proves the corollary.

Let U be a subset of G. We say that U is completely invariant (cf. [4(n), §3]) if it has the following property. If C is any compact subset of U, then $\mathrm{Cl}(C^G) \subset U$.

LEMMA 7. *Let U be a completely invariant subset of G and V an invariant subset of U which is closed in U. Then if V contains no semisimple element of U, V is empty.*

Suppose $x \in V$. Then it follows from the corollary of Lemma 6 that the semisimple component of x also lies in V. Hence the lemma.

Now assume that $\mathfrak{g}$ is semisimple. For any $c > 0$, let $\mathfrak{g}(c)$ denote the set of all $X \in \mathfrak{g}$ such that(6) $|\operatorname{Im} \lambda| < c$ for every eigenvalue λ of $\operatorname{ad} X$. Clearly $\mathfrak{g}(c)$ is an open and completely invariant neighborhood of zero in $\mathfrak{g}$ and $\mathcal{N} \subset \mathfrak{g}(c)$. Moreover, if $X \in \mathfrak{g}(c)$ then $tX \in \mathfrak{g}(c)$ for $0 \leqq t \leqq 1$. Hence $\mathfrak{g}(c)$ is connected.

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(5) As usual $\mathrm{Cl}\, X$ and ${}^cX$, respectively, denote the closure and the complement of a subset X,

(6) Im λ denotes, as usual, the imaginary part of λ.

LEMMA 8. *Assume that $c \leqq \pi$. Then the exponential mapping from $\mathfrak{g}$ into G is everywhere regular and univalent on $\mathfrak{g}(c)$. Moreover,* $\exp \mathfrak{g}(c)$ *is completely invariant in G.*

The proof of the first part is the same as that of [4(h), Lemma 11]. In order to obtain the second part we use the notation of [4(h), Lemma 12] and first prove the following lemma.

LEMMA 9. *Assume $c \leqq \pi$ and let X_r $(r \geqq 1)$ be a sequence in $\mathfrak{g}(c)$. Then if $\|\exp X_r\|$ remains bounded, the same holds for $\|X_r\|$.*

We keep to the notation of the proof of [4(h), Lemma 12]. Then $X_r = \mathrm{Ad}(u_r)Y_r$, $Y_r = H_r + Z_r$ and therefore $|\mathrm{Im}\,\alpha(H_r)| < c \leqq \pi$ for any $\alpha \in P$. On the other hand $\|\exp X_r\| = \|\exp Y_r\|$ and $\mathrm{ad}\,Y_r$ has the same eigenvalues as $\mathrm{ad}\,H_r$. This shows that $|e^{\alpha(H_r)}|$ remains bounded for every $\alpha \in P$. In view of the above result this implies that $\|H_r\|$ itself remains bounded. The rest of the proof now goes through in the same way as for Lemma 12 of [4(h)].

Now fix a compact set C in $V = \exp \mathfrak{g}(c)$. We have to show that $\mathrm{Cl}(C^G) \subset V$. Let X_k and x_k $(k \geqq 1)$ be sequences in $\mathfrak{g}(c)$ and G respectively such that $\exp X_k \in C$ and $(\exp X_k)^{x_k}$ converges to some point y in G. We have to verify that $y \in V$. Let log denote the inverse mapping from V to $\mathfrak{g}(c)$. Then $\log C$ is compact. Hence, in view of Lemma 9, we can, by choosing suitable subsequences, arrange that $X_k \to X$ and $x_k X_k \to Y$, where $X \in \log C$ and $Y \in \mathfrak{g}$. But it is obvious that $\mathrm{ad}\,X$ and $\mathrm{ad}\,Y$ have the same eigenvalues. Hence $Y \in \mathfrak{g}(c)$ and therefore $y = \exp Y \in V$. This completes the proof of Lemma 8.

4. **Some algebraic results**(7). We return again to the case when $\mathfrak{g}$ is reductive. Fix a semisimple element $a \in G$ and let $\mathfrak{z} = \mathfrak{z}(a)$ denote the centralizer of a in $\mathfrak{g}$ and $\Xi = \Xi(a)$ the analytic subgroup of G corresponding to $\mathfrak{z}$. Put

$$\nu_a(y) = \det(\mathrm{Ad}(ay)^{-1} - 1)_{\mathfrak{g}/\mathfrak{z}} \quad (y \in \Xi).$$

Then ν_a is an analytic function on Ξ and $\nu_a(1) \neq 0$. Let $\Xi' = \Xi'(a)$ be the set of all points $y \in \Xi$, where $\nu_a(y) \neq 0$. Then Ξ' is an open neighborhood of 1 in Ξ.

In view of Lemma 5, $\mathfrak{z}$ satisfies the conditions of [4(l), §2]. Define $\mathfrak{q}$ as in [4(l), §2] and put $\mathfrak{Q} = \mathfrak{S}(\mathfrak{q}_c)$ and $\mathfrak{Q}_+ = \mathfrak{S}_+(\mathfrak{q}_c)$ in the notation of [4(k), §7].

LEMMA 10. *Fix $y \in \Xi'$. Then Γ_{ay} defines a bijective mapping of $\mathfrak{Q} \otimes \mathfrak{S}(\mathfrak{z}_c)$ onto $\mathfrak{G}$. Moreover,*

$$\sum_{d_1 + d_2 \leqq d} \Gamma_{ay}(\mathfrak{S}_{d_1}(\mathfrak{q}_c) \otimes \mathfrak{S}_{d_2}(\mathfrak{z}_c)) = {}_d\mathfrak{G} \qquad (d \geqq 0).$$

Put $W_d = \sum_{d_1 + d_2 \leqq d} \mathfrak{S}_{d_1}(\mathfrak{q}_c) \otimes \mathfrak{S}_{d_2}(\mathfrak{z}_c)$. Since $\mathfrak{g}$ is the direct sum of $\mathfrak{q}$ and $\mathfrak{z}$, it is clear that $\dim W_d = \dim {}_d\mathfrak{G}$. Hence it would be sufficient to prove that

(7) The results of this section are similar to those of [4(l), §2]. See also [4(e)].

$\Gamma_{ay}(W_d) = {}_d\mathfrak{G}$. We do this by induction on d. It is obvious that $\Gamma_{ay}(W_d) \subset {}_d\mathfrak{G}$. Hence it would be sufficent to show that

$$_d\mathfrak{G} \subset \Gamma_{ay}(W_d) + {}_{(d-1)}\mathfrak{G}.$$

Fix two integers $d_1, d_2 \geqq 0$ such that $d_1 + d_2 = d$ and suppose $Y_i \in \mathfrak{q}$ $(1 \leqq i \leqq d_1)$ and $Z_j \in \mathfrak{z}$ $(1 \leqq j \leqq d_2)$. Let $q = Y_1 Y_2 \cdots Y_{d_1} \in S(\mathfrak{q}_c)$ and $z = Z_1 Z_2 \cdots Z_{d_2} \in S(\mathfrak{z}_c)$. (Here we have to take $q = 1$ if $d_1 = 0$ and $z = 1$ if $d_2 = 0$.) Define λ as in §2. Then it would be enough to verify that

$$\lambda(qz) \in \Gamma_{ay}(W_d) + {}_{(d-1)}\mathfrak{G}.$$

If $d_1 = 0$, this is obvious since $\Gamma_{ay}(1 \otimes \lambda(z)) = \lambda(z)$. Hence we may assume $d_1 > 0$. Now ay commutes with a and therefore $\mathfrak{z}^{ay} = \mathfrak{z}$ and $\mathfrak{q}^{ay} = \mathfrak{q}$ (see [4(l), §2]). Therefore since $v_a(y) \neq 0$, we can choose $Y_i' \in \mathfrak{q}$ such that $(\mathrm{Ad}(ay)^{-1} - 1)Y_i' = Y_i$ $(1 \leqq i \leqq d_1)$. Put $q' = Y_1' \cdots Y_r' \in S(\mathfrak{q}_c)$. Then

$$\Gamma_{ay}(\lambda(q') \otimes \lambda(z)) \equiv \lambda(qz) \mod {}_{(d-1)}\mathfrak{G}$$

from Lemma 2. This proves the lemma.

COROLLARY 1. *Fix $g \in \mathfrak{G}$. Then for any $y \in \Xi'$, there exist unique elements $\alpha_y(g) \in \mathfrak{S}(\mathfrak{z}_c)$ and $\beta_y(g) \in \mathfrak{Q}_+ \otimes \mathfrak{S}(\mathfrak{z}_c)$ such that*

$$g = \alpha_y(g) + \Gamma_{ay}(\beta_y(g)).$$

Moreover, if $g \in {}_d\mathfrak{G}$, then $d^0\alpha_y(g) \leqq d$ and

$$\beta_y(g) \in \sum_{d_2 \geqq 0} \sum_{1 \leqq d_1 \leqq d - d_2} \mathfrak{S}_{d_1}(\mathfrak{q}_c) \otimes \mathfrak{S}_{d_2}(\mathfrak{z}_c).$$

This is obvious from Lemma 10.

Let M be an analytic manifold and f a mapping of M into a complex vector space V. We say that f is analytic if the subspace U of V spanned by the image $f(M)$ is of finite dimension and f, viewed as a mapping of M into U, is analytic in the usual sense.

COROLLARY 2. *Given $g \in \mathfrak{G}$, we can choose an integer $r \geqq 0$ such that the mappings $y \to v_a(y)^r \alpha_y(g)$ and $y \to v_a(y)^r \beta_y(g)$ $(y \in \Xi')$ can be extended to analytic mappings of Ξ into $\mathfrak{S}(\mathfrak{z}_c)$ and $\mathfrak{Q}_+ \otimes \mathfrak{S}(\mathfrak{z}_c)$, respectively.*

Let $d = d^0 g$. If $d = 0$ our statement is obvious. So we assume that $d \geqq 1$ and use induction. We may obviously assume that $g = \lambda(qz)$ in the notation of the proof of Lemma 10. Let $A(y)$ denote the restriction of $(\mathrm{Ad}(ay)^{-1} - 1)$ on $\mathfrak{q}$ $(y \in \Xi)$. Then if t is an indeterminate,

$$\det(t - A(y)) = \sum_{0 \leqq k \leqq m} D_k(y) t^k.$$

Here $m = \dim \mathfrak{q}$, D_k $(0 \leqq k \leqq m)$ are analytic functions on Ξ, $D_m = 1$ and

$$D_0(y) = (-1)^m \det A(y) = (-1)^m \nu_a(y).$$

Therefore

$$\nu_a(y) = B(y)A(y) = A(y)B(y),$$

where

$$B(y) = (-1)^{m+1} \sum_{0 \leqq k < m} D_{k+1}(y) A(y)^k.$$

Put $Y_i(y) = B(y)Y_i$ $(1 \leqq i \leqq d_1)$ and $q(y) = \prod_{1 \leqq i \leqq d_1} Y_i(y) \in S(\mathfrak{q}_c)$ $(y \in \Xi)$. Then it follows from Lemma 2 that

$$v(y) = \Gamma_{ay}(\lambda(q(y)) \otimes z) - \nu_a(y)^{d_1}\lambda(qz) \in {}_{(d-1)}\mathfrak{G}.$$

Therefore

$$v(y) = \sum_{1 \leqq i \leqq r} a_i(y) v_i$$

where a_i are analytic functions on Ξ and $d^0 v_i < d$. The required result now follows immediately by applying the induction hypothesis to v_i $(1 \leqq i \leqq r)$.

COROLLARY 3. *Let Z_a denote the centralizer of a in G. Then if $x \in Z_a$, $y \in \Xi'$ and $g \in \mathfrak{G}$, we have*

$$\alpha_{xyx^{-1}}(g^x) = (\alpha_y(g))^x.$$

This follows immediately from Lemma 1.

5. **The mapping** $\delta_{a,G/\Xi}$. We keep to the notation of §4. Let U_G be an open neighborhood of a in G. Put $U_\Xi = \Xi' \cap (a^{-1}U_G)$. Then U_Ξ is an open neighborhood of 1 in Ξ. For any differential operator D on U_G, we define a differential operator $\Delta(D)$ on U_Ξ as follows:

$$(\Delta(D))_y = \alpha_y(D_{ay}) \qquad (y \in U_\Xi).$$

Here D_{ay} and $(\Delta(D))_y$ denote, as usual, the local expressions (see [4(e), p. 112]) of D at ay and $\Delta(D)$ at y, respectively. Corollary 2 of Lemma 10 insures that there does exist a differential operator $\Delta(D)$ on U_Ξ satisfying the above relation and it is analytic if D is analytic. Finally, if we assume that U_G and D are invariant under G (see [4(j), §2]), it follows from Corollary 3 of Lemma 10 that U_Ξ and $\Delta(D)$ are invariant under Z_a. We shall denote the mapping $D \to \Delta(D)$ by δ_a or, if necessary, by $\delta_{a,G/\Xi}$.

Let b be an element of U_Ξ which is regular in Ξ and let $\mathfrak{h}$ denote the centralizer of b in $\mathfrak{z}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{z}$ and therefore also of $\mathfrak{g}$ (see Lemma 5). Let $A_\mathfrak{h}$ denote the Cartan subgroup of G corresponding to $\mathfrak{h}$ (see [4(m), §5]). Then a, b are in $A_\mathfrak{h}$. Let A be the connected component of 1 in $A_\mathfrak{h}$.

LEMMA 11. *The following two conditions on an element c of A are mutually equivalent.*

(1) $c \in b^{-1}U_\Xi$ *and* $\det(\mathrm{Ad}(bc)^{-1} - 1)_{\mathfrak{z}/\mathfrak{h}} \neq 0$.

(2) $c \in (ab)^{-} U_G$ *and* $\det(\mathrm{Ad}(abc)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \neq 0$.

Since $\mathrm{Ad}(a) = 1$ on $\mathfrak{z}$, it is clear that

$$\det(\mathrm{Ad}(abc)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} = \det(\mathrm{Ad}(abc)^{-1} - 1)_{\mathfrak{g}/\mathfrak{z}} \det(\mathrm{Ad}(bc)^{-1} - 1)_{\mathfrak{z}/\mathfrak{h}}$$

for $c \in A$. Now suppose (1) holds. Then $bc \in U_\Xi$ and therefore $abc \in U_G$ and $\det(\mathrm{Ad}(abc)^{-1} - 1)_{\mathfrak{g}/\mathfrak{z}} \neq 0$. Therefore (1) implies (2). Conversely, assume that (2) holds. Then $bc \in a^{-1} U_G$ and

$$\nu_a(bc) \det(\mathrm{Ad}(bc)^{-1} - 1)_{\mathfrak{z}/\mathfrak{h}} \neq 0.$$

Hence $bc \in U_\Xi$ and (1) holds.

Corollary. *ab is regular in G and $\mathfrak{h}$ is the centralizer of ab in $\mathfrak{g}$.*

Take $c = 1$ in Lemma 11. Then condition (1) obviously holds and therefore $\det(\mathrm{Ad}(ab)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \neq 0$ by (2). Since $\mathrm{Ad}(ab) = 1$ on $\mathfrak{h}$, it follows that $\mathfrak{h}$ is the centralizer of ab in $\mathfrak{g}$. Therefore ab is regular in G by Lemma 5.

Let U_A be the set of all $c \in A$ satisfying the conditions of Lemma 11.

Lemma 12. *Let D be a differential operator on U_G. Then*(8)

$$\delta_{ab,G/A}(D) = \delta_{b,\Xi/A}(\delta_{a,G/\Xi}(D)).$$

It follows from Lemma 11 that both sides are differential operators on U_A. Let $\Delta_1 = \delta_{a,G/\Xi}(D)$, $\Delta_2 = \delta_{b,\Xi/A}(\Delta_1)$ and $\Delta = \delta_{ab,G/A}(D)$. We have to prove that $\Delta_2 = \Delta$. Let $\mathfrak{m} = [\mathfrak{h}, \mathfrak{z}]$ and $\mathfrak{p} = \mathfrak{q} + \mathfrak{m}$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and $\mathfrak{z} = \mathfrak{h} + \mathfrak{m}$ where both sums are direct.

Fix $h \in U_A$. Then

$$(\Delta_2)_h - (\Delta_1)_{bh} \in \Gamma_{bh}(\mathfrak{S}_+(\mathfrak{m}_c) \otimes \mathfrak{S}(\mathfrak{h}_c)).$$

On the other hand $bh \in U_\Xi$ and therefore

$$(\Delta_1)_{bh} - D_{abh} \in \Gamma_{abh}(\mathfrak{S}_+(\mathfrak{q}_c) \otimes \mathfrak{S}(\mathfrak{z}_c)).$$

Since $\mathrm{Ad}(a) = 1$ on $\mathfrak{z}$, it is clear that

$$\sigma_{abh}(z) = \sigma_{bh}(z) \quad (z \in \mathfrak{S}(\mathfrak{z}_c))$$

and therefore

$$(\Delta_2)_h - D_{abh} \in \Gamma_{abh}(\mathfrak{G}_+ \otimes \mathfrak{S}(\mathfrak{z}_c)),$$

where $\mathfrak{G}_+ = \mathfrak{S}_+(\mathfrak{g}_c)$. But

$$\Gamma_{abh}(\mathfrak{S}(\mathfrak{m}_c) \otimes \mathfrak{S}(\mathfrak{h}_c)) = \Gamma_{bh}(\mathfrak{S}(\mathfrak{m}_c) \otimes \mathfrak{S}(\mathfrak{h}_c)) = \mathfrak{S}(\mathfrak{z}_c)$$

from Lemma 10 (applied to (Ξ, b) instead of (G, a)), since $\det(\mathrm{Ad}(bh)^{-1} - 1)_{\mathfrak{z}/\mathfrak{h}} \neq 0$. Hence

$$\begin{aligned} \Gamma_{abh}(\mathfrak{G}_+ \otimes \mathfrak{S}(\mathfrak{z}_c)) &= \sigma_{abh}(\mathfrak{G}_+)\mathfrak{S}(\mathfrak{z}_c) \\ &= \sigma_{abh}(\mathfrak{G}_+)\sigma_{abh}(\mathfrak{S}(\mathfrak{m}_c))\mathfrak{S}(\mathfrak{h}_c) \\ &= \sigma_{abh}(\mathfrak{G}_+)\mathfrak{S}(\mathfrak{h}_c). \end{aligned}$$

(8) Cf. [4(I), Lemma 11].

But $\mathfrak{G} = \mathfrak{S}(\mathfrak{p}_c)\mathfrak{S}(\mathfrak{h}_c)$ and since $\mathfrak{h}$ is abelian, it is clear that

$$\sigma_{abh}(\mathfrak{S}_+(\mathfrak{h}_c))\mathfrak{S}(\mathfrak{h}_c) = \{0\}.$$

Therefore since

$$\mathfrak{G}_+ = \mathfrak{S}_+(\mathfrak{p}_c) + \mathfrak{S}(\mathfrak{p}_c)\mathfrak{S}_+(\mathfrak{h}_c),$$

we conclude that

$$\Gamma_{abh}(\mathfrak{G}_+ \otimes \mathfrak{S}(\mathfrak{z}_c)) = \Gamma_{abh}(\mathfrak{S}_+(\mathfrak{p}_c) \otimes \mathfrak{S}(\mathfrak{h}_c)).$$

This shows that

$$(\Delta_2)_h - D_{abh} \in \Gamma_{abh}(\mathfrak{S}_+(\mathfrak{p}_c) \otimes \mathfrak{S}(\mathfrak{h}_c))$$

and therefore $(\Delta_2)_h = \Delta_h$ from the definition of Δ.

6. **The case when a is regular.** Let $\mathfrak{Z}$ be the algebra of all differential operators on G which are invariant under both left and right translations of G. It is obvious that $\mathfrak{Z}$ consists of the center of $\mathfrak{G}$ and therefore $\mathfrak{Z}$ is abelian.

Let G' be the set of all regular elements of G. Fix $a \in G'$ and let $\mathfrak{h}$ denote the centralizer of a in $\mathfrak{g}$ and A the analytic subgroup of G corresponding to $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and

$$\nu_a(h) = \det(\mathrm{Ad}(ah)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \qquad (h \in A).$$

Hence $A' = A \cap (a^{-1}G')$ is the set of all points $h \in A$ where $\nu_a(h) \neq 0$. Let W be the Weyl group of $(\mathfrak{g}_c, \mathfrak{h}_c)$. Then W operates on $\mathfrak{S}(\mathfrak{h}_c)$. Let $I(\mathfrak{h}_c)$ be the algebra of all invariants of W in $\mathfrak{S}(\mathfrak{h}_c)$. We have a canonical isomorphism γ of $\mathfrak{Z}$ onto $I(\mathfrak{h}_c)$ (see [4(e), Lemma 19]). Thus for every $z \in \mathfrak{Z}$, $\gamma(z)$ is a differential operator on A which is invariant under the translations of A.

