ON THE DISTRIBUTION OF THE SUPREMUM FOR STOCHASTIC PROCESSES WITH INTERCHANGEABLE INCREMENTS[1]

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1. Introduction. In the theory and applications of stochastic processes frequently arises the problem of finding the distribution of $\sup_{0 \leq u \leq t} \xi(u)$ where $\{\xi(t), 0 \leq t < \infty\}$ is a separable stochastic process. For stochastic processes with stationary independent increments G. Baxter and M. D. Donsker [3] solved this problem in principle. They determined the double Laplace-Stieltjes transform of $P\{\sup_{0 \leq u \leq t} \xi(u) \leq x\}$ for such processes. However, even in simple cases, it seems too complicated to invert the transforms. For example, if $\{v(t), 0 \leq t < \infty\}$ is a Poisson process Baxter and Donsker found that they were unable to invert the Laplace-Stieltjes transform of $P\{\sup_{0 \leq u \leq t}[v(u) - cu] \leq x\}$ where c is a positive constant. Actually, in this particular case, R. Pyke [15] found the distribution of $\sup_{0 \leq u \leq t}[v(u) - cu]$ by a direct method.

In this paper we shall show that for a wide class of stochastic processes the distribution of $\sup_{0 \leq u \leq t} \xi(u)$ can be found in a simple and elementary way by making use of only a generalization of the classical ballot theorem.

Throughout this paper we suppose that $\{\chi(t), 0 \leq t \leq T\}$ is a separable stochastic process with nonnegative increments, that $\chi(t)$ increases only in jumps and is continuous on the right, that $\chi(0) = 0$, and that for all $n = 2, 3, \cdots,$ and $0 < t \leq T$

$$(1) \qquad \chi\left(\frac{r\,t}{n}\right) - \chi\left(\frac{rt - t}{n}\right), \qquad r = 1, 2, \cdots, n,$$

are interchangeable random variables. In the following considerations the parameter range will be either a finite or an infinite interval. In what follows we shall refer to the process $\{\chi(t), 0 \leq t \leq T\}$ as a process with *nonnegative interchangeable increments*. In particular, the random variables (1) are interchangeable if they are mutually independent, identically distributed random variables. In this particular case we shall refer to the process $\{\chi(t), 0 \leq t \leq T\}$ as a process with *nonnegative, stationary independent increments*.

We mention here that for $0 \leqq t \leqq T$

$$(2) \qquad\qquad E\{\chi(t)\} = \rho t$$

where ρ is a nonnegative number (possibly $\rho = \infty$). If $\rho = 0$, then $P\{\chi(t) = 0\} = 1$ for all $t \in [0, T]$.

We define the following two processes with interchangeable increments: $\xi_1(t) = \chi(t) - t$ and $\xi_2(t) = t - \chi(t)$ for $0 \leqq t \leqq T$, and we shall find the distribution of $\sup_{0 \leqq u \leqq t} \xi_1(u)$ and that of $\sup_{0 \leqq u \leqq t} \xi_2(u)$ for $0 \leqq t \leqq T$. The results of this paper have many applications in the theory of queues, dams, storage, order statistics, mathematical risk, physics and elsewhere.

2. **Generalizations of the classical ballot theorem.** The classical ballot theorem which is linked to the names of J. Betrand [5], D. André [1] and É. Barbier [2] asserts that if in a ballot candidate A scores a votes and candidate B scores b votes, if all the possible voting records are equally probable, and if $a \geqq \mu b$, where $\mu \geqq 0$ is an integer, then the probability that throughout the counting the number of votes registered for A is always greater than μ times the number of votes registered for B is given by

$$(3) \qquad\qquad P = (a - b\mu)/(a + b).$$

As a generalization of this theorem we arrive at the following theorem.

THEOREM 1. *If $v_1, v_2, \cdots, v_n$ are interchangeable random variables taking on nonnegative integer values, and $N_r = v_1 + \cdots + v_r$, $r = 1, \cdots, n$, then*

$$(4) \qquad P\{N_r < r \text{ for all } r = 1, \cdots, n \mid N_n = k\} = \begin{cases} \left(1 - \dfrac{k}{n}\right) & \text{for } 0 \leqq k \leqq n, \\ \\ 0 & \text{otherwise.} \end{cases}$$

provided that the left-hand side is defined.