LEMMA 13. *$\delta_{a,G/A}(z) = |\nu_a|^{-1/2}\gamma(z) \circ |\nu_a|^{1/2}$ on A' for any $z \in \mathfrak{Z}$.*

This is substantially the same as the first statement of [4(e), Theorem 2, p. 125].

7. **Application to invariant distributions**(9). Fix a semisimple element $a \in G$ and define Ξ and Ξ' as in §4.

LEMMA 14. *Consider the mapping ϕ: $(x, y) \to (ay)^x$ of $G \times \Xi$ into G. Then if $n = \dim G$, ϕ is everywhere of* rank *n on $G \times \Xi'$.*

We identify the tangent space of $G \times \Xi$ at a point (x, y) with $\mathfrak{g} \times \mathfrak{z}$ in the usual way. Then a simple calculation shows that

$$(d\phi)_{x,y}(X, Z) = (Z + (\mathrm{Ad}(ay)^{-1} - 1)X)^x$$

for $X \in \mathfrak{g}$ and $Z \in \mathfrak{z}$. But

$$\mathfrak{z} + (\mathrm{Ad}(ay)^{-1} - 1)\mathfrak{q} = \mathfrak{z} + \mathfrak{q} = \mathfrak{g}$$

if $y \in \Xi'$ and therefore our assertion is obvious.

(9) The results of this section are similar to those of [4(l), §7].

Let dx denote the Haar measure on G. We orient G and fix a left-invariant differential form $\omega_G > 0$ of degree n on G, corresponding to the measure dx. Then the set up of [4(k), §5] is applicable to $M = G$, if we define $y^x = xyx^{-1}$ $(x, y \in G)$ as above.

Let U be an open neighborhood of 1 in Ξ' which is invariant under Ξ (i.e., $U^y = U$ for $y \in \Xi$). Put $\Omega = \phi(G \times U) = (aU)^G$. Then by Lemma 14, Ω is open in G. Let dx, dy denote the Haar measures on G and Ξ, respectively. Now take $M = G \times U$, $N = \Omega$, $\pi = \phi$ in Theorem 1 of [4(k)] and let ω_M and ω_N be the differential forms corresponding to the measures $dxdy$ and dx, respectively. Let $\alpha \to f_\alpha$ denote the corresponding mapping of $C_c^\infty(G \times U)$ onto $C_c^\infty(\Omega)$.

LEMMA 15. *Let T be an invariant distribution on Ω. Then there exists a unique distribution σ_T on U such that $T(f_\alpha) = \sigma_T(\beta_\alpha)$ $(\alpha \in C_c^\infty(G \times U))$, where*

$$\beta_\alpha(y) = \int \alpha(x:y)\,dx \qquad (y \in U).$$

Moreover, σ_T is invariant under Ξ and $\sigma_T = 0$ implies that $T = 0$.

Define $T'(\alpha) = T(f_\alpha)$ $(\alpha \in C_c^\infty(G \times U))$. Then (see [4(k), Lemma 5]) T' is a distribution on $G \times U$. Fix $x_0 \in G$ and let α_{x_0} denote the function $(x, y) \to \alpha(x_0 x : y)$ on $G \times U$. We claim that $T'(\alpha) = T'(\alpha_{x_0})$. For if $F \in C_c^\infty(\Omega)$, we have

$$\int f_{\alpha_{x_0}} F\,dx = \int \alpha_{x_0}(x:y)\,F((ay)^x)\,dxdy = \int \alpha(x:y)\,F^{x_0}((ay)^x)\,dxdy$$

$$= \int f_\alpha F^{x_0} dx = \int f_\alpha^{x_0^{-1}} \quad F\,dx.$$

Hence $f_{\alpha_{x_0}} = f_\alpha^{x_0^{-1}}$ and therefore $T'(\alpha_{x_0}) = T(f_{\alpha_{x_0}}) = T(f_\alpha) = T'(\alpha)$. Now fix $\beta \in C_c^\infty(U)$ and put $T_\beta'(\gamma) = T'(\gamma \times \beta)$ $(\gamma \in C_c^\infty(G))$. Then T_β' is a distribution on G which is invariant under the left translations of G. Hence $T_\beta' = c(\beta)$, where $c(\beta)$ is a constant (see [4(k), Lemmas 6 and 7]). Now select $\gamma_0 \in C_c^\infty(G)$ such that $\int \gamma_0\,dx = 1$. Then

$$c(\beta) = T_\beta'(\gamma_0) = T'(\gamma_0 \times \beta) \qquad (\beta \in C_c^\infty(U)).$$

This shows that the mapping $\beta \to c(\beta)$ is a distribution on U which we denote by σ_T. Then

$$T'(\gamma \times \beta) = \sigma_T(\beta) \int \gamma\,dx \qquad (\gamma \in C_c^\infty(G),\ \beta \in C_c^\infty(U))$$

and therefore we conclude from [4(k), Lemma 3] that $T'(\alpha) - \sigma_T(\beta_\alpha) = 0$ for $\alpha \in C_c^\infty(G)$. Since $\beta_\alpha = \beta$ for $\alpha = \gamma_0 \times \beta$, the mapping $\alpha \to \beta_\alpha$ of $C_c^\infty(G \times U)$ into $C_c^\infty(U)$ is surjective. Finally the mapping $\alpha \to f_\alpha$ of $C_c^\infty(G \times U)$ into $C_c^\infty(\Omega)$ is also surjective (see [4(k), Theorem 1]) and so all the statements of the lemma, except the invariance of σ_T under Ξ, are now obvious.

Fix $\xi \in \Xi$ and define $\alpha^{\xi}(x:y) = \alpha(x:y^{\xi^{-1}})$. Then we claim that $T'(\alpha) = T'(\alpha^{\xi})$ for $\alpha \in C_c^{\infty}(G \times U)$. This is seen as follows.

$$\int f_{\alpha^{\xi}} F\,dx = \int \alpha(x:y^{\xi^{-1}}) F((ay)^x)\,dxdy = \int \alpha(x:y) F((ay)^{x\xi})\,dxdy$$
$$= \int \alpha(x\xi^{-1}:y) F((ay)^x)\,dxdy$$

for any $F \in C_c^{\infty}(\Omega)$. Hence if $\alpha'(x:y) = \alpha(x\xi^{-1}:y)$, it is clear that $f_{\alpha^{\xi}} = f_{\alpha'}$. Therefore

$$T'(\alpha^{\xi}) = T'(\alpha') = \sigma_T(\beta_{\alpha'}).$$

But

$$\beta_{\alpha'}(y) = \int \alpha(x\xi^{-1}:y)\,dx = \beta_{\alpha}(y) \qquad (y \in U)$$

by the right-invariance of dx. Hence $T'(\alpha^{\xi}) = \sigma_T(\beta_{\alpha}) = T'(\alpha)$. On the other hand

$$\beta_{\alpha^{\xi}}(y) = \int \alpha(x:y^{\xi^{-1}})\,dx = \beta_{\alpha}(y^{\xi^{-1}}) \qquad (y \in U).$$

Therefore $\beta_{\alpha^{\xi}} = (\beta_{\alpha})^{\xi}$. Now for a given $\beta \in C_c^{\infty}(U)$, we can choose $\alpha \in C_c^{\infty}(G \times U)$ such that $\beta = \beta_{\alpha}$. Then

$$\sigma_T(\beta) = T'(\alpha) = T'(\alpha^{\xi}) = \sigma_T(\beta_{\alpha^{\xi}}) = \sigma_T(\beta^{\xi}).$$

This shows that σ_T is invariant under Ξ.

COROLLARY. *Let D be an invariant differential operator on Ω. Then $\sigma_{DT} = \Delta\sigma_T$, where $\Delta = \delta_a(D)$.*

It follows from Corollary 2 of Lemma 10 and the definition of Δ, that we can select $q_i \in \mathfrak{S}_+(\mathfrak{q}_c)$, $v_i \in \mathfrak{S}(\mathfrak{z}_c)$ and $a_i \in C^{\infty}(U)$ $(1 \leqq i \leqq r)$ such that

$$D_{ay} = \Delta_y + \sum_{1 \leqq i \leqq r} a_i(y)\Gamma_{ay}(q_i \otimes v_i) \qquad (y \in U).$$

Moreover, $\sigma_{DT}(\beta_{\alpha}) = T(D^* f_{\alpha})$ for $\alpha \in C_c^{\infty}(G \times U)$, where the star denotes the adjoint as usual. Fix $F \in C_c^{\infty}(\Omega)$. Then

$$\int D^* f_{\alpha} \cdot F\,dx = \int f_{\alpha} DF\,dx = \int \alpha(x:y) F((ay)^x; D)\,dxdy.$$

Put $F(x:u) = F(u^x)$ for any pair $(x,u) \in G \times G$ such that $u^x \in \Omega$. Then it is clear from Lemma 4 that

$$F((ay)^x; D) = F((ay)^x; D^x) = F(x:ay; D)$$
$$= F(x:ay; \Delta_y) + \sum_{1 \leqq i \leqq r} a_i(y) F(x; q_i : ay; v_i)$$

for $x \in G$ and $y \in U$. Put

$$\alpha_0(x:y) \;=\; \alpha(x:y;\Delta^*),$$
$$\alpha_i(x:y) \;=\; \alpha(x;q_i^*:y;(a_iv_i)^*) \qquad (1 \leqq i \leqq r).$$

Then

$$\int D^*f_\alpha \cdot F\,dx \;=\; \sum_{0\leqq i\leqq r} \int \alpha_i(x:y)\,F((ay)^x)\,dxdy$$
$$=\; \sum_{0\leqq i\leqq r} \int f_{\alpha_i} F\,dx.$$

This proves that

$$D^*f_\alpha \;=\; \sum_{0\leqq i\leqq r} f_{\alpha_i}$$

and therefore

$$T(D^*f_\alpha) \;=\; \sum_{0\leqq i\leqq r} \sigma_T(\beta_{\alpha_i}).$$

Now $\beta_{\alpha_0} = \Delta^*\beta_\alpha$ and if j denotes the distribution on G corresponding to the constant function 1, it is obvious that $q_ij = 0$ since $q_i \in \mathfrak{G}_+$. Hence it follows that $\beta_{\alpha_i} = 0$ $(1 \leqq i \leqq r)$ and therefore

$$T(D^*f_\alpha) \;=\; \sigma_T(\Delta^*\beta_\alpha).$$

This proves that $\sigma_{DT} = \Delta\sigma_T$.

For any $X \in \mathfrak{g}$, let $\tau_G(X)$ denote the vector-field on G defined by

$$\tau_G(X)f = \;(df^{\exp tX}/dt)_{t=0} \quad (f \in C^\infty(G)).$$

Let V be an open subset of G. Then the local invariance of a differential operator, a distribution or a C^∞-function on V is defined as in [4(k), §5]. Since $[\tau_G(X),\, \tau_G(Y)] = \tau_G([X,Y])$ $(X, Y \in \mathfrak{g})$, τ_G can be extended (uniquely) to a homomorphism of $\mathfrak{G}$ into the algebra of all differential operators on G.

Let G_0 and U_0 be open neighborhoods of 1 in G and Ξ', respectively, and put $\Omega_0 = (aU_0)^{G_0}$. Define the mapping $\alpha \to f_\alpha$ of $C_c^\infty(G_0 \times U_0)$ onto $C_c^\infty(\Omega_0)$ as above. Then the following result is proved in the same way as [4(l), Lemma 17] and [4(k), Theorem 3].

LEMMA 16. *Assume that G_0 is connected and T is a locally invariant distribution on Ω_0. Then there exists a unique distribution σ_T on U_0 such that $T(f_\alpha) = \sigma_T(\beta_\alpha)$ $(\alpha \in C_c^\infty(G_0 \times U_0))$, where*

$$\beta_\alpha(y) = \int \alpha(x:y)\,dx \qquad (y \in U_0).$$

Moreover, σ_T is locally invariant (with respect to Ξ) and $\sigma_T = 0$ implies that

$T_1 = 0$. *Finally,* $\sigma_{DT} = \delta_a(D)\sigma_T$ *for any locally invariant differential operator* D *on* Ω_0.

8. **Some preparation for Theorem 1.** Let G_0, Ω_0 and Ω be three open subsets of G such that $\Omega_0{}^{G_0} \subset \Omega$. For any $f \in C^\infty(\Omega)$, we write $f(x:y) = f(y^x)$ $(x \in G_0, y \in \Omega_0)$ as in §2.

LEMMA 17. *Let* $f \in C^\infty(\Omega)$. *Then*

$$f(x;g:y) = f(x:y;\tau_G(g^*))$$

for $x \in G_0$, $y \in \Omega_0$ *and* $g \in \mathfrak{G}$.

The proof is the same as that of [4(k), Lemma 11].

LEMMA 18. *Let* D *be a differential operator and* f *a locally invariant* C^∞-*function on an open subset* Ω *of* G. *Fix a semisimple element* $a \in \Omega$ *and define* $\Omega_\Xi = a^{-1}\Omega \cap \Xi'$ *in the notation of* §4. *Then*

$$f(ay;D) = f(ay;\delta_a(D)) \qquad (y \in \Omega_\Xi).$$

Fix $y_0 \in \Omega_\Xi$ and choose open neighborhoods G_0 and Ω_0 of 1 and ay_0, respectively, in G such that $\Omega_0^{G_0} \subset \Omega$. Put $\Delta = \delta_a(D)$. Then it follows from the definition of Δ that

$$D_{ay_0} - \Delta_{y_0} = \sum_{1 \leq i \leq r} \Gamma_{ay_0}(g_i \otimes v_i),$$

where $g_i \in \mathfrak{G}_+$ and $v_i \in \mathfrak{S}(\mathfrak{z}_c)$. Therefore we conclude from Lemma 4 that

$$f(ay_0\,;D_{ay_0} - \Delta_{y_0}) = \sum_{1 \leq i \leq r} f(1;g_i:ay_0;v_i).$$

But

$$f(1;g_i:x) = f(x;\tau_G(g_i^*)) = 0 \qquad (x \in \Omega_0)$$

from Lemma 17 since f is locally invariant and $g_i^* \in \mathfrak{G}_+$. Therefore $f(1;g_i:x;v_i) = 0$ for $x \in \Omega_0$ and hence

$$f(ay_0;D_{ay_0} - \Delta_{y_0}) = 0.$$

This proves the lemma.

LEMMA 19. *Let* D *and* Ω *be as above. Then the following two conditions on* D *are equivalent.*

(1) $\delta_a(D) = 0$ *for every regular element* a *in* Ω.

(2) *For any open subset* Ω_0 *of* Ω *and a locally invariant* C^∞-*function* f *on* Ω_0, $Df = 0$.

Suppose (1) holds and Ω_0 and f are given as in condition (2). Fix a regular element $a \in \Omega_0$. Then it follows from Lemma 18 that

$$f(a;D) = f(a;\delta_a(D)) = 0.$$

Therefore $Df = 0$ on $\Omega_0' = \Omega_0 \cap G'$. Since Ω_0' is obviously dense in Ω_0, we conclude that $Df = 0$.

Conversely, assume that (2) holds and fix $a \in \Omega \cap G'$. Let $\mathfrak{h}$ be the centralizer of a in $\mathfrak{g}$ and A the analytic subgroup of G corresponding to $\mathfrak{h}$. Put $\Omega_A = a^{-1}\Omega \cap A'$ where A' is the set of all $h \in A$ where

$$\nu_a(h) = \det(\mathrm{Ad}(ah)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \neq 0.$$

Then $\delta_a(D)$ is a differential operator on Ω_A. Let $x \to x^*$ denote the natural projection of G on $G^* = G/A$. Since A is abelian, $(ah)^x$ ($x \in G$, $h \in A$) depends only on x^* and so we may denote it by $(ah)^{x^*}$. It follows from Lemma 14 that the mapping $\psi: (x^*, h) \to (ah)^{x^*}$ of $G^* \times \Omega_A$ into G is everywhere regular. Fix a point $h_0 \in \Omega_A$. Then we can choose open neighborhoods G_0^* and U of 1^* and h_0 in G^* and Ω_A, respectively, such that $\Omega_0 = \psi(G_0^* \times U) \subset \Omega$ and ψ defines an analytic diffeomorphism of $G_0^* \times U$ onto the open neighborhood Ω of ah_0 in Ω. Fix $\beta \in C_\infty(U)$ and define $f \in C^\infty(\Omega_0)$ by $f(\psi(x^*, h)) = \beta(h)$ ($x^* \in G_0^*$, $h \in U$). Then it is obvious that f is locally invariant and therefore $Df = 0$ by (2). On the other hand we know from Lemma 18 that

$$f(ah_0; u) = f(ah_0; D) = 0,$$

where u is the local expression of $\delta_a(D)$ at h_0. Since $u \in \mathfrak{S}(\mathfrak{h}_c)$ and $f(ah) = \beta(h)$ ($h \in U$), it is obvious that $\beta(h_0; u) = 0$. This being true for every $\beta \in C^\infty(U)$, we conclude that $u = 0$. Since h_0 was an arbitrary point of Ω_A, this proves that $\delta_a(D) = 0$. Therefore (2) implies (1).

9. **First part of the proof of Theorem 1.** We shall now begin the proof of the following theorem (cf. [4(n), Theorem 5]).

THEOREM 1. *Let Ω be a completely invariant open set in G and D an analytic differential operator on Ω. Assume that:*

(1) *D is invariant under G,*

(2) *$\delta_a(D) = 0$ for every regular element $a \in \Omega$.*

Then $DT = 0$ for every invariant distribution T on Ω.

We use induction on $\dim G$. By replacing (U, V) in Lemma 7 with $(\Omega, \mathrm{Supp}\, DT)$, it becomes obvious that it would be enough to verify that no semisimple element of Ω lies in $\mathrm{Supp}\, DT$. Let Z denote the center of G. Fix a semisimple element a in Ω and first assume that $a \notin Z$. Put $\Omega_\Xi = a^{-1}\Omega \cap \Xi'$ in the notation of §4. Then it is obvious that Ω_Ξ is a completely invariant open neighborhood of 1 in Ξ. Corresponding to Lemma 15, we get an invariant distribution σ_T on Ω_Ξ. Moreover, $\sigma_{DT} = \Delta\sigma_T$, where $\Delta = \delta_a(D)$ (see the corollary of Lemma 15). Fix an element $b \in \Omega_\Xi$ which is regular in Ξ. Then $ab \in \Omega' = \Omega \cap G'$ (see the corollary of Lemma 11) and therefore

$$\delta_{b,\Xi/A}(\Delta) = \delta_{ab}(D) = 0$$

in the notation of Lemma 12. Moreover, as we have seen in §5, Δ is analytic and invariant under Ξ. Now $\dim \Xi < \dim G$, since $a \notin Z$. Therefore we conclude from the induction hypothesis that $\Delta\sigma_T = 0$. But then $a \notin \operatorname{Supp} DT$ by Lemma 15.

So now we may assume that $a \in \Omega \cap Z$. It follows from its definition (see §2) that the mapping Γ_x depends only on $\mathrm{Ad}(x)$ $(x \in G)$. Therefore if we apply the translation by a^{-1} to the whole problem, we are reduced to the case $a = 1$. So we may assume that $1 \in \Omega$ and it remains to show that $1 \notin \operatorname{Supp} DT$.

Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. Choose an open and relatively compact neighborhood $\mathfrak{c}_0$ of zero in $\mathfrak{c}$ such that the exponential mapping is univalent on $\mathfrak{c}_0$. Moreover, select a number c $(0 < c \leqq \pi)$ and define $\mathfrak{g}_1(c)$ as in Lemma 8. Put $\mathfrak{g}_0 = \mathfrak{c}_0 + \mathfrak{g}_1(c)$. Then $\mathfrak{g}_0$ is an open and completely invariant neighborhood of zero in $\mathfrak{g}$ and the exponential mapping is everywhere regular on $\mathfrak{g}_0$. Now suppose $\exp(C_1 + X_1) = \exp(C_2 + X_2)$, where $C_i \in \mathfrak{c}_0$ and $X_i \in \mathfrak{g}_1(c)$ $(i = 1,2)$. Then $\exp(\operatorname{ad} X_1) = \exp(\operatorname{ad} X_2)$ and so it follows from Lemma 8 that $X_1 = X_2$. Hence $\exp C_1 = \exp C_2$ and therefore $C_1 = C_2$ from the definition of $\mathfrak{c}_0$. This proves that the exponential mapping defines an analytic diffeomorphism of $\mathfrak{g}_0$ onto the open set $\exp \mathfrak{g}_0$ in G. Let log denote its inverse and put $U = \log \Omega_0$, where $\Omega_0 = \exp \mathfrak{g}_0 \cap \Omega$. Let V be a compact subset of U. Since $\mathfrak{g}_0$ is completely invariant, $\mathrm{Cl}(V^G) \subset \mathfrak{g}_0$. Moreover,

$$\exp(\mathrm{Cl}(V^G)) \subset \mathrm{Cl}(\exp V^G) = \mathrm{Cl}((\exp V)^G) \subset \Omega$$

since Ω is completely invariant. Hence it follows that $\mathrm{Cl}(V^G) \subset U$ and this shows that U is completely invariant.