Formula (4) can be proved by mathematical induction on n as is given in reference [18] or in a direct combinatorial way as is given in reference [20].

If, in particular, we define v_r as 0 when the rth vote is cast for A and as $(\mu + 1)$ when the rth vote is cast for B, then $v_1, \cdots, v_{a+b}$ are interchangeable random variables taking on nonnegative integer values and $v_1 + \cdots + v_{a+b} = b(\mu + 1)$. If we put $n = a + b$ and $k = b(\mu + 1)$ in (4), then we get (3).

From (4) it follows that

$$(5) \quad P\{N_r < r \text{ for all } r = 1, \cdots, n \text{ and } N_n \leqq l\} = \sum_{k=0}^{l} \left(1 - \frac{k}{n}\right) P\{N_n = k\}$$

for $0 \leqq l \leqq n$.

From Theorem 1 we can deduce the following more general theorem.

THEOREM 2. *If $\{\chi(t),\ 0 \leq t \leq T\}$ has nonnegative interchangeable increments, then for $0 < t \leq T$*

$$(6) \qquad P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t \mid \chi(t) = y\} = \begin{cases} \left(1 - \dfrac{y}{t}\right) & \text{for } 0 \leq y \leq t, \\ 0 & \text{otherwise,} \end{cases}$$

where the conditional probability is defined up to an equivalence.

Proof. Define

$$(7) \qquad v_r = \left[\frac{2^m}{t}\left(\chi_a\left(\frac{rt}{2^m}\right) - \chi_a\left(\frac{rt - t}{2^m}\right)\right)\right], \qquad r = 1, 2, \cdots, 2^m,$$

where $\chi_a(t)$ is the total amount of jumps of magnitude $\geq a > 0$ occurring in the interval $(0, t]$ in the process $\{\chi(t),\ 0 \leq t \leq T\}$ and $[x]$ is the greatest integer $\leq x$. The random variables $v_r, r = 1, 2, \cdots, 2^m$, are interchangeable random variables taking on nonnegative integer values. If we put $N_r = v_1 + \cdots + v_r,\ r = 1, 2, \cdots, 2^m$, $n = 2^m$ and $l = [2^m z / t]$, where $0 \leq z \leq t$, into (4) and let $m \to \infty$ and $a \to 0$, then by the continuity theorem of probability we get

$$(8) \qquad P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t \text{ and } \chi(t) \leq z\} = \int_0^z \left(1 - \frac{y}{t}\right) d_y P\{\chi(t) \leq y\}$$

for $0 \leq z \leq t$. In proving (8) we also used the facts that with probability 1 at most a finite number of jumps of magnitude $\geq a > 0$ occur in the interval $(0, t]$, that $\chi_a(t)$ is monotone nondecreasing if $a \to 0$, that $\lim_{a \to 0} \chi_a(u) = \chi(u)$ and that $\lim_{a \to 0} \sup_{0 \leq u \leq t}[\chi_a(u) - u] = \sup_{0 \leq u \leq t}[\chi(u) - u]$. Theorem 2 immediately follows from (8).

If $z = t$ in (8), then we get

$$(9) \qquad P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t\} = E\left\{\left[1 - \frac{\chi(t)}{t}\right]^+\right\}$$

where $[x]^+ = x$ if $x \geq 0$ and $[x]^+ = 0$ if $x \leq 0$.

3. **The distribution of the supremum for the process** $\{\chi(t) - t\}$. First we consider the case when $\{\chi(t) - t,\ 0 \leq t \leq T\}$ has interchangeable increments, and then we shall show that further results can be obtained if $\{\chi(t) - t,\ 0 \leq t \leq T\}$ has stationary independent increments.

THEOREM 3. *If $\{\chi(t),\ 0 \leq t \leq T\}$ has nonnegative interchangeable increments, then*

$$(10) \qquad \begin{aligned} P&\left\{\sup_{0 \leq u \leq t} [\chi(u) - u] \leq x\right\} \\ &= P\{\chi(t) \leq t + x\} - \iint\limits_{0 \leq y \leq z \leq t} \left(\frac{t - z}{t - y}\right) d_y d_z P\{\chi(y) \leq y + x, \chi(t) \leq z + x\} \end{aligned}$$

for all x and for all finite $t \in [0, T]$.

Proof. It is sufficient to prove that the subtrahend on the right-hand side of (10) is the probability that $\chi(t) \leqq t + x$ and $\chi(u) > u + x$ for some $u \in [0, t]$. Let $y = \sup\{u : \chi(u) > u + x \text{ and } 0 \leqq u \leqq t\}$. Then $\chi(y) = y + x$ and $\chi(u) \leqq u + x$ for $y \leqq u \leqq t$ or equivalently $\chi(u) - \chi(y) \leqq u - y$ for $y \leqq u \leqq t$. Under the condition that $\chi(y) = y + x$ and $\chi(t) = z + x$, the probability that $\chi(u) - \chi(y) \leqq u - y$ for $y \leqq u \leqq t$ is $(t - z)/(t - y)$ for $0 \leqq y \leqq z \leqq t$. This follows from (6) if we apply it to the process $\chi^*(u) = \chi(y + u) - \chi(y)$, $0 \leqq u \leqq t - y$. If we integrate $(t - z)/(t - y)$ with respect to $d_y d_z P\{\chi(y) \leqq y + x, \chi(t) \leqq z + x\} = P\{y + x \leqq \chi(y) \leqq y + x + dy, z - y \leqq \chi(t) - \chi(y) \leqq z - y + dz\}$ over the domain $0 \leqq y \leqq z \leqq t$, then we get the subtrahend on the right-hand side of (10). This proves the theorem.

If the process $\{\chi(t), 0 \leqq t \leqq T\}$ has stationary independent increments, then in (10) $d_y d_z P\{\chi(y) \leqq y + x, \chi(t) \leqq z + x\} = P\{y + x \leqq \chi(y) \leqq y + x + dy\} \cdot P\{z - y \leqq \chi(t) - \chi(y) \leqq z - y + dz\}$.

Now we shall consider the case $T = \infty$ and $t = \infty$. Let

$$(11) \qquad W(x) = P\left\{ \sup_{0 \leqq u < \infty} [\chi(u) - u] \leqq x \right\}$$

and

$$(12) \qquad \Omega(s) = \int_0^\infty e^{-sx} dW(x).$$

Theorem 4. *If* $\{\chi(t), 0 \leqq t < \infty\}$ *has nonnegative, stationary independent increments, then for all* x

$$(13) \qquad W(x) = 1 - (1 - \rho) \int_0^\infty d_y P\{\chi(y) \leqq y + x\}$$

whenever $0 \leqq \rho < 1$ *and* $W(x) \equiv 0$ *whenever* $\rho \geqq 1$.

Proof. Let

$$(14) \qquad W(t, x) = P\left\{ \sup_{0 \leqq u \leqq t} [\chi(u) - u] \leqq x \right\}.$$

Then by the continuity theorem for probabilities it follows that

$$(15) \qquad W(x) = \lim_{t \to \infty} W(t, x).$$

Now by (9)

$$(16) \qquad W(t, 0) = E\left\{ \left[1 - \frac{\chi(t)}{t} \right]^+ \right\}$$

and by the strong law of large numbers (cf. A. Kolmogorov [12])

$$(17) \qquad P\left\{ \lim_{t \to \infty} \frac{\chi(t)}{t} = \rho \right\} = 1.$$

By (16) and (17)

$$(18) \qquad W(0) = \lim_{t \to \infty} W(t,0) = \begin{cases} 1 - \rho & \text{if } 0 \leq \rho < 1, \\ \\ 0 & \text{if } \rho > 1. \end{cases}$$

By (9) and (10) we have

$$(19) \qquad W(t,x) = P\{\chi(t) \leq t + x\} - \int_0^t W(t-u,0)\, d_u P\{\chi(u) \leq u + x\}$$

for all x.