Now, in order to complete the proof, we need some preparation which will be undertaken in the next section.

10. **Reduction to** $\mathfrak{g}$. Put

$$\xi(X) = \left|\det\{(e^{\operatorname{ad} X/2} - e^{-\operatorname{ad} X/2})/\operatorname{ad} X\}\right|^{1/2} \qquad (X \in \mathfrak{g}).$$

Then ξ is analytic around every point $X_0 \in \mathfrak{g}$, where $\xi(X_0) \neq 0$. Moreover, the exponential mapping of $\mathfrak{g}$ into G is regular at X_0 if and only if $\xi(X_0) \neq 0$ (see, for example, [5, p. 95]).

Let U be an open subset of $\mathfrak{g}$ such that the exponential mapping is regular and univalent on U and put $U_G = \exp U$. Then U_G is open in G and the exponential mapping defines an analytic diffeomorphism of U onto U_G. For any function ϕ on U, let f_ϕ denote the function on U_G given by

$$f_\phi(\exp X) = \xi(X)^{-1}\phi(X) \qquad (X \in U).$$

Then f_ϕ is C^∞ or analytic if and only if the same holds for ϕ. In particular, $f \to f_\phi$ defines a linear topological mapping of $C_c^\infty(U)$ onto $C_c^\infty(U_G)$. Moreover, it is obvious that, for any differential operator D on U_G, there exists a unique differential operator $\Delta(D)$ on U such that $Df_\phi = f_{\Delta(D)\phi}$ for $\phi \in C^\infty(U)$. Finally, D is analytic if and only if $\Delta(D)$ is analytic.

As usual let dX denote the Euclidean measure on $\mathfrak{g}$ and dx the Haar measure on G. Then if dX is suitably normalized, we have the relation (see [5, p. 95])

$$dx = \xi(X)^2 dX \qquad (x = \exp X,\ X \in U).$$

Hence it follows that

$$\int \phi_1 \phi_2 \, dX = \int f_{\phi_1} f_{\phi_2} \, dx$$

for $\phi_1 \in C^\infty(U)$ and $\phi_2 \in C_c^\infty(U)$.

LEMMA 20. $\Delta(D^*) = \Delta(D)^*$ *for any differential operator D on U_G.*

Fix D and write Δ for $\Delta(D)$. Then if $\phi_1, \phi_2 \in C_c^\infty(U)$, we have

$$\begin{aligned}\int D^* f_{\phi_1} \cdot f_{\phi_2} \, dx &= \int f_{\phi_1} \cdot D f_{\phi_2} \, dx = \int f_{\phi_1} f_{\Delta\phi_2} \, dx \\ &= \int \phi_1 \cdot \Delta\phi_2 \, dX = \int \Delta^*\phi_1 \cdot \phi_2 \, dX \\ &= \int f_{\Delta^*\phi_1} \cdot f_{\phi_2} \, dx.\end{aligned}$$

This shows that $D^* f_{\phi_1} = f_{\Delta^*\phi_1}$ and from this our assertion follows immediately.

For any distribution T on U_G, let τ_T denote the distribution on U given by $\tau_T(\phi) = T(f_\phi)$ ($\phi \in C_c^\infty(U)$). Then it follows from Lemma 20 that $\tau_{DT} = \Delta(D)\tau_T$.

Now assume that U is invariant under G. Since $\exp(X^x) = (\exp X)^x$ ($x \in G, X \in \mathfrak{g}$), U_G is also invariant. Moreover, since ξ is obviously invariant under G, it is clear that $(f_\phi)^x = f_{\phi^x}$ and $\Delta(D^x) = (\Delta(D))^x$ ($x \in G$) for $\phi \in C^\infty(U)$ and any differential operator D on U. Similarly $\tau_{T^x} = (\tau_T)^x$.

11. **Completion of the proof of Theorem 1.** We are now ready to finish the proof of Theorem 1. Define U as in §9. Then $U_G = \exp U = \Omega_0$ and, corresponding to T, we get an invariant distribution τ_T on U. Since D is an invariant and analytic differential operator on U_G, $\Delta = \Delta(D)$ is also invariant and analytic on U. Let ϕ be any invariant C^∞-function on U. Then

$$f_{\Delta\phi} = Df_\phi = 0$$

from Lemma 19. Hence $\Delta\phi = 0$. However, since U is completely invariant (see §9), we conclude from [4(n), Theorem 5] that $\tau_{DT} = \Delta\tau_T = 0$. Obviously this implies that $DT = 0$ on $U_G = \Omega_0$ and therefore $1 \notin \operatorname{Supp} DT$. This completes the proof of Theorem 1.

12. **Two isomorphisms.** Let $\mathfrak{m}$ be a subalgebra of $\mathfrak{g}$ such that (1) $\mathfrak{m}$ is reductive in $\mathfrak{g}$ and (2) rank $\mathfrak{m}$ = rank $\mathfrak{g}$. As before, let $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{g})$ be the center of $\mathfrak{G} = \mathfrak{S}(\mathfrak{g}_c)$

and $\mathfrak{Z}(\mathfrak{m})$ the center of $\mathfrak{S}(\mathfrak{m}_c)$. We shall now define a homomorphism $\mu = \mu_{\mathfrak{g}/\mathfrak{m}}$ of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{m})$.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{m}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let W and $W(\mathfrak{m})$ denote the Weyl groups of $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{m},\mathfrak{h})$, respectively. Then $W(\mathfrak{m})$ is a subgroup of W. Let $I(\mathfrak{h}_c)$ and $I_{\mathfrak{m}}(\mathfrak{h}_c)$ denote the algebras of invariants of W and $W(\mathfrak{m})$, respectively, in $\mathfrak{S}(\mathfrak{h}_c)$. Then $I_{\mathfrak{m}}(\mathfrak{h}_c) \supset I(\mathfrak{h}_c)$. Let γ: $\mathfrak{z} \to I(\mathfrak{h}_c)$ and $\gamma_{\mathfrak{m}}$: $\mathfrak{Z}(\mathfrak{m}) \to I_{\mathfrak{m}}(\mathfrak{h}_c)$ denote the canonical isomorphisms (see [4(e), Lemma 19]). We define $\mu(z) = \gamma_{\mathfrak{m}}^{-1}(\gamma(z))$ $(z \in \mathfrak{Z})$. Since any two Cartan subalgebras of $\mathfrak{m}_c$ are conjugate under the connected complex adjoint group of $\mathfrak{m}_c$, it follows easily from [4(e), §6] that μ is independent of the choice of $\mathfrak{h}$.

Lemma 21. *$\mathfrak{Z}(\mathfrak{m})$ is a free abelian module over $\mu_{\mathfrak{g}/\mathfrak{m}}(\mathfrak{Z})$ of* rank $[W: W(\mathfrak{m})]$.

It is enough to show that $I_{\mathfrak{m}}(\mathfrak{h}_c)$ is a free abelian module over $I(\mathfrak{h}_c)$ of rank $[W: W(\mathfrak{m})]$. The proof of this is substantially the same as that of Lemma 8 of [4(i)].

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we can take $\mathfrak{m} = \mathfrak{h}$. Then it is clear that $\mu_{\mathfrak{g}/\mathfrak{h}} = \gamma$. As usual let $I(\mathfrak{g}_c)$ denote the algebra of all invariants of G in $S(\mathfrak{g}_c)$. Then we have the Chevalley isomorphism j: $p \to p_{\mathfrak{h}}$ of $I(\mathfrak{g}_c)$ onto(10) $I(\mathfrak{h}_c)$ (see [4(l), §9]). For any $z \in \mathfrak{Z}$, let p_z denote the element $j^{-1}(\gamma(z)) \in I(\mathfrak{g}_c)$. Then $z \to p_z$ is an isomorphism of $\mathfrak{Z}$ onto $I(\mathfrak{g}_c)$. It follows again from the results of [4(e), §6] that this isomorphism is independent of the choice of $\mathfrak{h}$. We shall call it the canonical isomorphism of $\mathfrak{Z}$ onto $I(\mathfrak{g}_c)$.

13. **A consequence of Theorem 1.** We use the notation of §5.

Lemma 22. *Let U_G be a completely invariant open set in G. Fix a semisimple element $a \in U_G$ and define $U_\Xi = \Xi' \cap (a^{-1} U_G)$ as in §5. Then U_Ξ is completely invariant under Ξ. Let σ be an invariant distribution on U_Ξ. Then*(11)

$$\delta_a(z)\sigma = |\nu_a|^{-1/2}\mu_{\mathfrak{g}/\mathfrak{z}}(z)(|\nu_a|^{1/2}\sigma)$$

for $z \in \mathfrak{Z}$.

It is obvious that U_Ξ is an open and completely invariant subset of Ξ. Therefore, in view of Theorem 1 and Lemma 19, it is enough to prove the following result.

Lemma 23. *Let V be an open subset of U_Ξ and f a C^∞-function on V which is locally invariant under Ξ. Then*

$$\delta_a(z)f = |\nu_a|^{-1/2}\mu_{\mathfrak{g}/\mathfrak{z}}(z)(|\nu_a|^{1/2}f) \qquad (z \in \mathfrak{Z}).$$

Let V' be the set of those elements of V which are regular in Ξ. Since V' is

(10) Since $\mathfrak{h}$ is abelian, we may identify $S(\mathfrak{h}_c)$ with $\mathfrak{S}(\mathfrak{h}_c)$ under the canonical mapping λ of $S(\mathfrak{g}_c)$ onto $\mathfrak{G}$.

(11) Cf. [4(l), Theorem 2].

dense in V, it is enough to verify that the above equation holds on V'. Fix $b \in V'$ and $z \in \mathfrak{Z}$. Then we have to show that

$$f(b; \delta_a(z)) = f(b; |v_a|^{-1/2}\mu(z) \circ |v_a|^{1/2}),$$

where $\mu = \mu_{\mathfrak{g}/\mathfrak{z}}$. Let $\mathfrak{h}$ be the centralizer of b in $\mathfrak{z}$ and A the analytic subgroup of G corresponding to $\mathfrak{h}$. Let A' denote the set of all points $h \in A$ where

$$\det(\mathrm{Ad}(bh)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \neq 0$$

and put $U_A = A' \cap b^{-1}U_\Xi$ and $V_A = (b^{-1}V) \cap U_A$. Then V_A is an open neighborhood of 1 in A'. Moreover,

$$\begin{aligned} f(bh; \delta_a(z)) &= f(bh; \delta_{b,\Xi/A}(\delta_a(z))) \\ &= f(bh; \delta_{ab}(z)) \qquad (h \in V_A) \end{aligned}$$

from Lemmas 18 and 12. But since ab is regular in G (see the corollary of Lemma 11), it follows from Lemma 13 that

$$\delta_{ab}(z) = |v_{ab}|^{-1/2}\gamma(z) \circ |v_{ab}|^{1/2}$$

on V_A. Therefore

$$f(bh; \delta_a(z)) = |v_{ab}(h)|^{-1/2} F(h; \gamma(z)),$$

where

$$F(h) = |v_{ab}(h)|^{1/2} f(bh) \qquad (h \in V_A).$$

Now put $f_1(y) = |v_a(y)|^{1/2} f(y)$ for $y \in V$. Then

$$\begin{aligned} f(bh; |v_a|^{-1/2}\mu(z) \circ |v_a|^{1/2}) &= |v_a(bh)|^{-1/2} f_1(bh; \mu(z)) \\ &= |v_a(bh)|^{-1/2} f_1(bh; \delta_{b,\Xi/A}(\mu(z))) \qquad (h \in V_A) \end{aligned}$$

from Lemma 18. On the other hand it follows from Lemma 13 (applied to Ξ) and the definition of $\mu(z)$ that

$$\delta_{b,\Xi/A}(\mu(z)) = |v_{b,\Xi}|^{-1/2}\gamma(z) \circ |v_{b,\Xi}|^{1/2}$$

on U_A, where

$$v_{b,\Xi}(h) = \det(\mathrm{Ad}(bh)^{-1} - 1)_{\mathfrak{z}/\mathfrak{h}} \qquad (h \in A).$$

Therefore

$$f_1(bh; \delta_{b,\Xi/A}(\mu(z))) = |v_{b,\Xi}(h)|^{-1/2} F_1(h; \gamma(z)),$$

where

$$F_1(h) = |v_{b,\Xi}(h)|^{1/2} f_1(bh) = |v_{b,\Xi}(h) v_a(bh)|^{1/2} f(bh) \qquad (h \in V_A).$$

But since

$$v_{b,\Xi}(h) v_a(bh) = v_{ab}(h),$$

we have $F = F_1$. This shows that

$$f(bh;|\nu_a|^{-1/2}\mu(z)\circ|\nu_a|^{1/2}) = |\nu_a(bh)\nu_{b,\Xi}(h)|^{-1/2}F_1(h;\gamma(z))$$
$$= |\nu_{ab}(h)|^{-1/2}F(h;\gamma(z)) = f(bh;\delta_a(z))$$

for $h \in V_A$. Putting $h = 1$, we get the required result.

14. **The relation between $\mathfrak{Z}$ and $\partial(I(\mathfrak{g}_c))$.** We now use the notation of §10. For any open subset V of U and a function ϕ on V, we define, as before, the function f_ϕ on $V_G = \exp V$ by $f_\phi(\exp X) = \xi(X)^{-1}\phi(X)$ $(X \in V)$. Let $z \to p_z$ denote the canonical isomorphism of $\mathfrak{Z}$ onto $I(\mathfrak{g}_c)$ (see §12).

LEMMA 24. *Let ϕ be a locally invariant C^∞-function on an open subset V of U. Then*

$$zf_\phi = f_{\partial(p_z)\phi}$$

for $z \in \mathfrak{Z}$.

Fix $z \in \mathfrak{Z}$ and let V' be the set of all regular points in V. Consider the differential operator $\Delta(z)$ on U corresponding to z (see §12). Then it is enough to prove that $\Delta(z)\phi = \partial(p_z)\phi$ on V'. Fix a point $H_0 \in V'$ and let $\mathfrak{h}$ denote the centralizer of H_0 in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}_0$ be an open and connected neighborhood of H_0 in $\mathfrak{h} \cap V'$. Then it would be sufficient to show that

$$\phi(H;\Delta(z)) = \phi(H;\partial(p_z)) \qquad (H \in \mathfrak{h}_0).$$

Since ϕ is locally invariant, it follows from [4(l), Lemma 14] and [4(f), Theorem 1] that

$$\phi(H;\partial(p_z)) = \phi(H;\delta_{\mathfrak{g}/\mathfrak{h}}'(\partial(p_z))) = \pi(H)^{-1}\phi(H;\partial(q)\circ\pi) \qquad (H \in \mathfrak{h}_0).$$

Here π denotes, as usual, the product of all the positive roots of $(\mathfrak{g},\mathfrak{h})$ and $q = (p_z)_{\mathfrak{h}}$ in the notation of [4(l), §8]. Let A be the analytic subgroup of G corresponding to $\mathfrak{h}$ and put $A' = A \cap G'$ and

$$\nu(h) = \det(\mathrm{Ad}(h)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \qquad (h \in A).$$

Since $\mathfrak{h}_0 \subset U$, it follows that $\xi(H) \neq 0$ and therefore $\exp H \in A'$ for $H \in \mathfrak{h}_0$. Moreover, it is clear that f_ϕ is locally invariant with respect to G. Therefore we conclude from Lemma 17 and [4(e), Theorem 2] that

$$f_\phi(\exp H;z) = |\nu(\exp H)|^{-1/2} f_\phi(\exp H;\gamma(z)\circ|\nu|^{1/2})$$

for $H \in \mathfrak{h}_0$. But it is obvious that

$$|\nu(\exp H)|^{1/2} = \xi(H)|\pi(H)|$$

and therefore

$$|\nu(\exp H)|^{1/2}f_\phi(\exp H) = |\pi(H)|\phi(H) \qquad (H \in \mathfrak{h}_0).$$

If r is the number of positive roots of $(\mathfrak{g},\mathfrak{h})$, we know that $\det(\mathrm{ad}H) = (-1)^r\pi(H)^2$ $(H \in \mathfrak{h})$. This shows that $\pi(H)^2$ is real. Therefore since $\mathfrak{h}_0$ is connected and π is nowhere zero on $\mathfrak{h}_0$, it follows that $|\pi(H)| = \varepsilon\pi(H)(H \in \mathfrak{h}_0)$, where $\varepsilon = |\pi(H_0)|/\pi(H_0)$ Moreover, $j(p_z) = \gamma(z)$ in the notation of §12. Hence it is clear that

$$\begin{aligned} f_\phi(\exp H; z) &= \pi(H)^{-1}\xi(H)^{-1}\phi(H; \partial(q) \circ \pi) \\ &= \xi(H)^{-1}\phi(H; \partial(p_z)) \qquad (H \in \phi_0). \end{aligned}$$

On the other hand

$$\phi(X; \Delta(z)) = \xi(X) f_\phi(\exp X; z) \qquad (X \in V)$$

from the definition of $\Delta(z)$. Therefore

$$\phi(H; \Delta(z)) = \phi(H; \partial(p_z)) \qquad (H \in \mathfrak{h}_0)$$

and this proves the lemma.

COROLLARY. *Assume that U is completely invariant. Then if T is an invariant distribution on U_G,*

$$\tau_{zT} = \partial(p_z)\tau_T \qquad (z \in \mathfrak{Z}).$$

We know (see §10) that $\tau_{zT} = \Delta(z)\tau_T$ and it follows from Lemma 24 and [4(n), Theorem 5] that $\Delta(z)\tau_T = \partial(p_z)\tau_T$. Hence the corollary.

15. **Proof of Theorem 2.** We now come to one of the main results of this paper (cf. [4(j), Theorem 1]).

THEOREM 2. *Let Ω be a completely invariant open set in G and T a distribution on Ω. We assume that:*

(1) T is invariant;

(2) there exists an ideal $\mathfrak{U}$ in $\mathfrak{Z}$ such that $\dim \mathfrak{Z}/\mathfrak{U} < \infty$ and $uT = 0$ for $u \in \mathfrak{U}$. Then T is a locally summable function which is analytic on $\Omega' = \Omega \cap G'$.

We shall use induction on $\dim G$. Let Ω_0 be the set of all points $a \in \Omega$ with the following property. There exists an open neighborhood U of a in Ω and a locally summable function F on U such that F is analytic on $U \cap G'$ and $T = F$ on U. Clearly Ω_0 is an open and invariant subset of Ω. It would be enough to prove that $\Omega_0 = \Omega$. But then, in view of Lemma 7, we have only to verify that Ω_0 contains all semisimple points of Ω.

LEMMA 25. $\Omega' \subset \Omega_0$.

Fix $a \in \Omega'$ and let $\mathfrak{h}$ denote the centralizer of a in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Consider the analytic subgroup A of G corresponding to $\mathfrak{h}$ and put $\Omega_A = a^{-1}\Omega \cap A'$, where A' is the set of all $h \in A$ such that

$$v_a(h) = \det(\mathrm{Ad}(ah)^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}} \neq 0.$$

Let σ_T denote the distribution on Ω_A corresponding to T under Lemma 15. Put $\sigma = |v_a|^{1/2}\sigma_T$. Then we conclude from Lemma 13 and the corollary of Lemma 15 that $\gamma(u)\sigma = 0$ for $u \in \mathfrak{U}$. Since $\mathfrak{S}(\mathfrak{h}_c)$ is a finite module over $\gamma(\mathfrak{Z}) = I(\mathfrak{h}_c)$ (Lemma 21), it follows that $\mathfrak{V} = \mathfrak{S}(\mathfrak{h}_c)\gamma(\mathfrak{U})$ has finite codimension in $\mathfrak{S}(\mathfrak{h}_c)$. Fix a base $H_1, \cdots, H_l$ for $\mathfrak{h}$ over $\boldsymbol{R}$ and put

$$\square = H_1^2 + \cdots + H_l^2 .$$

Then if $N = \dim \mathfrak{S}(\mathfrak{h}_c)/\mathfrak{V}$, it is obvious that we can choose $c_i \in \boldsymbol{C}$ $(1 \leqq i \leqq N)$ such that

$$v = \square^N + \sum_{1 \leqq k \leqq N} c_k \square^{N-k} \in \mathfrak{V}.$$

Now v is an analytic differential operator on A which is obviously elliptic. Therefore since $v\sigma = 0$, we conclude that σ coincides with an analytic function g on Ω_A. Put $G^* = G/A$ and define the mapping ψ: $G^* \times \Omega_A \to \Omega$ as in the proof of Lemma 19. Then we can choose open neighborhoods G_0 and V of 1 in G and Ω_A, respectively, such that ψ defines an analytic diffeomorphism of $G_0^* \times V$ onto the open subset $U = \psi(G_0^* \times V)$ of Ω. Define the analytic function F on U by

$$F((ah)^{x^*}) = |v_a(h)|^{-1/2} g(h) \qquad (x^* \in G_0^*,\ h \in V).$$

Then by Lemma 15, we get

$$T(f_\alpha) = \sigma_T(\beta_\alpha) = \int \beta_\alpha |v_a|^{-1/2} g\, dh \qquad (\alpha \in C_c^\infty(G_0 \times V)),$$

where dh is the Haar measure on A. On the other hand, it follows from the definition of f_α that

$$\begin{aligned} \int f_\alpha F\, dx &= \int \alpha(x:h) F((ah)^x) dx\, dh \\ &= \int \beta_\alpha |v_a|^{-1/2} g\, dh. \end{aligned}$$

This shows that $T = F$ on U and therefore $a \in \Omega_0$.