If $\rho < 1$ and $t \to \infty$ in (19), then by using (18) we get (13). Now $W(x)$ is a proper distribution function because by (17)

$$(20) \qquad P\left\{ \sup_{0 \leq u < \infty} [\chi(u) - u] < \infty \right\} = 1.$$

We have seen that $W(0) = 1 - \rho$ and obviously $W(x) = 0$ if $x < 0$.

If $\rho \geq 1$, then $W(x) \equiv 0$ because

$$(21) \qquad P\left\{ \sup_{0 \leq u < \infty} [\chi(u) - u] = \infty \right\} = 1.$$

For $\rho > 1$ (21) follows from (17) and for $\rho = 1$ by a theorem of K. L. Chung, and W. H. J. Fuchs [6]. This completes the proof of the theorem.

If $\{\chi(t),\ 0 \leq t < \infty\}$ has nonnegative, stationary independent increments then for $\Re(s) \geq 0$

$$(22) \qquad E\{e^{-s\chi(t)}\} = e^{-t\Phi(s)}$$

with an appropriate $\Phi(s)$ and $\rho = \lim_{s \to 0} \Phi(s)/s$. If ρ is a finite positive number, then we can define a distribution function $H^*(x)$ by supposing that

$$(23) \qquad \int_0^\infty e^{-sx} dH^*(x) = \Phi(s)/\rho s$$

for $\Re(s) \geq 0$ and $H^*(x) = 0$ for $x \leq 0$. The representation (22) makes it possible to find the distribution function $W(x)$ also in another way.

THEOREM 5. *If $\{\chi(t),\ 0 \leq t < \infty\}$ has nonnegative, stationary independent increments, and $0 \leq \rho < 1$, then*

$$(24) \qquad \int_0^\infty e^{-sx} dW(x) = \frac{1 - \rho}{1 - \dfrac{\Phi(s)}{s}}$$

for $\Re(s) \geq 0$ and

$$(25) \qquad W(x) = (1 - \rho) \sum_{n=0}^\infty \rho^n H^*(x)$$

where $H_n^*(x)$ denotes the nth iterated convolution of $H^*(x)$ with itself; $H_0^*(x) = 1$ if $x \geqq 0$ and $H_0^*(x) = 0$ if $x < 0$.

Proof. Let

$$(26) \qquad \Omega(t, s) = \int_0^\infty e^{-sx} d_x W(t, x).$$

Then by (19)

$$(27) \qquad \Omega(t, s) = e^{[s - \Phi(s)]t} - s \int_0^t W(t - u, 0) e^{[s - \Phi(s)]u} \, du$$

for $\Re(s) \geqq 0$. Forming the derivative of (27) with respect to t we get

$$(28) \qquad \frac{\partial \Omega(t, s)}{\partial t} = [s - \Phi(s)] \Omega(t, s) - s W(t, 0).$$

If $0 \leqq \rho < 1$, then by (17) $\lim_{t \to \infty} W(t, x) = W(x)$ is a proper distribution function and consequently $\lim_{t \to \infty} \Omega(t, s) = \Omega(s)$ for $\Re(s) \geqq 0$. Using (18) we obtain from (28) that

$$(29) \qquad \lim_{t \to \infty} \Omega(t, s) = \Omega(s) = \frac{1 - \rho}{1 - \dfrac{\Phi(s)}{s}}$$

for $\Re(s) \geqq 0$, which was to be proved. Formula (25) follows by inversion.

4. The distribution of the supremum for the process $\{t - \chi(t)\}$. First we consider the case when $\{t - \chi(t), \ 0 \leqq t \leqq T\}$ has interchangeable increments, and then we shall show that further results can be obtained if $\{t - \chi(t), \ 0 \leqq t \leqq T\}$ has stationary independent increments.