It is clear from the above lemma that there exists an analytic function F on Ω' such that $T = F$ on Ω'. Now fix a semisimple element $a \in \Omega$ and let us use the notation of §4. Z being the center of G, first assume that $a \notin Z$ so that $\dim \mathfrak{z} < \dim \mathfrak{g}$. Put $\Omega_\Xi = a^{-1}\Omega \cap \Xi'$. Then Ω_Ξ is an open and completely invariant neighborhood of 1 in Ξ. Let σ_T denote the distribution on Ω_Ξ which corresponds to T under Lemma 15. Then σ_T is invariant under Ξ and it follows from the corollary of Lemma 15 that $\delta_a(u)\sigma_T = 0$ for $u \in \mathfrak{U}$. But then we conclude from Lemma 22 that

$$\mu(u)\sigma = 0 \qquad (u \in \mathfrak{U}).$$

Here $\sigma = |v_a|^{1/2}\sigma_T$ and $\mu = \mu_{\mathfrak{g}/\mathfrak{z}}$. Since $\mathfrak{Z}(\mathfrak{z})$ is a finite module over $\mu(\mathfrak{Z})$ (Lemma 21), it is clear that $\mathfrak{V} = \mathfrak{Z}(\mathfrak{z})\mu(\mathfrak{U})$ has finite codimension in $\mathfrak{Z}(\mathfrak{z})$. Let Ω_Ξ' be the set of those elements in Ω_Ξ which are regular in Ξ. Then it follows by the induction hypothesis that $\sigma = g$, where g is a locally summable function on Ω_Ξ which is analytic on Ω_Ξ'.

Let ϕ denote the mapping $(x,y) \to (ay)^x$ of $G \times \Omega_\Xi$ into Ω. Then $U = \phi(G \times \Omega_\Xi)$ is an open neighborhood of a in Ω (Lemma 14). Moreover, it is easy to verify that $\phi(G \times \Omega_\Xi') = U'$, where $U' = U \cap G' = U \cap \Omega'$. Since $T = F$ on Ω', we have

$$T(f_\alpha) = \int f_\alpha F \, dx = \int \alpha(x:y) F((ay)^x) \, dxdy$$

for $\alpha \in C_c^\infty(G \times \Omega_\Xi')$ in the notation of Lemma 15. However,

$$T(f_\alpha) = \sigma_T(\beta_\alpha) = \int \alpha(x:y) \left| v_a(y) \right|^{-1/2} g(y) \, dxdy.$$

This shows that the analytic function

$$(x,y) \to F((ay)^x) - \left| v_a(y) \right|^{-1/2} g(y)$$

is zero on $G \times \Omega_\Xi'$ and therefore $F \circ \phi$ is locally summable on $G \times \Omega_\Xi$. Hence F is locally summable on U (see [4(k), Corollary 2 of Theorem 1]) and

$$\begin{aligned} \int f_\alpha F \, dx &= \int \alpha(x:y) F((ay)^x) \, dxdy \\ &= \int \alpha(x:y) \left| v_a(y) \right|^{-1/2} g(y) \, dxdy \\ &= \sigma_T(\beta_\alpha) = T(f_\alpha) \end{aligned}$$

for $\alpha \in C_c^\infty(G \times \Omega_\Xi)$. This proves that $T = F$ on U and therefore $a \in \Omega_0$.

It remains to consider the case when $a \in Z$. Then by a translation by a^{-1}, we are reduced to the case $a = 1$. Then, as we have seen in §9, there exists an open and completely invariant neighborhood U of zero in $\mathfrak{g}$ such that the exponential mapping of $\mathfrak{g}$ into G is univalent and regular on U and $U_G = \exp U \subset \Omega$. Let τ_T be the distribution on U corresponding to T (see §10). Then we know from the corollary of Lemma 24 that $\partial(p_u)\tau_T = 0$ for $u \in \mathfrak{U}$. Let $\mathfrak{V}$ denote the image of $\mathfrak{U}$ in $I(\mathfrak{g}_c)$ under the canonical isomorphism $z \to p_z$ of $\mathfrak{Z}$ onto $I(\mathfrak{g}_c)$. Then

$$\dim I(\mathfrak{g}_c)/\mathfrak{V} = \dim \mathfrak{Z}/\mathfrak{U} < \infty$$

and so we conclude from [4(n), Theorem 1] that $\tau_T = \Phi$, where Φ is a locally summable function on U. Define the function f_Φ on U_G as in §10. Then it is obvious that f_Φ is locally summable on U_G and $T = f_\Phi$ on U_G. But since $T = F$ on $U_G \cap \Omega'$, it follows that $f_\Phi = F$ almost everywhere on U_G. Hence F is locally summable on U_G and $T = F$ on U_G. This shows that $1 \in \Omega_0$ and so the proof of Theorem 2 is now complete.

The above theorem shows that F is locally summable on Ω and $T = F$ on Ω. Fix $z \in \mathfrak{Z}$. Then the distribution zT also satisfies all the conditions of Theorem 2 and it is obvious that $zT = zF$ on Ω'. Hence zF is also locally summable on Ω and $zT = zF$ on Ω. Thus we obtain the following corollary (cf. [4(n), Lemma 16]).

COROLLARY. *For any $z \in \mathfrak{Z}$, the function zF on Ω' is locally summable on Ω and $zT = zF$. Hence*

$$\int f \cdot zF\,dx = \int z^* f \cdot F\,dx$$

for $f \in C_c^\infty(\Omega)$.

16. **Some elementary facts about reductive groups.** As before let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{g}_1$. We can choose a Cartan involution θ of $\mathfrak{g}_1$ such that $\theta(\mathfrak{h}_1) = \mathfrak{h}_1$ [4(e), p. 100]. We extend θ to an automorphism of $\mathfrak{g}$ by defining $\theta(C) = C$ for $C \in \mathfrak{c}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the subspaces of $\mathfrak{g}$ corresponding to the eigenvalues 1 and -1 of θ. Then $\mathfrak{c} \subset \mathfrak{k}$ and $\mathfrak{p} \subset \mathfrak{g}_1$. Moreover, since $\theta(\mathfrak{h}) = \mathfrak{h}$, it is clear that $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p}$.

Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$ and Z the center of G.

LEMMA 26. *The mapping $\phi:(k,X) \to k \exp X$ $(k \in K,\ X \in \mathfrak{p})$ is an analytic diffeomorphism of $K \times \mathfrak{p}$ onto G. Moreover, $Z \subset K$ and K/Z is compact.*

It is easy to verify (see [4(d), p. 614]) that ϕ is everywhere regular. Let C, G_1 and K_1 be the analytic subgroups of G corresponding to $\mathfrak{c}$, $\mathfrak{g}_1$ and $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1$, respectively. Then $G = CG_1$ and $G_1 = K_1 \exp \mathfrak{p}$ (see e.g. [5, pp. 214–215]). Therefore since $CK_1 = K$, it follows that ϕ is surjective. Now suppose

$$k_1 \exp X_1 = k_2 \exp X_2 \quad (k_i \in K, X_i \in \mathfrak{p},\ i = 1,2).$$

Put $k = k_2^{-1} k_1$. Then $k \exp X_1 = \exp X_2$ and therefore

$$\mathrm{Ad}(k)\exp(\mathrm{ad}\, X_1) = \exp(\mathrm{ad}\, X_2).$$

Since $\mathrm{Ad}(G)$ is semisimple, we conclude [5, pp. 214–215] that $X_1 = X_2$. Hence $k_1 = k_2$. This proves that ϕ is univalent and so it is an analytic diffeomorphism.

Let Z_1 be the center of G_1. Then we know that $Z_1 \subset K_1$ and K_1/Z_1 is compact [5,p.214]. Since $K = C K_1$ and $Z = C Z_1$, it follows that $Z \subset K$ and K/Z is compact.

COROLLARY. 1. *θ can be extended to an automorphism of G such that*

$$\theta(k \exp X) = k \exp(-X) \qquad (k \in K,\ X \in \mathfrak{p}).$$

First assume that G is simply connected. Then our statement is obvious. Moreover, θ leaves Z pointwise fixed since $Z \subset K$. Therefore if Z_0 is any closed subgroup of Z, it defines an automorphism of G/Z_0. From this our assertion follows immediately in the general case.

COROLLARY 2(12). *Let $Y' = \mathrm{Ad}(k \exp X)Y$, where $Y, Y' \in \mathfrak{g}$, $k \in K$ and $X \in \mathfrak{p}$. Then if Y and Y' are both eigenvectors of θ, $[X, Y] = 0$.*

(12) This result was pointed out to me by A. Borel.

Since

$$\theta(Y') = \mathrm{Ad}(k\exp(-X))\theta(Y),$$

it is clear that

$$e^{2\,\mathrm{ad}X}\,Y = \varepsilon Y,$$

where $\varepsilon = \pm 1$. Moreover, it follows from [4(k), Lemma 27] that ad X is semisimple and all its eigenvalues are real. Therefore it is obvious that $\varepsilon = 1$ and $[X, Y] = 0$.

COROLLARY 3. *Let* $\mathfrak{a}$ *be a subset of* $\mathfrak{h}$ *such that* $\mathfrak{a} = \theta(\mathfrak{a})$ *and let* Ξ *and* $\mathfrak{z}$ *be the centralizers of* $\mathfrak{a}$ *in* G *and* $\mathfrak{g}$, *respectively. Then they are both invariant under* θ, $\mathfrak{z}$ *is reductive in* $\mathfrak{g}$ *and*

$$\Xi = \Xi_K \exp(\mathfrak{z} \cap \mathfrak{p}),$$

where $\Xi_K = \Xi \cap K$.

For the proof we can obviously replace $\mathfrak{a}$ by the linear subspace of $\mathfrak{g}$ spanned by it. Then $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{k} + \mathfrak{a} \cap \mathfrak{p}$. The invariance of Ξ and $\mathfrak{z}$ under θ is obvious and therefore (see [4(g), Lemma 10]) $\mathfrak{z}$ is reductive in $\mathfrak{g}$. The last statement follows from Corollary 1.

Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}$.

COROLLARY 4. $A = A_K A_{\mathfrak{p}}$, *where* $A_K = A \cap K$ *and* $A_{\mathfrak{p}} = \exp(\mathfrak{h} \cap \mathfrak{p})$. *Moreover,* $Z \subset A_K$ *and* A_K/Z *is compact.*

The first statement follows from Corollary 3 if we take $\mathfrak{a} = \mathfrak{h}$. It is obvious from Lemma 26 that $Z \subset A_K$. Moreover, since K/Z is compact and A_K is closed in K, it follows that A_K/Z is compact.

COROLLARY 5. *Suppose every root of* $(\mathfrak{g}, \mathfrak{h})$ *is imaginary* (*see* [4(m), §4]). *Then* A *is connected and contained in* K.

For then it is obvious that $\mathfrak{h} \cap \mathfrak{p} = \{0\}$ and therefore $A = A_K$. Since $\mathfrak{k}$ is reductive and its derived algebra is compact, the connected component of 1 in A is maximal abelian in K. This shows that A is connected.

17. **Complex semisimple groups.** Let $\mathfrak{g}_c$ be a complex semisimple Lie algebra and G_c a complex analytic group corresponding to it. Fix a Cartan subalgebra $\mathfrak{h}_c$ of $\mathfrak{g}_c$. Then we can choose a compact real form $\mathfrak{u}$ of $\mathfrak{g}_c$ such that $\mathfrak{h} = \mathfrak{h}_c \cap \mathfrak{u}$ is a Cartan subalgebra of $\mathfrak{u}$ (see [5, p. 155]). Let η denote the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{u}$. Then if we regard $\mathfrak{g}_c$ as a Lie algebra over $\boldsymbol{R}$, η is a Cartan involution of $\mathfrak{g}_c$ and $\mathfrak{g}_c = \mathfrak{u} + (-1)^{1/2}\mathfrak{u}$ is the corresponding Cartan decomposition. Let U be the real analytic subgroup of G_c corresponding to $\mathfrak{u}$. Then U is compact and by Lemma 26, the mapping

$$(u, X) \to u\exp(-1)^{1/2}X \qquad (u \in U,\ X \in \mathfrak{u})$$

is an analytic diffeomorphism of $U \times \mathfrak{u}$ onto G_c.

LEMMA 27. *Let* $\mathfrak{a}$ *be a subset of* $\mathfrak{h}_c$ *such that* $\eta(\mathfrak{a}) = \mathfrak{a}$ *and let* $\mathfrak{z}_c$ *and* Ξ_c *denote the centralizers of* $\mathfrak{a}$ *in* $\mathfrak{g}_c$ *and* G_c, *respectively. Then* $\mathfrak{z}_c$ *is reductive in* $\mathfrak{g}_c$ *and* Ξ_c *is connected.*

We may obviously replace $\mathfrak{a}$ by the subspace $\mathfrak{a}_c$ spanned by it over C. It follows from Corollary 3 of Lemma 26 that $\mathfrak{z}_c$ is reductive in $\mathfrak{g}_c$ and

$$\Xi_c = \Xi \exp((-1)^{1/2}\mathfrak{z}),$$

where $\Xi = \Xi_c \cap U$ and $\mathfrak{z} = \mathfrak{z}_c \cap \mathfrak{u}$. It is clear that Ξ is the centralizer of $\mathfrak{a}_c \cap \mathfrak{u}$ in U and therefore it is connected (see [5, p. 247]). This proves that Ξ_c is connected.

COROLLARY. *Let* A_c *be the Cartan subgroup of* G_c *corresponding to* $\mathfrak{h}_c$. *Then* A_c *is connected.*

By definition A_c is the centralizer of $\mathfrak{h}_c$ in G_c. Hence the corollary follows by taking $\mathfrak{a} = \mathfrak{h}_c$.

The following lemma, together with its proof, was pointed out to me by Borel.

LEMMA 28. $\mathfrak{z}_c$ *being as above, put* $\mathfrak{z}_{1c} = [\mathfrak{z}_c, \mathfrak{z}_c]$ *and let* Ξ_{1c} *be the complex analytic subgroup of* G_c *corresponding to* $\mathfrak{z}_{1c}$. *Then if* G_c *is simply connected, the same holds for* Ξ_{1c}.

Put $\mathfrak{h}_R = (-1)^{1/2}(\mathfrak{h}_c \cap \mathfrak{u})$ and $\mathfrak{a}_R = (-1)^{1/2}(\mathfrak{a}_c \cap \mathfrak{u})$. Introduce compatible orders (see [4(g), p. 195]) in the spaces of linear functions on $\mathfrak{h}_R$ and $\mathfrak{a}_R$. Let P be the set of all positive roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$ under this order. Let $\bar{\alpha}$ denote the restriction of α on $\mathfrak{a}_c$ for any root α and let P_0 denote the set of those $\alpha \in P$ for which $\bar{\alpha} = 0$. Consider the set $(\alpha_1, \alpha_2, \cdots, \alpha_l)$ of simple roots in P and assume that $\alpha_i \in P_0$ $(1 \leqq i \leqq m)$ and $\alpha_i \notin P_0$ $(m < i \leqq l)$. We claim that $(\alpha_1, \cdots, \alpha_m)$ is a set of fundamental roots for $(\mathfrak{z}_c, \mathfrak{h}_c)$. Fix $\alpha \in P_0$. Then $\alpha = \sum_{1 \leqq i \leqq l} r_i \alpha_i$, where r_i are integers $\geqq 0$. Hence

$$\sum_{1 \leqq i \leqq l} r_i \bar{\alpha}_i = \bar{\alpha} = 0.$$

Now $\bar{\alpha}_i = 0$ $(1 \leqq i \leqq m)$ and $\bar{\alpha}_i > 0$ $(m < i \leqq l)$ by the compatibility of our orders. So it is obvious that $r_i = 0$ for $m < i \leqq l$. Since $(\alpha_1, \cdots, \alpha_m)$ are linearly independent, this proves our assertion.

For any root α, let H_α denote, as usual, the element in $\mathfrak{h}_R$ such that $\operatorname{tr}(\operatorname{ad} H \operatorname{ad} H_\alpha) = \alpha(H)$ for $H \in \mathfrak{h}_c$. Put

$$H_i = 2\alpha_i(H_{\alpha_i})^{-1} H_{\alpha_i} \qquad (1 \leqq i \leqq l).$$

Then it is clear that H_i $(1 \leqq i \leqq m)$ form a base for $\mathfrak{h}_c \cap \mathfrak{z}_{1c}$ over C. Now suppose t_i $(1 \leqq i \leqq l)$ are complex numbers such that

$$\exp\left(2\pi(-1)^{1/2} \sum_{1 \leqq i \leqq m} t_i H_i\right) = 1$$

in Ξ_{1c}. Then since G_c is simply connected, we can conclude (see Weyl [8]) that t_i are rational integers. This implies that Ξ_{1c} is simply connected.

LEMMA 29. *Assume that G_c is simply connected and let λ be a linear function on $\mathfrak{h}_c$. Then there exists a character ξ_λ of A_c such that*

$$\xi_\lambda(\exp H) = e^{\lambda(H)} \qquad (H \in \mathfrak{h}_c)$$

if and only if $2\lambda(H_\alpha)/\alpha(H_\alpha)$ is a rational integer for every root α. Put

$$\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha,$$

where P is the set of positive roots under some order. Then the above condition is fulfilled for $\lambda = \rho$.

This is well known (see Weyl [8]).

18. **Acceptable groups.** Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ which we assume, as before, to be reductive. Let j be the inclusion mapping of $\mathfrak{g}$ into $\mathfrak{g}_c$ and G_c a complex analytic group corresponding to $\mathfrak{g}_c$. We say that G_c is a complexification of G if j can be extended to a homomorphism of G into G_c.

Define $\mathfrak{c}$ and $\mathfrak{g}_1$ as in §9 and let C and G_1, respectively, be the corresponding analytic subgroups of G. We call G_1 the semisimple part of G. Similarly let C_c and G_{1c} denote the complex analytic subgroups of G_c corresponding to $\mathfrak{c}_c$ and $\mathfrak{g}_{1c}$, respectively. We say that G_c is quasisimply connected (q. s. c.) if $C_c \cap G_{1c} = \{1\}$ and G_{1c} is simply connected. Moreover, G itself is called q. s. c. if it has a q. s. c. complexification. Assume that $G_1 \cap C$ is finite. Then G always has a complexification. Moreover, since the center of a complex semisimple group is finite, it is clear that there exists a q. s. c. covering group $\tilde{G}$ which covers G only finitely many times.

Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Consider a complexification G_c of G and let A_c denote its Cartan subgroup corresponding to $\mathfrak{h}_c$. Then A_c is connected (corollary of Lemma 27) and it is obvious that $j(A) \subset A_c$. Let λ be a linear function on $\mathfrak{h}_c$. Then there exsts at most one complex-analytic homomorphism ξ_λ of A_c into $\boldsymbol{C}$ such that

$$\xi_\lambda(\exp H) = e^{\lambda(H)} \qquad (H \in \mathfrak{h}_c).$$

Then $\xi_\lambda \circ j$ is a homomorphism of A into $\boldsymbol{C}$, which is easily seen to be independent of the particular choice of G_c (so long as it can be defined by means of G_c at all). We shall write ξ_λ instead of $\xi_\lambda \circ j$. If α is a root of $(\mathfrak{g}, \mathfrak{h})$, it is obvious that ξ_α always exists.

Let P be the set of all positive roots of $(\mathfrak{g}, \mathfrak{h})$ in some order and put

$$\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha.$$

If G is q. s. c., we can take G_c to be a q. s. c. complexification of G. Then it follows from Lemma 29 that ξ_ρ exists.

Let W denote the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Then it is well known (see [8]) that $s\rho - \rho$ $(s \in W)$ is an integral linear combination of the roots. Therefore the condition that ξ_ρ should be defined is independent of the order of roots. Moreover, since any two Cartan subalgebras of $\mathfrak{g}_c$ are conjugate under the (connected) adjoint group of $\mathfrak{g}_c$, it follows that the above condition also does not depend on the choice of $\mathfrak{h}$. We shall say that G is *acceptable* if this condition is satisfied. Similarly a complexification G_c of G is called acceptable if ξ_ρ can be defined on A_c.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{h} \cap \mathfrak{p}$ in $\mathfrak{g}$ and M the analytic subgroup of G corresponding to $\mathfrak{m}$. Introduce compatible orders (see [4(g), p. 195]) on the spaces of real linear functions on $\mathfrak{h}\cap\mathfrak{p}$ and $\mathfrak{h} \cap \mathfrak{p} + (-1)^{1/2}\ \mathfrak{h} \cap \mathfrak{k}$ respectively. We assume that P is the set of positive roots under this order. Let P_M denote the set of those $\alpha \in P$ which vanish identically on $\mathfrak{h} \cap \mathfrak{p}$. Put

$$\rho_M = \frac{1}{2} \sum_{\alpha \in P_M} \alpha.$$

LEMMA 30. *Suppose G is acceptable. Then the same holds for M and in fact*

$$\xi_{\rho_M}(h) = \xi_\rho(h_1) \qquad (h \in A \cap M),$$

where $h = h_1 h_2$ $(h_1 \in A_K, h_2 \in A_{\mathfrak{p}})$ in the notation of §16.

Let P_+ be the complement of P_M in P. Then it is easy to verify that if $\alpha \in P_+$, the same holds for $-\theta\alpha$. This shows that $\rho - \rho_M = 0$ on $\mathfrak{h} \cap \mathfrak{k}$. Let $\mathfrak{m}_1$ be the set of all $X \in \mathfrak{m}$ such that $\mathrm{tr}(\mathrm{ad}\, H \,\mathrm{ad}\, X) = 0$ for $H \in \mathfrak{h}\cap\mathfrak{p}$. Then $\theta(\mathfrak{m}_1) = \mathfrak{m}_1$ and $(\mathfrak{h} \cap \mathfrak{p}) \cap \mathfrak{m}_1 = \{0\}$. Hence if M_1 is the analytic subgroup of G corresponding to $\mathfrak{m}_1$, it is clear that $M = A_{\mathfrak{p}} M_1$ and $A_{\mathfrak{p}} \cap M_1 = \{1\}$. Now $\mathfrak{m}$ is reductive (Corollary 3 of Lemma 26) and $\mathfrak{h} \cap \mathfrak{p}$ lies in the center of $\mathfrak{m}$. Therefore since $\rho = \rho_M$ on $\mathfrak{h} \cap \mathfrak{k}$ and $A_{\mathfrak{p}}$ is simply connected, the statement of the lemma follows immediately by considering an acceptable complexification of G.

19. **Behavior of F around singular points.** From now on we assume that G is acceptable. Put

$$\Delta_A(h) = \xi_\rho(h) \prod_{\alpha \in P} (1 - \xi_\alpha(h)^{-1}) \qquad (h \in A).$$

(We shall often drop the subscript A if there is no risk of confusion.) Then $A' = A \cap G'$ is the set of all points $h \in A$, where $\Delta(h) \neq 0$. Put

$$\Delta_R'(h) = \prod_{\alpha \in P_R} (1 - \xi_\alpha(h)^{-1}) \qquad (h \in A),$$

where P_R is the set of all real roots (see [4(m), §4]) in P. Let $A'(R)$ be the set of those $h \in A$ where $\Delta_R'(h) \neq 0$. We now use the notation of §15.

LEMMA 31 [13]. *Put $\Phi_A(h) = \Delta_A(h) F(h)$ $(h \in A' \cap \Omega)$. Then Φ_A can be extended to an analytic function on $A'(R) \cap \Omega$.*

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[13] Cf. [4(n), Theorem 2] and [4(e), Theorem 8].

Fix a point $a \in A \cap \Omega$. Then a is semisimple. We now use the notation of §4 and define $\Omega_\Xi = a^{-1}\Omega \cap \Xi'$. Put $\sigma = |v_a|^{1/2}\sigma_T$ as in §15, and let Ω_Ξ' be the set of all elements in Ω_Ξ which are regular in Ξ. We denote by g, as before, the analytic function on Ω_Ξ' such that g is locally summable on Ω_Ξ and $\sigma = g$ on Ω_Ξ. Since T is invariant, the same holds for F and, as we have seen during the proof of Theorem 2,

$$F(ay) = |v_a(y)|^{-1/2} g(y) \qquad (y \in \Omega_\Xi').$$

Now Ω_Ξ is an open and completely invariant neighborhood of 1 in Ξ. Hence (see §9) we can choose an open and completely invariant neighborhood U of zero in $\mathfrak{z}$ such that the exponential mapping defines an analytic diffeomorphism of U onto an open subset U_Ξ of Ω_Ξ. Consider the function $\xi_\mathfrak{z}$ on $\mathfrak{z}$ (see §10) and, for any $\phi \in C_c^\infty(U)$, define $f_\phi \in C_c^\infty(U_\Xi)$ by

$$f_\phi(\exp Z) = \xi_\mathfrak{z}(Z)^{-1}\phi(Z) \qquad (Z \in U).$$

Let τ be the distribution on U given by $\tau(\phi) = \sigma(f_\phi)$ $(\phi \in C_c^\infty(U))$. Define $\mathfrak{V} = \mathfrak{Z}(\mathfrak{z}) \cdot \mu(\mathfrak{U})$, where $\mu = \mu_{\mathfrak{g}/\mathfrak{z}}$ (in the notation of §12). Then we know from Lemmas 21 and 22 that $\mathfrak{Z}(\mathfrak{z})$ is a finite module over $\mu(\mathfrak{Z})$ and $v\sigma = 0$ $(v \in \mathfrak{V})$. Let $z \to p_z$ denote the canonical isomorphism of $\mathfrak{Z}(\mathfrak{z})$ onto $I(\mathfrak{z}_c)$ (see §12). Then $\partial(p_v)\tau = 0$ $(v \in \mathfrak{V})$ from the corollary of Lemma 24. Hence Theorem 1 of [4(n)] is applicable to $(\mathfrak{z}, U, \tau)$. Let $\mathfrak{z}'$ denote the set of all elements of $\mathfrak{z}$ which are regular in $\mathfrak{z}$ and let ψ be the analytic function on $U' = U \cap \mathfrak{z}'$ such that $\tau = \psi$.

Lemma 32. $\psi(Z) = \xi_\mathfrak{z}(Z)|v_a(\exp Z)|^{1/2}F(a \exp Z)$ $(Z \in U')$.

Fix $\phi \in C_c^\infty(U)$. Then

$$\int \phi\psi\, dZ = \tau(\phi) = \sigma(f_\phi) = \int f_\phi g\, dy$$

$$= \int \phi(Z)\xi_\mathfrak{z}(Z)g(\exp Z)\, dZ.$$

Here dy is the Haar measure of Ξ and dZ the Euclidean measure on $\mathfrak{z}$ and they are related (see §10) by the equation

$$dy = \xi_\mathfrak{z}(Z)^2 dZ \qquad (y = \exp Z,\ Z \in U).$$

Since $\exp(U') \subset \Omega_\Xi'$, we have

$$g(\exp Z) = |v_a(\exp Z)|^{1/2} F(a \exp Z) \qquad (Z \in U')$$

and so our assertion is now obvious.

Let $P_\mathfrak{z}$ be the set of all roots $\alpha \in P$ such that $\xi_\alpha(a) = 1$. Put $P_{\mathfrak{z},R} = P_\mathfrak{z} \cap P_R$ and let $\mathfrak{h}'(R)$ be the set of all points $H \in \mathfrak{h}$, where $\prod_{\alpha \in P_{\mathfrak{z},R}} \alpha(H) \neq 0$. Then we know from [4(n), Theorem 2] that there exists an analytic function u on $\mathfrak{h}'(R) \cap U$ such that

$$u(H) = \pi_{\mathfrak{z}}(H)\psi(H) \qquad (H \in \mathfrak{h} \cap U'),$$

where $\pi_{\mathfrak{z}} = \prod_{\alpha \in P_{\mathfrak{z}}} \alpha$.

LEMMA 33. *Let $\mathfrak{h}_0$ be an open and connected neighborhood of zero in $U \cap \mathfrak{h}$. Then*

$$\pi_{\mathfrak{z}}(H)\xi_{\mathfrak{z}}(H)\left|v_a(\exp H)\right|^{1/2} = c\Delta(a \exp H) \qquad (H \in \mathfrak{h}_0),$$

where c is a constant. Let P' be the complement of $P_{\mathfrak{z}}$ in P and p the number of roots in P'. Then $p = (\dim \mathfrak{g} - \dim \mathfrak{z})/2$ and

$$c^2 = (-1)^p \operatorname{sign} v_a(1).$$

Finally

$$c = \left|v_a(1)\right|^{1/2}\left\{\xi_\rho(a)\prod_{\alpha \in P'}(1 - \xi_\alpha(a)^{-1})\right\}^{-1}.$$

Put $\rho' = (1/2)\sum_{\alpha \in P'} \alpha$. Then it is clear that

$$\begin{aligned} v_a(\exp H) &= \prod_{\alpha \in P'}\{(\xi_\alpha(a \exp H)^{-1} - 1)(\xi_\alpha(a \exp H) - 1)\} \\ &= (-1)^p \xi_{2\rho'}(a)\left\{e^{\rho'(H)}\prod_{\alpha \in P'}(1 - \xi_\alpha(a \exp H)^{-1})\right\}^2 \end{aligned}$$

for $H \in \mathfrak{h}$. Since $v_a(\exp H)$ is real and $\neq 0$ for $H \in \mathfrak{h}_0$, it is clear that

$$\left|v_a(\exp H)\right|^{-1/2} e^{\rho'(H)}\prod_{\alpha \in P'}(1 - \xi_\alpha(a \exp H)^{-1})$$

is an analytic function on $\mathfrak{h}_0$ whose fourth power is a constant. Therefore since $\mathfrak{h}_0$ is connected, we conclude that

$$\left|v_a(\exp H)\right|^{1/2} = c_1 e^{\rho'(H)}\prod_{\alpha \in P'}(1 - \xi_\alpha(a \exp H)^{-1}) \qquad (H \in \mathfrak{h}_0),$$

where

$$c_1 = \left|v_a(1)\right|^{1/2}\prod_{\alpha \in P'}(1 - \xi_\alpha(a)^{-1})^{-1}.$$

A similar argument shows that

$$\pi_{\mathfrak{z}}(H)\xi_{\mathfrak{z}}(H) = \prod_{\alpha \in P_{\mathfrak{z}}}(e^{\alpha(H)/2} - e^{-\alpha(H)/2}) \qquad (H \in \mathfrak{h}_0).$$

Hence

$$\pi_{\mathfrak{z}}(H)\xi_{\mathfrak{z}}(H)\left|v_a(\exp H)\right|^{1/2} = c\Delta(a \exp H) \qquad (H \in \mathfrak{h}_0),$$

where $c = c_1\xi_\rho(a)^{-1}$.

It is obvious that $\dim \mathfrak{g} - \dim \mathfrak{z} = 2p$. Since $\xi_\alpha(a) = 1$ for $\alpha \in P_{\mathfrak{z}}$, it is clear that $\xi_{2\rho'}(a) = \xi_{2\rho}(a)$. Now

$$\begin{aligned} c_1^2\prod_{\alpha \in P'}(1 - \xi_\alpha(a^{-1}))^2 &= \left|v_a(1)\right| = v_a(1)\operatorname{sign} v_a(1) \\ &= (-1)^p(\operatorname{sign} v_a(1))\xi_{2\rho'}(a)\prod_{\alpha \in P'}(1 - \xi_\alpha(a^{-1}))^2. \end{aligned}$$

This shows that $c^2 = (-1)^p \operatorname{sign} v_a(1)$.

It follows from Lemmas 32 and 33 that

$$u(H) = c\Phi_A(a \exp H) \qquad (H \in \mathfrak{h}_0 \cap U').$$

Now put $V = a \exp \mathfrak{h}_0$ and $v(a \exp H) = c^{-1}u(H)$ $(H \in \mathfrak{h}_0 \cap \mathfrak{h}'(R))$. Then V is an open neighborhood of a in $A \cap \Omega$, v is an analytic function on $V \cap A'(R)$ and $v = \Phi_A$ on V. This proves Lemma 31.

For any root α, let s_α denote the Weyl reflexion corresponding to α. The Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ operates on $\mathfrak{h}_c$ and therefore also on $\mathfrak{S}(\mathfrak{h}_c)$.

Lemma 34. *Fix a point $a \in A \cap \Omega$ and suppose v is an element in $\mathfrak{S}(\mathfrak{h}_c)$ such that $v^{s_\alpha} = -v$ for every real root α for which $\xi_\alpha(a) = 1$. Then $v\Phi_A$ can be extended to a continuous function around a.*

We keep to the above notation. Then by Lemma 19 of [4(n)] $\partial(v)u$ can[10] be extended to a continuous function around zero. Since $\Phi_A(a \exp H) = c^{-1}u(H)$ $(H \in \mathfrak{h}_0 \cap U')$, our assertion is now obvious.

For any root α, define H_α as in [4(m), §4] and put $\varpi = \prod_{\alpha \in P} H_\alpha \in \mathfrak{S}(\mathfrak{h}_c)$. Then, by Lemma 34, $\varpi\Phi_A$ can be extended to a continuous function Ψ_A on A.

Lemma 35. *Let A and B be two Cartan subgroups of G. Then $\Psi_A = \Psi_B$ on $A \cap B \cap \Omega$.*

Fix $a \in A \cap B \cap \Omega$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be the Cartan subalgebras corresponding to $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Define $\mathfrak{z}$, U and ψ as in Lemma 32. Then $\mathfrak{a}$, $\mathfrak{b}$ are Cartan subalgebras of $\mathfrak{z}$. Put $\mathfrak{h} = \mathfrak{a}$ or $\mathfrak{b}$ and define[14]

$$\varpi_{\mathfrak{z}}{}^{\mathfrak{h}} = \varpi_{\mathfrak{z}} = \prod_{\alpha \in P_{\mathfrak{z}}} H_\alpha, \qquad \varpi_{\mathfrak{g}/\mathfrak{z}}{}^{\mathfrak{h}} = \varpi_{\mathfrak{g}/\mathfrak{z}} = \prod_{\alpha \in P'} H_\alpha$$

in the notation introduced above. Then $\varpi = \varpi_{\mathfrak{z}} \cdot \varpi_{\mathfrak{g}/\mathfrak{z}}$. Since $\varpi^{s_\alpha} = -\varpi$, $\varpi_{\mathfrak{z}}^{s_\alpha} = -\varpi_{\mathfrak{z}}$ for any $\alpha \in P_{\mathfrak{z}}$, it is clear that $\varpi_{\mathfrak{g}/\mathfrak{z}}$ is invariant under the Weyl group of $(\mathfrak{z}, \mathfrak{h})$. Therefore, by Chevalley's theorem [4(f), Lemma 9], there exists an element $\eta \in I(\mathfrak{z}_c)$ such that the projection $\eta_{\mathfrak{a}}$ of η in $\mathfrak{S}(\mathfrak{a}_c) = S(\mathfrak{a}_c)$ (see [4(l), §8]) is $\varpi_{\mathfrak{g}/\mathfrak{z}}{}^{\mathfrak{a}}$.

Let G_c be an acceptable complexification of G and Ξ_c the analytic subgroup of G_c corresponding to $\mathfrak{z}_c$. Then we can choose $y \in \Xi_c$ such that $(\mathfrak{a}_c)^y = \mathfrak{b}_c$. Thus we have an isomorphism $D \to D^y$ of $\mathfrak{D}(\mathfrak{a}_c)$ onto $\mathfrak{D}(\mathfrak{b}_c)$ (see [4(l), §3]). Since the definition of Ψ_B is obviously independent of the order of roots, we may assume that the positive roots of $\mathfrak{a}$ are mapped into positive roots of $\mathfrak{b}$ under this isomorphism. Define j as in §18. Then it is obvious that $yj(a)y^{-1} = j(a)$. Therefore it follows from Lemma 33 that $c_A = c_B$ and $\eta_{\mathfrak{b}} = \varpi_{\mathfrak{g}/\mathfrak{z}}{}^{\mathfrak{b}}$. (Here c_A and c_B are the constants which correspond to c of Lemma 33 for the cases $\mathfrak{h} = \mathfrak{a}$ and $\mathfrak{h} = \mathfrak{b}$, respectively.)

(14) We use a similar notation in other cases. For example $\pi_{\mathfrak{z}}{}^{\mathfrak{h}} = \pi_{\mathfrak{z}}$ and $\varpi^{\mathfrak{h}} = \varpi$.

Now put

$$u^{\mathfrak{h}}(H) = \pi_{\mathfrak{z}}^{\mathfrak{h}}(H)\psi(H) \qquad (H \in U' \cap \mathfrak{h}).$$

Then it follows from the corollary of Theorem 3 of [4(n)] and [4(f), Theorem 1] (both applied to $\mathfrak{z}$) that

$$\partial(\varpi_{\mathfrak{z}}^{\mathfrak{a}} \cdot \eta_{\mathfrak{a}})u^{\mathfrak{a}} = \partial(\varpi_{\mathfrak{z}}^{\mathfrak{b}} \cdot \eta_{\mathfrak{b}})u^{\mathfrak{b}}$$

on $\mathfrak{a} \cap \mathfrak{b} \cap U$. This proves that $\partial(\varpi^{\mathfrak{a}})u^{\mathfrak{a}} = \partial(\varpi^{\mathfrak{b}})u^{\mathfrak{b}}$ on $\mathfrak{a} \cap \mathfrak{b} \cap \Omega$. But if U_0 is an open convex neighborhood of zero in U, we know that

$$u^{\mathfrak{a}}(H) = c_A\Phi_A(a \exp H) \qquad (H \in U_0' \cap \mathfrak{a}),$$
$$u^{\mathfrak{b}}(H) = c_B\Phi_B(a \exp H) \qquad (H \in U_0' \cap \mathfrak{b}),$$

where $U_0' = U_0 \cap U'$. Therefore since $c_A = c_B \neq 0$, we conclude that $\Psi_A(a) = \Psi_B(a)$.

20. **The function** $\nabla_G F$. We write $\varpi = \varpi_A$ for a given Cartan subgroup A.

LEMMA 36. *There exists a unique differential operator ∇_G on G' with the following properties.*

(1) *∇_G is invariant under G.*

(2) *Let A be a Cartan subgroup of G. Then*

$$f(h; \nabla_G) = f(h; \varpi_A \circ \Delta_A)$$

for $f \in C^\infty(G)$ and $h \in A \cap G'$.

Moreover, ∇_G is analytic.

The proof is similar to that of [4(n), Lemma 24]. Since two distinct Cartan subgroups cannot have a regular element in common, the uniqueness is obvious. The existence is proved as follows. Fix a Cartan subgroup A of G and define $G_A = \bigcup_{x \in G} xA'x^{-1}$, where $A' = A \cap G'$. Let $\mathfrak{h}$ be Cartan subalgebra of A, $\tilde{A}$ the normalizer of $\mathfrak{h}$ in G and $\tilde{A}_K = \tilde{A} \cap K$ in the notation of §16. Then by Corollary 2 of Lemma 26, $\tilde{A} = \tilde{A}_K A_{\mathfrak{p}}$ and if A_0 is the center of A, it follows (see §16) that

$$W_A = \tilde{A}/A_0 \simeq \tilde{A}_K/A_0 \cap K$$

is both compact and discrete and therefore it is finite. Let $x \to x^*$ denote the natural projection of G on $G^* = G/A_0$. Define $h^{x^*} = h^x$ ($h \in A$, $x \in G$). Then the mapping ϕ: $(x^*, h) \to h^{x^*}$ of $G^* \times A$ into G is everywhere regular on $G^* \times A'$. Hence $G_A = \phi(G^* \times A')$ is open in G. Now W_A operates on G^* and A as follows. Let y be an element in A whose image in W_A is s. Then

$$x^*s = (xy)^* \ (x \in G), \ \ h^s = yhy^{-1}.$$

Define

$$(x^*, h)s = (x^*s, h^{s^{-1}}) \qquad (x^* \in G^*, \ h \in A').$$

In this way W_A operates on the right on $G^* \times A'$ without fixed points and the quotient space $(G^* \times A')/W_A$ may be identified with G_A by means of ϕ. By making use of the homomorphism $j: G \to G_c$ (see §18) one proves without difficulty that the differential operator $\varpi_A \circ \Delta_A$ on A is invariant under W_A. The rest of the proof now goes through exactly as in [4(n), §9].

LEMMA 37. *For any $z \in \mathfrak{Z}$, $(\nabla_G \circ z)F$ can be extended to a continuous function on Ω.*

Since the distribution zT also satisfies all the conditions of Theorem 2, it is enough to consider the case $z = 1$. Let Ω_0 be the set of all points $x_0 \in \Omega_0$ for which there exists an open neighborhood V of x_0 in Ω and a continuous function v on V such that $v = \nabla_G F$ on $V \cap G'$. Obviously Ω_0 is an open and invariant subset of Ω. Hence, in view of Lemma 7, it would be enough to prove that every semisimple element of Ω is contained in Ω_0.

Fix a semisimple element $a \in \Omega$ and let us use the notation of Lemma 32. Let $\mathfrak{z}'$ be the set of those elements of $\mathfrak{z}$ which are regular in $\mathfrak{z}$. Define the differential operator $\nabla_{\mathfrak{z}}$ on $\mathfrak{z}'$ as in [4(n), §9] and fix an open and convex neighborhood U_0 of zero in U and put $U_0' = U_0 \cap U'$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{z}$. Then, as we have seen in §19, there exists a unique element $\eta \in I(\mathfrak{z}_c)$ such that $\eta_{\mathfrak{a}} = \varpi_{\mathfrak{g}/\mathfrak{z}}{}^{\mathfrak{a}}$. Let c denote the constant of Lemma 33 corresponding to $\mathfrak{h} = \mathfrak{a}$.

LEMMA 38. $F(a \exp Z;\ \nabla_G) = c\psi(Z;\ \nabla_{\mathfrak{z}} \circ \partial(\eta))\ (Z \in U_0')$.

Fix $H_0 \in U_0'$ and let $\mathfrak{h}$ be the centralizer of H_0 in $\mathfrak{z}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{z}$ and therefore also of $\mathfrak{g}$. Moreover, $a \exp H_0 \in \Omega \cap G'$. Let A be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Then

$$F(a \exp H_0;\ \nabla_G) = F(a \exp H_0;\ \varpi_A \circ \Delta_A)\ ,$$

from the definition of ∇_G. Put $\mathfrak{h}_0 = \mathfrak{h} \cap U_0$. Then we have seen in §19 that

$$\Delta_A(a \exp H)F(a \exp H) = c_A \pi_{\mathfrak{z}}(H)\psi(H) \qquad (H \in \mathfrak{h}_0 \cap U'),$$

where c_A is a constant. Moreover, by a suitable choice of positive roots of $(\mathfrak{g},\mathfrak{h})$ we can arrange (see the proof of Lemma 35) that $c_A = c$ and $\eta_{\mathfrak{h}} = \varpi_{\mathfrak{g}/\mathfrak{z}}{}^{\mathfrak{h}}$. Then it follows from [4(f), Theorem 1] and the definition of $\nabla_{\mathfrak{z}}$ that

$$\psi(H; \nabla_{\mathfrak{z}} \circ \partial(\eta)) = \psi(H; \partial(\varpi) \circ \pi_{\mathfrak{z}}) \qquad (H \in \mathfrak{h}_0 \cap U'),$$

where $\varpi = \varpi_A$. Therefore it is clear that

$$F(a \exp H_0; \nabla_G) = c\psi(H_0; \nabla_{\mathfrak{z}} \circ \partial(\eta))$$

and this proves our assertion.

It follows from Lemma 38 and [4(n), Lemma 25] that there exists an open neighborhood V_Ξ of 1 in Ω_Ξ and a continuous function g_0 on V_Ξ such that

$$F(ay; \nabla_G) = g_0(y) \qquad (y \in V_\Xi' = V_\Xi \cap \Omega_\Xi')$$

in the notation of §19. Let $x \to \bar{x}$ denote the natural mapping of G on $\bar{G} = G/\Xi$. Select open neighborhoods $\bar{G}_0$ and V_0 of $\bar{1}$ and 1 in $\bar{G}$ and V_Ξ, respectively. If they are sufficiently small the following conditions hold. There exists an analytic mapping ϕ of $\bar{G}_0$ into G such that: (1) $\overline{\phi(\bar{x})} = \bar{x}$ for $\bar{x} \in \bar{G}_0$ and (2) the mapping $\alpha:(\bar{x}, y) \to (ay)^{\phi(\bar{x})}$ of $\bar{G}_0 \times V_0$ into G is univalent and regular and $V = \alpha(\bar{G}_0 \times V_0) \subset V_\Xi$. Then V is an open neighborhood of a in G and α defines an analytic diffeomorphism of $\bar{G}_0 \times V_0$ on V. Define a function F_0 on V by

$$F_0(\alpha(\bar{x}, y)) = g_0(y) \qquad (\bar{x} \in \bar{G}_0, y \in V_0).$$

Then F_0 is continuous and since $\nabla_G F$ is invariant under G, it is obvious that $F_0 = \nabla_G F$ on $V \cap G'$. This shows that $a \in \Omega_0$ and therefore Lemma 37 is proved.

21. **An elementary result.** Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

LEMMA 39. *Let λ be a linear function on $\mathfrak{h}_c$. Then there exists an invariant analytic function f_λ on $\mathfrak{g}$ such that*

$$\pi(H) f_\lambda(H) = \sum_{s \in W} \varepsilon(s) e^{\lambda(sH)} \qquad (H \in \mathfrak{h}).$$

Moreover, f_λ is unique.

Let $\mathfrak{h}'$ be the set of all elements $H \in \mathfrak{h}$, where $\pi(H) \neq 0$. Since $(\mathfrak{h}')^G$ is an open subset of $\mathfrak{g}$, the uniqueness of f_λ is obvious. Therefore it remains to prove its existence. For this we may obviously assume that $\mathfrak{g}$ is semisimple and G is the connected adjoint group of $\mathfrak{g}$. Now we use the notation of §16. Let G_c be the (connected) complex adjoint group of $\mathfrak{g}_c$ and U the real analytic subgroup of G_c corresponding to the compact real form $\mathfrak{u} = \mathfrak{k} + (-1)^{1/2}\mathfrak{p}$ of $\mathfrak{g}_c$. Then U is compact. Put $B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y)$ $(X, Y \in \mathfrak{g}_c)$ as usual and consider

$$f(X:Y) = \int_U \exp B(uX, Y)\, du \qquad (X, Y \in \mathfrak{g}_c),$$

where du is the normalized Haar measure on U. Then f is obviously a holomorphic function on $\mathfrak{g}_c \times \mathfrak{g}_c$ and it is clear that

$$f(X; \tau(Z): Y) = 0 \qquad (Z \in \mathfrak{u})$$

in the notation of [4(l), §4]. Since f is holomorphic in X, this implies that $f(xX:Y) = f(X:Y)$ for $x \in G_c$.

Let H_λ denote the element in $\mathfrak{h}_c$ such that $B(H, H_\lambda) = \lambda(H)$ for all $H \in \mathfrak{h}_c$. Then we know from [4(f), Theorem 2] that

$$\pi(H_\lambda)\pi(H) f(H:H_\lambda) = c \sum_{s \in W} \varepsilon(s) e^{\lambda(sH)} \qquad (H \in \mathfrak{h}_c),$$

where c is a number $\neq 0$ independent of H and λ. Therefore we can take

$$f_\lambda(X) = c^{-1}\pi(H_\lambda) f(X:H_\lambda) \qquad (X \in \mathfrak{g}).$$

22. **The invariant integral on** G. We now return to the notation of §19. Let A_0 denote the center of A and $x \to x^*$ the natural projection of G on $G^* = G/A_0$. Put $h^{x^*} = h^x$ $(h \in A, x \in G)$ and let dx^* denote the invariant measure on G^*. For any $f \in C_c^{\infty}(G)$, put

$$F_f(h) = \varepsilon_R(h)\Delta(h) \int_{G^*} f(h^{x^*})\, dx^* \qquad (h \in A'),$$

where $A' = A \cap G'$ and $\varepsilon_R(h) = \operatorname{sign} \Delta_R'(h)$. Then F_f is a C^{∞}-function on A' and if γ is the canonical isomorphism of $\mathfrak{Z}$ onto $I(\mathfrak{h}_c)$ (see §6), we have [4(h), Theorem 3]

$$F_{zf} = \gamma(z) F_f \qquad (z \in \mathfrak{Z},\ f \in C_c^{\infty}(G)).$$

Let S_I denote the set of all positive singular imaginary roots of $(\mathfrak{g}, \mathfrak{h})$ (see [4(m), §4]). Define

$$\Delta_I'(h) = \prod_{\alpha \in S_I} (1 - \xi_\alpha(h)^{-1}) \qquad (h \in A)$$

and let $A'(I)$ be the set of those points $h \in A$ where $\Delta_I'(h) \neq 0$.

LEMMA 40. *Fix $f \in C_c^{\infty}(G)$. Then F_f can be extended to a C^{∞}-function on $A'(I)$. Let a be a point in A and v an element in $\mathfrak{S}(\mathfrak{h}_c)$ such that $v^{s_\alpha} = -v$ for every singular imaginary root α for which $\xi_\alpha(a) = 1$. Then vF_f can be extended to a continuous function around a.*

Let $\mathfrak{z}$ and Ξ_1 denote the centralizers of a in $\mathfrak{g}$ and G, respectively, and Ξ the connected component of 1 in Ξ_1. Then if Z is the center of G, $\Xi_1/Z\Xi$ is finite (see [4(g_2), Lemma 15]). Choose an open neighborhood B of 1 in A with the following property (see [4(h), Theorem 1]). If $h \in B$ and $x \in G$ vary in such a way that $(ah)^x$ stays inside some compact subset of G, then the coset $\bar{x} = x\Xi_1$ remains within a compact subset of $\bar{G} = G/\Xi_1$. Let $x \to \bar{x}$ denote the natural projection of G on $\bar{G}$. Since $\mathfrak{z}$ is reductive and $\Xi_1/Z\Xi$ is finite, it follows that the group Ξ_1 is unimodular. Hence we have an invariant measure $d\bar{x}$ on $\bar{G}$. Let dy^* denote the invariant measure on $\Xi_1^* = \Xi_1/A_0$. Then if $d\bar{x}$ and dy^* are suitably normalized, we have

$$F_f(ah) = \varepsilon_R(ah)\Delta(ah) \int_{\bar{G}} d\bar{x} \int_{\Xi_1^*} f(x(ah)^{y^*} x^{-1})\, dy^* \qquad (h \in B')$$

for $f \in C_c^{\infty}(G)$. Here $B' = B \cap a^{-1}A'$. Now fix an open and relatively compact subset G_0 of G and choose a compact set $\bar{\Omega}$ in $\bar{G}$ such that $(aB)^x \cap G_0 = \emptyset$ $(x \in G)$ unless $\bar{x} \in \bar{\Omega}$. Let dy denote the Haar measure of Ξ_1 and choose $\gamma \in C_c^{\infty}(G)$ such that

$$\int_{\Xi Z} \gamma(xy)\, dy = 1$$

if $\bar{x} \in \bar{\Omega}$ $(x \in G)$. Then if dy is suitably normalized, we have

$$F_f(ah) = \varepsilon_R(ah)\Delta(ah) \int_G \gamma(x)\, dx \int_{\Xi^*} f(x(ah^{y^*})x^{-1}) dy^* \qquad (h \in B')$$

for $f \in C_c^\infty(G_0)$. Here $\Xi^* = \Xi/\Xi \cap A_0$. Now fix $f \in C_c^\infty(G_0)$ and put

$$g_0(y) = \int_G \gamma(x) f(x(ay)x^{-1})\, dx \qquad (y \in \Xi).$$

Then $g_0 \in C_c^\infty(\Xi)$.

We now use the notation of §19. Select an open and connected neighborhood $\mathfrak{h}_0$ of zero in $\mathfrak{h}$ such that $\exp \mathfrak{h}_0 \subset B$, $\xi_\alpha(a \exp H) \neq 1$ $(\alpha \in P')$ and

$$(1 - e^{-\alpha(H)})/\alpha(H) \neq 0 \qquad (\alpha \in P_{\mathfrak{z}})$$

for $H \in \mathfrak{h}_0$. Then if $\mathfrak{h}_0'$ is the set of all points $H \in \mathfrak{h}_0$, where $\pi_{\mathfrak{z}}(H) \neq 0$, it is clear that $\exp \mathfrak{h}_0' \subset B'$ and

$$F_f(a \exp H) = \varepsilon_R(a \exp H) \Delta(a \exp H) \int_{\Xi^*} g_0((\exp H)^{y^*}) dy^* \qquad (H \in \mathfrak{h}_0').$$

A simple argument shows (see §21) that there exists an analytic function D_a on $\mathfrak{z}$ such that: (1) D_a is invariant under Ξ and (2) $\Delta(a \exp H) = \pi_{\mathfrak{z}}(H) D_a(H)$ for $H \in \mathfrak{h}$. Fix an open and completely invariant neighborhood $\mathfrak{z}_0$ of zero in $\mathfrak{z}$ such that the exponential mapping (from $\mathfrak{z}$ to Ξ) is regular and univalent on $\mathfrak{z}_0$ and select a C^∞-function u on $\mathfrak{z}$ such that: (1) u is invariant under Ξ, (2) $\operatorname{Supp} u \subset \mathfrak{z}_0$, and (3) $u = 1$ around zero. This is possible (see §9 and [4(n), Corollary 1 of Lemma 45]). Now put

$$g(Z) = u(Z) D_a(Z) g_0(\exp Z) \qquad (Z \in \mathfrak{z}).$$

Then $g \in C_c^\infty(\mathfrak{z}_0)$. Since $\mathfrak{h}_0$ is connected and $\xi_\alpha(a \exp H) \neq 1$ for $\alpha \in P'$ and $H \in \mathfrak{h}_0$, it is clear that

$$\varepsilon_R(a \exp H) = \varepsilon_{\mathfrak{z},R}(H) \varepsilon_a \qquad (H \in \mathfrak{h}_0),$$

where

$$\varepsilon_{\mathfrak{z},R}(H) = \operatorname{sign} \prod_{\alpha \in P_Z \cap P_R} \alpha(H)$$

and ε_a is a constant. Therefore

$$F_f(a \exp H) = \varepsilon_a \varepsilon_{\mathfrak{z},R}(H) \pi_{\mathfrak{z}}(H) \int_{\Xi^*} g(y^* H)\, dy^* \qquad (H \in U \cap \mathfrak{h}_0'),$$

where U is an open neighborhood of zero in $\mathfrak{z}$ such that $u = 1$ on U. The second assertion of the lemma now follows by applying Theorem 1 of [4(m)] to $(\mathfrak{z}, \mathfrak{h})$ and g. Moreover, this obviously implies the first assertion.

COROLLARY. *ϖF_f can be extended to a continuous function on A.*

Since $\varpi^{s_\alpha} = -\varpi$ for every root α, this is an immediate consequence of Lemma 40.

23. **Statement of Theorem 3.** Define G_A as in §20. Since A_0 is abelian and A/A_0 is finite (see the proof of Lemma 36), the Haar measure dh of A is bi-invariant. We keep to the notation of §22.

LEMMA 41. *There exists a number $c > 0$ such that*

$$\int_G f(x)\,dx = c\int_A |\Delta(h)|^2\,dh\int_{G^*} f(h^{x^*})\,dx^*$$

for $f \in C_c(G_A)$.

We observe that

$$\det(\mathrm{Ad}(h)^{-1}-1)_{\mathfrak{g}/\mathfrak{h}} = (-1)^r\Delta(h)^2 \qquad (h \in A),$$

where r is the number of positive roots of $(\mathfrak{g},\mathfrak{h})$. From this our assertion follows in the usual way (see the proof of Lemma 36 and [4(c), p. 508]).

COROLLARY. *Let* $f \in C_c^{\infty}(G)$. *Then*

$$\int_A |\Delta(h)F_f(h)|\,dh \leqq c^{-1}\int_{G_A} |f(x)|\,dx.$$

This is obvious from the above lemma.

We now use the notation of §16. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{h}\cap\mathfrak{p}$ in $\mathfrak{g}$ and M the analytic subgroup of G corresponding to $\mathfrak{m}$. Let P_M be the set of all positive roots of $(\mathfrak{g},\mathfrak{h})$ which vanish identically on $\mathfrak{h}\cap\mathfrak{p}$. Then P_M is also the set of all positive imaginary roots of $(\mathfrak{g},\mathfrak{h})$ or, equivalently, the set of positive roots of $(\mathfrak{m},\mathfrak{h})$. Put

$$\Delta_M(h) = \xi_\rho(h_1)\prod_{\alpha\in P_M}(1-\xi_\alpha(h^{-1})) \qquad (h \in A),$$

where $h = h_1h_2$ $(h_1 \in A_K,\ h_2 \in A_{\mathfrak{p}})$. It follows from [4(h), Theorem 2] that

$$\int_A |\Delta_M(h)F_f(h)|\,dh < \infty \qquad (f \in C_c^{\infty}(G)).$$

THEOREM 3. *Let* ν *be a seminorm*(15) *on the complex vector space* $C_c^{\infty}(G)$ *and* $\mathfrak{Z}_0$ *a subalgebra of* $\mathfrak{Z}$ *containing* 1. *Assume that* $\mathfrak{Z}$ *is a finite module over* $\mathfrak{Z}_0$ *and*

$$\int_A |\Delta_M(h)F_f(h)|\,dh \leqq \nu(f) \qquad (f \in C_c^{\infty}(G)).$$

Then for any $u \in \mathfrak{S}(\mathfrak{h}_c)$, *we can choose a finite set of elements* $z_1,\cdots,z_N \in \mathfrak{Z}_0$ *such that*

$$\sup_{h\in A'} |F_f(h;u)| \leqq \sum_{1\leqq i\leqq N} \nu(z_if) \qquad (f \in C_c^{\infty}(G)).$$

REMARK. The above form of this theorem suggested itself to me after a conversation with R. P. Langlands. My original version was less comprehensive.

Let Z be the center of G and V a subset of A such that $VZ = A$. Put $V' = V\cap A'$. We claim that it would be sufficient to prove the following lemma for a conveniently chosen V.

LEMMA 42. *For any* $u \in \mathfrak{S}(\mathfrak{h}_c)$, *we can choose* $z_1,\cdots,z_N$ *in* $\mathfrak{Z}_0$ *such that*

(15) Here we ignore completely the topology of $C_c^{\infty}(G)$.

$$\sup_{h\in V'} |F_f(h;u)| \leqq \sum_{1\leqq i\leqq N} \nu(z_i f) \qquad (f \in C_c^{\infty}(G)).$$

Put:

$$\nu_0(f) = \int_A |\Delta_M(h)F_f(h)|\, dh \qquad (f \in C_c^{\infty}(G)).$$

Then ν_0 is a seminorm on $C_c^{\infty}(G)$ which satisfies the condition of Theorem 3. For any $y \in Z$, let f_y denote the function $x \to f(xy)$ on G. Then it is clear that $\xi_\rho(y)F_{f_y}(h) = F_f(hy)$ $(h \in A')$ and $\Delta_M(hy) = \xi_\rho(y)\Delta_M(h)$. Since $|\xi_\rho(y)| = 1$, it follows that $\nu_0(f_y) = \nu_0(f)$. Therefore if Lemma 42 holds for ν_0, we can conclude that

$$\begin{aligned}\sup_{h\in V'} |F_f(hy;u)| &= \sup_{h\in V'} |F_{f_y}(h;u)| \\ &\leqq \sum_{1\leqq i\leqq N} \nu_0(z_i f_y) = \sum_i \nu_0(z_i f) \qquad (y \in Z)\end{aligned}$$

since $z_i f_y = (z_i f)_y$. Now fix a seminorm ν as in Theorem 3. Then $\nu_0(g) \leqq \nu(g)$ for $g \in C_c^{\infty}(G)$. Hence

$$\sup_{h\in V'} |F_f(hy;u)| \leqq \sum_i \nu(z_i f) \qquad (y \in Z,\ f \in C_c^{\infty}(G)).$$

But since $V'Z = A'$, the assertion of Theorem 3 now follows immediately.

24. **Reduction to $\mathfrak{h}$ in a special case.** So now it remains to prove Lemma 42. First *assume that every root of* $(\mathfrak{g},\mathfrak{h})$ *is imaginary*. Then $M = G$ and it follows from Corollary 5 of Lemma 26 that A/Z is compact. So we can take V to be compact. Let $\mathscr{S}$ be the set of all seminorms σ on[15] $C_c^{\infty}(G)$ with the following property. We can choose a finite number of elements $z_1, \cdots, z_N$ in $\mathfrak{Z}_0$ such that

$$\sigma(f) \leqq \sum_{1\leqq i\leqq N} \nu(z_i f) \qquad (f \in C_c^{\infty}(G)).$$

Then since V is compact, it would obviously be enough to prove the following result.

LEMMA 43. *Given $h_0 \in A$, we can choose an open neighborhood U of h_0 in A with the following property. For any $u \in \mathfrak{S}(\mathfrak{h}_c)$, there exists an element $\sigma_u \in \mathscr{S}$ such that*

$$\sup_{h\in A'\cap U} |F_f(h;u)| \leqq \sigma_u(f) \quad (f \in C_c^{\infty}(G)).$$

Let $\mathfrak{c}$ be the center and $\mathfrak{g}_1$ the derived algebra of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{c} + \mathfrak{h}_1$, where $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$. Since every root of $(\mathfrak{g},\mathfrak{h})$ is imaginary, $-\mathrm{tr}(\mathrm{ad}\,H)^2$ $(H \in \mathfrak{h}_1)$ is a positive-definite quadratic form on $\mathfrak{h}_1$. We extend it to a positive-definite quadratic form Q on $\mathfrak{h}$ in such a way that $\mathfrak{c}$ and $\mathfrak{h}_1$ are orthogonal under Q and, moreover, regard $\mathfrak{h}$ as a real Hilbert space under the norm $\|H\|^2 = Q(H)$ $(H \in \mathfrak{h})$.

Let us now introduce the notation of §19 corresponding to $a = h_0$. Fix a number

c $(0 < c \leqq 1)$ and let V be the set of all $H \in \mathfrak{h}$ with $\|H\| < c$. We assume that c is so small that:

(1) $|(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\xi_\alpha(h_0^{-1}))| \geqq (1/2)|1 - \xi_\alpha(h_0^{-1})|$ for every root α of $(\mathfrak{g}, \mathfrak{h})$ and $H \in V$.

(2) The exponential mapping of V into A is univalent.

(3) $|\{(e^{\alpha(H)/2} - e^{-\alpha(H)/2})/\alpha(H)\}| \geqq 1/2$ for $\alpha \in P_{\mathfrak{z}}$ and $H \in V$.

Let $\mathfrak{h}'$ be the set of all $H \in \mathfrak{h}$ where $\pi_{\mathfrak{z}}(H) \neq 0$. Then $V' = V \cap \mathfrak{h}'$ consists of a finite number of connected components, say $V_1, \cdots, V_q$. Put

$$U = h_0 \exp(aV),$$

where a is a positive number $(0 < a \leqq 1)$. Then it is clear that

$$U \cap A' = \bigcup_{1 \leqq i \leqq q} U_i$$

where $U_i = h_0 \exp(aV_i)$. Then it would be sufficient to prove the following lemma.

Lemma 44. *Fix i $(1 \leqq i \leqq q)$. Then we can select a number a $(0 < a \leqq 1)$ with the following property. For any $u \in \mathfrak{S}(\mathfrak{h}_c)$ we can choose $\sigma \in \mathscr{S}$ such that*

$$\sup_{H \in aV_i} |F_f(h_0 \exp H; u)| \leqq \sigma(f) \qquad (f \in C_c^\infty(G)).$$

Put

$$\phi_f(H) = F_f(h_0 \exp H) \qquad (f \in C_c^\infty(G),\ H \in V').$$

Then it follows from [4(h), Theorem 3] that(10) $\phi_{zf} = \partial(\gamma(z))\phi_f$ for $z \in \mathfrak{Z}$. Moreover, it is obvious that Lemma 44 is equivalent to the following.

Lemma 45. *Fix i $(1 \leqq i \leqq q)$. Then we can select a $(0 < a \leqq 1)$ with the following property. For any $u \in S(\mathfrak{h}_c)$, we can choose $\sigma \in \mathscr{S}$ such that*

$$\sup_{H \in aV_i} |\phi_f(H; \partial(u))| \leqq \sigma(f)$$

for all $f \in C_c^\infty(G)$.

We may assume that $i = 1$. Let L be the rank of $\mathfrak{z}_1 = [\mathfrak{z}, \mathfrak{z}]$. Then we can choose L roots $\alpha_1, \cdots, \alpha_L$ of $(\mathfrak{z}, \mathfrak{h})$ with the following property. If α is a root of $(\mathfrak{z}, \mathfrak{h})$ such that $(-1)^{1/2}\alpha(H) > 0$ for $H \in V_1$, then $\alpha = \sum_{1 \leqq i \leqq L} m_i \alpha_i$, where m_i are rational integers $\geqq 0$. Put $t_i(H) = (-1)^{1/2}\alpha_i(H)$ $(1 \leqq i \leqq L,\ H \in \mathfrak{h})$ and choose a base H_j $(1 \leqq j \leqq l_1)$ for $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$ such that $t_i(H_j) = \delta_{ij}$ $(1 \leqq i \leqq L,\ 1 \leqq j \leqq l_1)$. Let H_j $(l_1 < j \leqq l)$ be an orthonormal base for $\mathfrak{c}$. Extend $(t_1, \cdots, t_L)$ to a Cartesian coordinate system $(t_1, \cdots, t_l)$ on $\mathfrak{h}$ by defining $t_i(H_j) = \delta_{ij}$ $(1 \leqq i, j \leqq l)$. Then a point $H \in V$ lies in V_1 if and only if $t_i(H) > 0$ $(1 \leqq i \leqq L)$. Define

$$\tau(H) = \begin{cases} \min_{1 \leqq i \leqq L} |t_i(H)| & \text{if } L > 0, \\ c & \text{if } L = 0 \qquad (H \in \mathfrak{h}). \end{cases}$$

Clearly $\|H\|^2 \geqq 2|\alpha(H)|^2$ for any root α of $(\mathfrak{g}, \mathfrak{h})$. Hence if $H_0 \in V_1/2$ and $\|H - H_0\| \leqq \tau(H_0)/2$ $(H \in \mathfrak{h})$, it is clear that $\|H\| \leqq \|H_0\| + \tau(H_0)/2 < c$. Therefore $H \in V$. Moreover,

$$|t_i(H - H_0)| = |\alpha_i(H - H_0)| \leqq \|H - H_0\| \leqq \frac{1}{2}\tau(H_0) \leqq \frac{1}{2}t_i(H_0) \qquad (1 \leqq i \leqq L).$$

Therefore $t_i(H) \geqq \tau(H_0)/2$ $(1 \leqq i \leqq L)$ and so $H \in V_1$. This also shows that $\tau(H) \geqq \tau(H_0)/2$. Thus we have obtained the following result.

LEMMA 46. *Fix $H_0 \in V_1/2$ and let H be an element in $\mathfrak{h}$ such that*

$$\|H - H_0\| \leqq \tau(H_0)/2.$$

Then $H \in V_1$ and $\tau(H) \geqq \tau(H_0)/2$.

Fix a function ψ on $\boldsymbol{R}$ of class C^∞ such that $\psi = 1$ on the interval $(-\infty, 0]$, $\psi = 0$ on the interval $[1, +\infty)$ and $0 \leqq \psi \leqq 1$ everywhere.

LEMMA 47. *For any real number ε $(0 < \varepsilon \leqq 1/2)$ define*

$$\Psi_\varepsilon(H) = \psi(\varepsilon^{-1}\|H\| - 2) \qquad (H \in \mathfrak{h}).$$

Then for any element $u \in S(\mathfrak{h}_c)$ of degree $\leqq d$, we can choose a number $b > 0$ such that

$$|\Psi_\varepsilon(H; \partial(u))| \leqq b\varepsilon^{-d}$$

for all $H \in \mathfrak{h}$ and $0 < \varepsilon \leqq 1/2$.

This is an immediate consequence of Lemma 55 of the Appendix (see also [6, p. 281]). Observe that $\Psi_\varepsilon(H) = 0$ unless $\|H\| \leqq 3\varepsilon$.

25. **Proof of a weaker result.** Now first we prove the following weaker form (16) of Lemma 45.

LEMMA 48. *Given $u \in S(\mathfrak{h}_c)$, we can choose an integer $q \geqq 0$ and $\sigma \in \mathscr{S}$ such that*

$$\sup_{H \in V_1/2} \left\{ \prod_{1 \leqq i \leqq L} t_i(H) \right\}^q |\phi_f(H; \partial(u))| \leqq \sigma(f)$$

for all $f \in C_c^\infty(G)$.

Let $\omega_1 \in \mathfrak{Z}$ be the Casimir operator (see [4(e), p. 140]) corresponding to $\mathfrak{g}_1$. Put

$$\omega = \omega_1 - \sum_{l_1 < j \leqq l} H_j^2 \in \mathfrak{Z}.$$

Then it is easy to verify (see [4(e), p. 144]) that (17) $\gamma(\omega) + \langle\rho, \rho\rangle$ is homo-

(16) Cf. [4(g), p. 206].

(17) $\langle\rho, \rho\rangle = \rho(H_\rho)$, where H_ρ is the unique element in $\mathfrak{h}_{1c}$ such that $\mathrm{tr}(\mathrm{ad}H\,\mathrm{ad}H_\rho) = \rho(H)$ for all $H \in \mathfrak{h}$.

geneous of degree 2 and $D = \partial(\gamma(\omega)) + <\rho, \rho>$ is an elliptic differential operator on $\mathfrak{h}$.

Now $\mathfrak{Z}$ being a finite module over $\mathfrak{Z}_0$, we can choose $v_1 = 1, v_2, \cdots, v_r$ in $\mathfrak{Z}$ such that

$$\mathfrak{Z} = \sum_{1 \leqq i \leqq r} \mathfrak{Z}_0 v_i.$$

Fix an integer $m \geqq 1$. Then we have an equation of the form

$$\omega_0^{mr} + \sum_{1 \leqq j \leqq r} z_j \omega_0^{m(r-j)} = 0,$$

where $\omega_0 = \omega + <\rho, \rho>$ and $z_j \in \mathfrak{Z}_0$.

For the proof of Lemma 48, we may obviously assume that $u \neq 0$. Let d be the degree of u. Fix a Euclidean measure dH on $\mathfrak{h}$ such that dH corresponds, locally, to the Haar measure dh on A, under the exponential mapping. Then if m is sufficiently large, there exists a function E_0 on $\mathfrak{h}$ of class $C^{2m(r-1)+d}$ such that

$$D^{mr} E_0 = \delta$$

in the sense of the theory of distributions on the Euclidean space $\mathfrak{h}$ (with respect to the measure dH). Here δ is the Dirac measure on $\mathfrak{h}$ concentrated at zero and E_0 is of class C^∞ everywhere except at the origin (see Lemma 57, §29). Put $E = \partial(u)^* E_0$, where the star denotes adjoint. It follows by applying the homomorphism γ to the relation above that

$$D^{mr} + \sum_{1 \leqq j \leqq r} \partial(\gamma(z_j)) D^{m(r-j)} = 0.$$

Since $D^* = D$, we find, by taking adjoints, that

$$D^{mr} + \sum_{1 \leqq j \leqq r} \partial(\gamma(z_j))^* D^{m(r-j)} = 0.$$

Put $E_j = -D^{m(r-j)} E$ $(1 \leqq j \leqq r)$. Then E_j is a function of class $C^{2m(j-1)}$ and

$$\partial(u)^* \delta = \sum_{1 \leqq j \leqq r} \partial(\gamma(z_j))^* E_j.$$

Clearly E_j is of class C^∞ everywhere except at zero. Put $E_{j,\varepsilon} = \Psi_\varepsilon E_j$ for any ε $(0 < \varepsilon \leqq 1/3)$ in the notation of Lemma 47. Then it is clear that

$$\sum_{1 \leqq j \leqq r} \partial(\gamma(z_j))^* E_{j,\varepsilon} = \partial(u)^* \delta + \beta_\varepsilon,$$

where $\beta_\varepsilon \in C_c^\infty(\mathfrak{h})$ and $\operatorname{Supp} \beta_\varepsilon \subset \operatorname{Supp} \Psi_\varepsilon$. Now $\Psi_\varepsilon(H) = 1$ if $\|H\| \leqq 2\varepsilon$. Hence $\beta_\varepsilon(H) = 0$ unless $2\varepsilon \leqq \|H\| \leqq 3\varepsilon$.
Therefore

$$\sup_H |\beta_\varepsilon(H)| = \sup_{2\varepsilon \leqq \|H\| \leqq 3\varepsilon} |\beta_\varepsilon(H)|.$$

Making use of Lemma 47 and the explicit formula for E_0 (see §29), we find that

$$\sup_H |\beta_\varepsilon(H)| \leqq b_1\varepsilon^{-p+1}|\log\varepsilon| \leqq b_2\varepsilon^{-p}$$

where b_1, b_2 are positive numbers and p an integer $\geqq 0$, all independent of ε $(0 < \varepsilon \leqq 1/3)$.

Now fix $H_0 \in V_1/2$ and put $\varepsilon_0 = \tau(H_0)/6$, $E_{j,H_0} = E_{j,\varepsilon_0}$ and $\beta_{H_0} = \beta_{\varepsilon_0}$ $(1 \leqq j \leqq r)$. Then

$$\sum_{1\leqq i\leqq r} \partial(\gamma(z_i))^* E_{i,H_0} = \partial(u)^*\delta + \beta_{H_0},$$

$$\sup |\beta_{H_0}| \leqq b_3\tau(H_0)^{-p},$$

where $b_3 = 6^p b_2$. Now $\operatorname{Supp} E_{i,H_0}$ and $\operatorname{Supp}\beta_{H_0}$ are both contained in $\operatorname{Supp}\Psi_{\varepsilon_0}$. Moreover, $\|H\| \leqq \tau(H_0)/2$ if $H \in \operatorname{Supp}\Psi_{\varepsilon_0}$. Hence if $H - H_0 \in \operatorname{Supp}\Psi_{\varepsilon_0}$, it follows from Lemma 46 that $H \in V_1$ and $\tau(H) \geqq \tau(H_0)/2$. Let $V(H_0)$ be the set of all $H \in V$ such that $\tau(H) \geqq \tau(H_0)/2$. Then it is clear that $V(H_0) \subset V_1$ and

$$\phi_f(H_0;\, \partial(u)) = \sum_{1\leqq i\leqq r} \int_{V(H_0)} \phi_f(H;\partial(\gamma(z_i)))E_{i,H_0}(H - H_0)\, dH$$
$$- \int_{V(H_0)} \phi_f(H)\beta_{H_0}(H - H_0)\, dH.$$

On the other hand it follows from the definition of V (see §24) that we can choose a number $c_1 > 0$ such that

$$|\Delta(H_0 \exp H)| \geqq c_1|\pi_3(H)| \qquad (H \in V).$$

Let q_1 be the number of roots in P_3. Then it follows from our definition of $t_1,\cdots,t_L$ that

$$|\pi_3(H)| \geqq \tau(H)^{q_1} \qquad (H \in V_1).$$

Therefore

$$|\Delta(h_0 \exp H)| \geqq c_1\tau(H)^{q_1} \qquad (H \in V_1).$$

Hence

$$\tau(H_0)^{q_1}\left|\int_{V(H_0)} \phi_f(H;\partial(\gamma(z_i)))\, E_{i,H_0}(H - H_0)\, dH\right|$$
$$\leqq c_2 \int_{V(H_0)} |\phi_{z_if}(H)\Delta(h_0\exp H)|\,|E_{i,H_0}(H - H_0)|\, dH,$$

where $c_2 = 2^{q_1}c_1^{-1}$. Moreover, since $|\Psi_\varepsilon| \leqq 1$ and E_i are continuous functions on $\mathfrak{h}$, it is clear that

$$\sup_{H\in V(H_0)} |E_{i,H_0}(H - H_0)| \leqq \sup_{\|H\|\leqq 2} |E_i(H)| \leqq c_3,$$

where c_3 is a positive number independent of H_0 or i. Hence

$$\tau(H_0)^{q_1} \left| \int_{V(H_0)} \phi_f(H;\partial(\gamma(z_i)))E_{i,H_0}(H-H_0)\,dH \right| \leqq c_2c_3\nu(z_if) \qquad (1 \leqq i \leqq r).$$

Similarly since $\sup|\beta_{H_0}| \leqq b_3\tau(H_0)^{-p}$, we get

$$|\tau(H_0)|^{p+q_1} \left| \int_{V(H_0)} \phi_f(H)\beta_{H_0}(H-H_0)\,dH \right| \leqq c_2b_3\nu(f).$$

Moreover, $\tau(H_0) \leqq c \leqq 1$. Hence

$$\tau(H_0)^{p+q_1}|\phi_f(H_0;\ \partial(u))| \leqq c_2b_3\nu(f) + c_2c_3 \sum_{1\leqq i\leqq r} \nu(z_if)$$

for $H_0 \in V_1/2$ and $f \in C_c^{\infty}(G)$. Now

$$\prod_{1\leqq i\leqq L} t_i(H) \leqq c^{L-1}\tau(H) \qquad (H \in V_1).$$

This is obvious if $L \geqq 1$ and is also true if $L = 0$. Therefore the statement of Lemma 48 follows immediately if we take $q = p + q_1$.

26. **Proof of Lemma 45**(18). Now we come to the proof of Lemma 45. If $L = 0$, it is an immediate consequence of Lemma 48. So we may assume that $L \geqq 1$.

By a monomial T we mean a function on $\mathfrak{h}$ of the form $t_1^{q_1}t_2^{q_2}\cdots t_L^{q_L}$, where $q_1,\cdots,q_L$ are integers $\geqq 0$. The degree of T is the integer $q_1+q_2+\cdots+q_L$ and we denote it by d^0T. Since $S(\mathfrak{h}_c)$ is a finite module over $I(\mathfrak{h}_c)=\gamma(\mathfrak{Z})$ (see [4(f), Lemma 11]), it is also a finite module over $\gamma(\mathfrak{Z}_0)$. Hence we can choose u_j $(1 \leqq j \leqq r)$ in $S(\mathfrak{h}_c)$ such that $u_1 = 1$ and

$$S(\mathfrak{h}_c) = \sum_{1\leqq j\leqq r} \gamma(\mathfrak{Z}_0)u_j.$$

We say that a monomial T has property (P) if there exists a number $a = a(T)$ $(0 < a \leqq 1)$ and $\sigma \in \mathscr{S}$ such that

$$\text{(P)} \qquad \sup_{H\in aV_1} T(H)|\phi_f(H;\partial(u_j))| \leqq \sigma(f) \qquad (1 \leqq j \leqq r)$$

for all $f \in C_c^{\infty}(G)$. Now suppose T has property (P) and put

$$\sigma_T(f) = \max_{1\leqq j\leqq r} \ \sup_{H\in aV_1} \ T(H)|\phi_f(H;\partial(u_j))| \qquad (f \in C_c^{\infty}(G)),$$

where $a = a(T)$. Then it is obvious that $\sigma_T \in \mathscr{S}$ and, for a given $u \in S(\mathfrak{h}_c)$, we can select $z_i \in \mathfrak{Z}_0$ $(1 \leqq i \leqq r)$ such that $u = \sum_{1\leqq i\leqq r}\gamma(z_i)u_i$. Hence

$$\phi_f(H;\partial(u)) = \sum_i \phi_{z_if}(H;\partial(u_i)) \qquad (H \in V',\ f \in C_c^{\infty}(G)).$$

(18) Cf. [4(g), pp. 208–211].

So it is clear that

$$\sigma_{T,u}(f) = \sup_{H \in aV_1} T(H)\left|\phi_f(H;\partial(u))\right| \leqq \sum_{1 \leqq j \leqq r} \sigma_T(z_j f)$$

for all $f \in C_c^\infty(G)$ and therefore $\sigma_{T,u} \in \mathscr{S}$.

Hence, in order to prove Lemma 45, it is obviously enough to obtain the following result.

LEMMA 49. *The monomial* 1 *has property* (P).

It is clear from Lemma 48 that monomials with property (P) actually do exist. Let T be a monomial with property (P) of the lowest possible degree. We claim that $T = 1$. For otherwise suppose $d^0 T > 0$. Then, without loss of generality, we may assume that $T = t_1^{q_1} t_2^{q_2} \cdots t_L^{q_L}$ and $q_1 \geqq 1$. Put $T_2 = t_2^{q_2} \cdots t_L^{q_L}$ so that $T = t_1^{q_1} T_2$ and $d^0 T_2 < d^0 T$. Let $a = a(T)$ and, for any $f \in C_c^\infty(G)$, put

$$\psi_{f,i}(H) = T_2(H)\phi_f(H;\partial(u_i)) \qquad (H \in aV_1,\ 1 \leqq i \leqq r).$$

We recall that $H_1, \cdots, H_l$ is a base for $\mathfrak{h}$ over $\boldsymbol{R}$ such that $t_i(H_j) = \delta_{ij}$ $(1 \leqq i, j \leqq l)$. Now choose $z_{ij} \in \mathfrak{Z}_0$ $(1 \leqq i, j \leqq r)$ such that

$$H_1 u_i = \sum_{1 \leqq j \leqq r} \gamma(z_{ij}) u_j \qquad (1 \leqq i \leqq r).$$

Then

$$\partial\psi_{f,i}/\partial t_1 = \sum_j \psi_{z_{ij}f,j}$$

on aV_1 and therefore

$$\left| t_1^{q_1}(\partial\psi_{f,i}/\partial t_1)\right| \leqq \sum_j \sigma_T(z_{ij}f) \qquad (f \in C_c^\infty(G))$$

on aV_1. Here

$$\sigma_T(f) = \max_j \sup_{H \in aV_1} T(H)\left|\phi_f(H;\partial(u_j))\right| \qquad (f \in C_c^\infty(G))$$

and it is clear that $\sigma_T \in \mathscr{S}$ since T has property (P). Put

$$\sigma(g) = \sum_{1 \leqq i, j \leqq r} \sigma_T(z_{ij}g) \qquad (g \in C_c^\infty(G)).$$

Then σ also lies in $\mathscr{S}$.

For any $b > 0$, let W_b denote the set of all $H \in \mathfrak{h}$ such that $\left|t_i(H)\right| \leqq b$ $(1 \leqq i \leqq l)$. Choose a_1 $(0 < a_1 \leqq 1)$ so small that $W_{a_1} \subset aV$ and a_2 $(0 < a_2 \leqq a_1)$ such that $a_2 V \subset W_{a_1}$. Suppose $H \in a_2 V_1$. Then $H' = H + (a_1 - t_1(H))H_1 \in W_{a_1} \subset aV$. Since $t_i(H) > 0$ $(1 \leqq i \leqq L)$ and $t_1(H) \leqq \|H\| < a_2 c \leqq a_2 \leqq a_1$, it follows that $t_i(H') > 0$ $(1 \leqq i \leqq L)$ and therefore $H' \in aV_1$. But aV_1 being convex, the whole line segment joining H to H' lies in aV_1. On the other hand, we have seen above that

$$\left|(\partial\psi_{f,i}/\partial t_1)\right| \leqq t_1^{-q_1}\sigma(f) \qquad (f \in C_c^\infty(G))$$

on aV_1. Hence, by integrating on this line segment, we get

$$|\psi_{f,i}(H') - \psi_{f,i}(H)| \leqq \sigma(f)\int_{t_1(H)}^{a_1} s^{-q_1}ds$$

since $t_1(H') = a_1$. Moreover,

$$|t_1(H')^{q_1}\psi_{f,i}(H')| = T(H')|\phi_f(H';\partial(u_i))| \leqq \sigma_T(f).$$

Therefore

$$|\psi_{f,i}(H')| \leqq a_1^{-q_1}\sigma_T(f).$$

This shows that

$$|\psi_{f,i}(H)| \leqq a_1^{-q_1}\,\sigma_T(f) + \sigma(f)\int_{t_1(H)}^{a_1} s^{-q_1}ds$$

for $H \in a_2V_1$ and $f \in C_c^{\infty}(G)$.

Now first suppose that $q_1 \geqq 2$. Then

$$\int_{t_1(H)}^{a_1} s^{-q_1}ds = (q_1-1)^{-1}(t_1(H)^{1-q_1} - a_1^{1-q_1}).$$

Hence if $T_1 = t_1^{q_1-1}T_2$, it is clear that

$$|T_1(H)\phi_f(H;\partial(u_i))| \leqq a_1^{-q_1}\sigma_T(f) + \sigma(f)$$

for $H \in a_2V_1$, $f \in C_c^{\infty}(G)$ and $1 \leqq i \leqq r$. This shows that T_1 has property (P). But since $d^0T_1 = d^0T - 1 < d^0T$, this gives a contradiction. So the case $q_1 \geqq 2$ is impossible.

Hence $q_1 = 1$. Then

$$\int_{t_1(H)}^{a_1} s^{-1}ds = \log(a_1/t_1(H))$$

and therefore

$$|\psi_{f,i}(H)| \leqq a_1^{-1}\sigma_T(f) + \sigma(f)\log(a_1/t_1(H)) \qquad (1 \leqq i \leqq r)$$

for $H \in a_2V_1$ and $f \in C_c^{\infty}(G)$. Put

$$\sigma'(g) = \sigma(g) + a_1^{-1}\,\sigma_T(g) \qquad (g \in C_c^{\infty}(G)).$$

Then $\sigma' \in \mathcal{S}$ and

$$|\psi_{f,i}| \leqq \sigma'(f)\{1 + \log(a_1/t_1)\}$$

on a_2V_1. Put

$$\sigma_2(g) = \sum_{1 \leqq i,\ \leqq r} \sigma'(z_{ij}g) \qquad (g \in C_c^{\infty}(G)).$$

Then $\sigma_2 \in \mathscr{S}$ and since

$$\partial\psi_{f,i}/\partial t_1 = \sum_j \psi_{z_{ij}f,j}$$

we conclude that

$$\left|(\partial\psi_{f,i}/\partial t_1)\right| \leqq \sigma_2(f)\{1+\log(a_1 t_1^{-1})\}$$

on $a_2 V_1$. Now choose numbers a_3, a_4 $(0 < a_4 \leqq a_3 \leqq a_2)$ such that $W_{a_3} \subset a_2 V$ and $a_4 V \subset W_{a_3}$. For any $H \in a_4 V_1$, define

$$H'' = H + (a_3 - t_1(H_1))H_1.$$

Then $H'' \in a_2 V_1$ and so again by integrating along the line segment joining H and H'', we conclude that

$$\begin{aligned}\left|\psi_{f,i}(H'') - \psi_{f,i}(H)\right| &\leqq \sigma_2(f)\int_{t_1(H)}^{a_3} (1+\log(a_1 s^{-1}))\,ds \\ &\leqq b\sigma_2(f)\end{aligned}$$

for $H \in a_4 V_1$, $f \in C_c^{\infty}(G)$ and $1 \leqq i \leqq r$. Here

$$b = \int_0^{a_3} (1+\log(a_1 s^{-1}))\,ds < \infty.$$

Now $t_1(H'') = a_3$ and

$$t_1(H'')\left|\psi_{f,i}(H'')\right| \leqq \sigma_T(f)$$

since $T = t_1 T_2$. Therefore

$$\left|\psi_{f,i}(H)\right| \leqq a_3^{-1}\sigma_T(f) + b\sigma_2(f) \qquad (1 \leqq i \leqq r)$$

for $H \in a_4 V_1$ and $f \in C_c^{\infty}(G)$. This shows that T_2 has property (P) and therefore again, since $d^0 T_2 = d^0 T - 1 < d^0 T$, we get a contradiction. This proves Lemma 49 and hence also Lemma 45.

27. **Proof of Theorem 3 in the general case.** Now we come to the general case and use of the notation of §16. Let ${}_0M$ be the centralizer of $\mathfrak{h} \cap \mathfrak{p}$ in G and M the connected component of 1 in ${}_0M$. Then $A \subset {}_0M$ and, by Lemma 30, M is acceptable.

Let G_c be a complexification of G and define j as in §18. Put

$$\Phi_0 = j(K) \cap \exp((-1)^{1/2}(\mathfrak{h} \cap \mathfrak{p})).$$

LEMMA 50. *Φ_0 is a finite group. Let Φ be a finite subset of G such that $j(\Phi) = \Phi_0$. Then $A = \Phi A^0 Z$, where A^0 is the connected component of* 1 *in A.*

Since $\mathfrak{h} \cap \mathfrak{p} \subset \mathfrak{g}_1$, we may obviously assume, for the proof of this lemma, that $\mathfrak{g}$ is semisimple and G_c is simply connected. Then $j(K)$ is compact and

$\Phi_0 \subset j(A) \cap j(K) = j(A_K)$. Extend θ to a complex-analytic automorphism of G_c. Then since $\theta = -1$ on $\mathfrak{p}$, it is clear that $a^2 = 1$ for every $a \in \Phi_0$. Therefore since $j(A_K)$ is a compact abelian group, it follows that Φ_0 is finite.

Now $A = A_K A_{\mathfrak{p}}$ and $A_{\mathfrak{p}} \subset A^0$ (see Corollary 4 of Lemma 26). Hence, in order to prove the second statement, it would be enough to verify that $j(A_K) = \Phi_0 j(A_K{}^0)$, where $A_K{}^0$ is the analytic subgroup of G corresponding to $\mathfrak{h} \cap \mathfrak{k}$.

Let A_c be the Cartan subgroup of G_c corresponding to $\mathfrak{h}_c$. Put $\mathfrak{u} = \mathfrak{k} + (-1)^{1/2}\mathfrak{p}$ and and define U and η as in §17. Then if $a \in j(A_K)$, it is clear that $a \in U \cap A_c = \exp(\mathfrak{h}_c \cap \mathfrak{u})$. But $\mathfrak{h}_c \cap \mathfrak{u} = \mathfrak{h} \cap \mathfrak{k} + (-1)^{1/2}(\mathfrak{h} \cap \mathfrak{p})$ and therefore $a = a_1 a_2$, where $a_1 \in j(A_K{}^0)$ and $a_2 \in \Phi_0$. This proves that $j(A_K) = \Phi_0 j(A_K{}^0)$.

LEMMA 51. *Let $a \in \Phi$ and $m \in M$. Then a and m commute.*

Since M is connected, this follows from the fact that its Lie algebra $\mathfrak{m}$ commutes with $\mathfrak{h} \cap \mathfrak{p}$.

Fix an order in the space of (real-valued) linear functions λ on $\mathfrak{h} \cap \mathfrak{p}$ and, for any such λ, let $\mathfrak{g}_\lambda$ denote the space of all $X \in \mathfrak{g}$ such that $[H, X] = \lambda(H)X$ for all $H \in \mathfrak{h} \cap \mathfrak{p}$. Put $\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_\lambda$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. Let N be the analytic subgroup of G corresponding to $\mathfrak{n}$. It is clear that ${}_0M$ normalizes $\mathfrak{n}$. Put

$$d(m) = \left|\det(\mathrm{Ad}(m))_{\mathfrak{n}}\right|^{1/2} \qquad (m \in {}_0M),$$

where the subscript $\mathfrak{n}$ denotes restriction on $\mathfrak{n}$. Put $G_0 = \mathrm{Ad}(G)$ and let K_0 denote the image of K in G_0 under the homomorphism $x \to \mathrm{Ad}(x)$. Then K_0 is compact. For any $x \in G$ and $y_0 \in G_0$, define $x^{y_0} = yxy^{-1}$ where y is any element of G such that $y_0 = \mathrm{Ad}(y)$. Put

$$\bar{f}(x) = \int_{K_0} f(x^{k_0})\, dk_0, \quad g_f(m) = d(m) \int_N \bar{f}(mn)\, dn$$

for $f \in C_c{}^\infty(G)$, $x \in G$ and $m \in {}_0M$. Here dk_0 and dn are the Haar measures on K_0 and N, respectively, and $\int_{K_0} dk_0 = 1$.

Introduce an order on the space of real-valued linear functions on $(-1)^{1/2}(\mathfrak{h} \cap \mathfrak{k}) + \mathfrak{h} \cap \mathfrak{p}$ which is compatible (see [4(g), p. 195]) with the one already chosen above. We may assume, without loss of generality, that the set P of positive roots of $(\mathfrak{g}, \mathfrak{h})$ is defined with respect to this order. Since every root of $(\mathfrak{m}, \mathfrak{h})$ is imaginary, it follows from Corollary 5 of Lemma 26 that $A \cap M = A^0$. Let $m \to m^*$ denote the natural projection of M on $M^* = M/A^0$ and define

$$F_g{}^M(h) = \Delta_M(h) \int_{M^*} g(h^{m^*})\, dm^* \qquad (h \in A^0 \cap M'),$$

$$\nu_M(g) = \int_{A^0} \left|\Delta_M(h) F_g{}^M(h)\right| dh \qquad (g \in C_c{}^\infty(M)),$$

where dm^* is the invariant measure on M^* and M' is the set of those elements of M which are regular in M.

Let $\mathfrak{M}$ be the subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{m}_c)$ and $\mathfrak{Z}_M$ the center of $\mathfrak{M}$. Then we have the isomorphism $\mu = \mu_{\mathfrak{g}/\mathfrak{m}}$ of $\mathfrak{Z}$ into $\mathfrak{Z}_M$ (see §12). Moreover, $G = KMN$ from [4(g), Lemma 11].

Lemma 52. *For any $a \in \Phi$, put*

$$g_{f,a}(m) = g_f(am) \qquad (m \in M,\ f \in C_c^{\infty}(G)).$$

Then $g_{f,a} \in C_c^{\infty}(M)$ and

$$g_{zf,a} = \mu(z) g_{f,a} \qquad (z \in \mathfrak{Z}).$$

Moreover, if dm^* and dn are suitably normalized, we have the relation

$$F_f(ah) = \xi_\rho(a) F_{g_{f,a}}{}^M(h)$$

for $f \in C_c^{\infty}(G)$, $h \in A^0 \cap (a^{-1}G')$ and $a \in \Phi$.

Although the proof of this lemma is not difficult, it is rather long. Hence we postpone it to another paper.

We can now complete the proof of Theorem 3. It is clear that

$$\Delta_M(ah) = \xi_\rho(a)\Delta_M(h) \qquad (a \in \Phi,\ h \in A).$$

Hence we conclude from Lemma 52 that

$$\nu_M(g_{f,a}) = \int_{A^0} \left| \Delta_M(h) F_f(ah) \right| dh \leqq \nu(f).$$

We have seen (Lemma 30) that M is acceptable and every root of $(\mathfrak{m}, \mathfrak{h})$ is imaginary. Moreover, $\mathfrak{Z}_M$ is a finite module over $\mu(\mathfrak{Z}_0)$ by Lemma 21. Hence Theorem 3 holds for $(M, A^0, \mu(\mathfrak{Z}_0), \nu_M)$ in place of $(G, A, \mathfrak{Z}_0, \nu)$. Therefore for any $u \in \mathfrak{S}(\mathfrak{h}_c)$, we can, in view of Lemma 52, choose a finite set of elements $z_1, \cdots, z_r \in \mathfrak{Z}_0$ such that

$$\sup_{h \in A'} \left| F_f(h; u) \right| \leqq \max_{a \in \Phi} \sum_{1 \leqq i \leqq r} \nu_M(\mu(z_i) g_{f,a}) \leqq \sum_{1 \leqq i \leqq r} \nu(z_i f)$$

for $f \in C_c^{\infty}(G)$. This proves Theorem 3.

28. **The local summability of $|D|^{-1/2}$.** Let $l = \operatorname{rank} G$ and put $D = D_l$ in the notation of §3. Then D is an analytic function on G and

$$D(h) = \det(1 - \operatorname{Ad}(h))_{\mathfrak{g}/\mathfrak{h}} = (-1)^p \Delta(h)^2 \qquad (h \in A),$$

where p is the number of positive roots of $(\mathfrak{g}, \mathfrak{h})$.

Lemma 53. *$|D|^{-1/2}$ is locally summable on G.*

Let C be a compact subset of G. Then we can choose $f \in C_c^{\infty}(G)$ such that $f \geqq 0$ everywhere and $f \geqq 1$ on C. Then it is clear that

$$\int_C |D|^{-1/2}\,dx \leqq \int_G |D|^{-1/2} f\,dx.$$

Let A_i $(1 \leqq i \leqq r)$ be a maximal set of Cartan subgroups of G no two of which are conjugate under G. Put

$$G_i = \bigcup_{x \in G} x A_i' x^{-1},$$

where $A_i' = A_i \cap G'$. Then G' is the disjoint union of $G_1, \cdots, G_r$ (see [4(e), Lemma 5]). Hence it would be enough to verify that

$$\int_{G_i} |D|^{-1/2} f\,dx < \infty \qquad (1 \leqq i \leqq r).$$

So fix i and put $A = A_i$. Then $G_i = G_A$ in the notation of §23 and it follows from Lemma 41 and [4(h), Theorem 2] that

$$\int_{G_A} |D|^{-1/2} f\,dx = c\int_A |F_f(h)|\,dh < \infty.$$

This proves the lemma.

29. **Appendix.** Put $\rho(x) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \geqq 0$ for $x \in \boldsymbol{R}^n$.

LEMMA 54. *Let α be a real number and $D = \partial^k/\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}$. Then*

$$D\rho^{\alpha} = \sum_{0 \leqq j \leqq k} p_j \rho^{\alpha-j-k},$$

$$D(\rho^{\alpha}\log\rho) = \sum_{0 \leqq j \leqq k} P_j\rho^{\alpha-j-k} + (\log\rho)\sum_{0 \leqq j \leqq k} Q_j\rho^{\alpha-j-k},$$

where p_j, P_j and Q_j are homogeneous polynomials in $(x_1, \cdots, x_n)$ of degree j.

This follows by an easy induction on k.

COROLLARY 1. *If $\alpha > k$, then ρ^{α} and $\rho^{\alpha}\log\rho$ are functions of class C^k on $\boldsymbol{R}^n$.*

This is obvious from the lemma.

COROLLARY 2. *$\rho^{k-1}D\rho$ remains bounded on $\boldsymbol{R}^n$.*

We know that

$$\rho^{k-1}D\rho = \sum_{0 \leqq j \leqq k} p_j\rho^{-j},$$

where p_j is a homogeneous polynomial of degree j in $(x_1, \cdots, x_n)$. Our assertion therefore follows from the obvious fact that $|p_j|\rho^{-j}$ is bounded on $\boldsymbol{R}^n$.

The following lemma is implicit in the paper of Morrey and Nirenberg [6, p. 281].

LEMMA 55. *Let h be a function in $C^\infty(\boldsymbol{R})$ which is constant on the intervals $(-\infty, 0]$ and $[1, +\infty)$. Choose two numbers r, δ such that $0 < \delta \leqq r \leqq 1$ and put*

$$H_{r,\delta}(x) = h(\delta^{-1}(\rho(x) - r)) \qquad (x \in \boldsymbol{R}^n).$$

Then for each integer $k \geqq 0$, there exists a number $c_k \geqq 0$, independent of r and δ, such that

$$|DH_{r,\delta}| \leqq c_k \delta^{-k}$$

for $D = \partial^k / \partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}$ $(1 \leqq i_1, \cdots, i_k \leqq n)$.

We use induction on k. If $k = 0$, we can take $c_0 = \sup |h|$. So let us assume that $k \geqq 1$ and put $h'(t) = dh/dt$ $(t \in \boldsymbol{R})$. Then h' also satisfies the conditions of the lemma and

$$\partial H_{r,\delta} / \partial x_i = \delta^{-1} H_{r,\delta}{}' \cdot \partial \rho / \partial x_i \qquad (1 \leqq i \leqq n),$$

where

$$H_{r,\delta}{}'(x) = h'(\delta^{-1}(\rho(x) - r)) \qquad (x \in \boldsymbol{R}^n).$$

Now $H_{r,\delta}{}'(x) = 0$ unless $r \leqq \rho(x) \leqq r + \delta$ and therefore

$$\sup_x |DH_{r,\delta}| \leqq \delta^{-1} \sup_{\rho \geqq \delta} |D'(H_{r,\delta}{}' \cdot \partial \rho / \partial x_{i_k})|,$$

where $D' = \partial^{k-1} / \partial x_{i_1} \cdots \partial x_{i_{k-1}}$. Hence if we expand

$$D'(H_{r,\delta}{}' \cdot \partial \rho / \partial x_{i_k})$$

by means of the Leibniz formula, make use of Corollary 2 of Lemma 54 and apply the induction hypothesis to $H_{r,\delta}{}'$, we get the required assertion.

Put $\Delta = \sum_{1 \leqq i \leqq n} (\partial / \partial x_i)^2$ and let δ denote the Dirac measure concentrated at the origin.

LEMMA 56. *If n is odd*

$$\Delta^{l + (n-1)/2} \rho^{2l-1} = c_l \delta \quad (l \geqq 1)$$

and if n is even

$$\Delta^{l + n/2} (\rho^{2l} \log \rho) = c_l{}' \delta \qquad (l \geqq 0).$$

Here c_l and $c_l{}'$ are nonzero numbers and the above relations are meant in the sense of the theory of distributions.

This is well known (see [7, p. 47]).

LEMMA 57. *Fix integers $d \geqq 0$ and $r \geqq 1$. Then we can choose an integer $m \geqq 1$ and a function e on $\boldsymbol{R}^n$ of class $C^{2m(r-1)+d}$ such that*

$$\Delta^{mr}e = \delta.$$

Choose m so large that $2m > d + n$. First suppose n is odd. Then $l = mr - (n-1)/2$ is an integer and

$$2l = 2mr - n + 1 > d + 1.$$

Hence $l \geqq 1$. Put $e = c_l^{-1}\rho^{2l-1}$. Then

$$\Delta^{mr}e = \Delta^{l+(n-1)/2}e = \delta$$

and $2l - 1 = 2mr - n = 2m(r-1) + 2m - n > 2m(r-1) + d$. Hence e is of class $C^{2m(r-1)+d}$ by Corollary 1 of Lemma 54.

On the other hand if n is even put $l = mr - n/2$. Then $2l = 2mr - n > d$ and therefore l is positive. Now put

$$e = (c_l')^{-1}\rho^{2l}\log\rho.$$

Then $2l = 2mr - n > 2m(r-1) + d$ and therefore again e is of class $C^{2m(r-1)+d}$.

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