Theorem 6. *If $\{\chi(t), \ 0 \leqq t \leqq T\}$ has nonnegative interchangeable increments, then*

$$(30) \qquad P\left\{ \sup_{0 \leqq u \leqq t} [u - \chi(u)] \leqq x \right\} = 1 - \int_x^t \frac{x}{y} d_y P\{\chi(y) \leqq y - x\}$$

for $x > 0$ and $0 < t \leqq T$.

Proof. We shall find the probability of the complementary event. The event that $u - \chi(u) > x$ for some $u \in (0, t]$ can occur in the following mutually exclusive ways: $\inf\{u : u - \chi(u) > x\} = y$ where $0 \leqq y \leqq t$. Then $\chi(y) = y - x$ and $u - \chi(u) \leqq x$ for $0 \leqq u \leqq y$ or equivalently $\chi(y) - \chi(u) \leqq y - u$ for $0 \leqq u \leqq y$. By Theorem 1

$$(31) \qquad P\{\chi(y) - \chi(u) \leqq y - u \text{ for } 0 \leqq u \leqq y \mid \chi(y) = y - x\} = \frac{x}{y}$$

for $0 < x \leqq y$. If we integrate (31) with respect to $d_y P\{\chi(y) \leqq y - x\}$

$= P\{y - x \leqq \chi(y) \leqq y - x + dy\}$ from x to t, then we get the probability of the complementary event. This proves (30).

If $\{\chi(t),\ 0 \leqq t < \infty\}$ has nonnegative, stationary independent increments and $t = \infty$, then (30) reduces to an exponential distribution function given by the next theorem.

THEOREM 7. *If* $\{\chi(t), 0 \leqq t < \infty\}$ *has nonnegative, stationary independent increments, then for* $x \geqq 0$

$$(32) \qquad P\left\{ \sup_{0 \leqq u < \infty} [u - \chi(u)] \leqq x \right\} = 1 - e^{-\omega x},$$

where ω *is the largest real root of the equation*

$$(33) \qquad \Phi(\omega) = \omega.$$

If $0 \leqq \rho \leqq 1$, *then* $\omega = 0$ *and if* $\rho > 1$, *then* $\omega > 0$.

Proof. Define

$$(34) \qquad \theta_x = \inf\{u : u - \chi(u) > x\}.$$

Then $\{\theta_x, 0 \leqq x < \infty\}$ is a stochastic process with nonnegative, stationary independent increments. Thus for $\Re(s) \geqq 0$

$$(35) \qquad E\{e^{-s\theta_x}\} = e^{-x\omega(s)}$$

with an appropriate $\omega(s)$. On the other hand if $\chi(x) = y$, then $\theta_x = x + \theta_y^*$ where θ_y^* has the same distribution as θ_y. Thus

$$(36) \qquad E\{e^{-s\theta_x}\} = e^{-sx - x\Phi(\omega(s))} \ .$$

If we compare (35) and (36) we get the equation

$$(37) \qquad \omega(s) = s + \Phi(\omega(s))$$

found earlier by D. G. Kendall [11].

If $s > 0$ is real, then

$$(38) \qquad z = s + \Phi(z)$$

has one and only one nonnegative real root $z = \omega(s)$ and $\omega(s)$ is a continuous function of s. If $s \to 0$, then $\omega(s) \to \omega$ where ω is the largest real root of $\Phi(\omega) = \omega$. If $\rho > 1$, then $\omega > 0$, whereas if $0 \leqq \rho \leqq 1$, then $\omega = 0$.

By the definition of θ_x we have

$$(39) \qquad P\left\{ \sup_{0 \leqq u \leqq t} [u - \chi(u)] \leqq x \right\} = 1 - P\{\theta_x \leqq t\}$$

and by (35)

$$(40) \qquad \lim_{t \to \infty} P\{\theta_x \leqq t\} = \lim_{s \to 0} e^{-x\omega(s)} = e^{-x\omega}$$

which proves (32). If $0 \leq \rho < 1$, then it follows immediately from (17) that the right-hand aside of (32) is 0.

5. **Examples.** In what follows we shall consider examples from the theory of queues, dams, and order statistics.

1. QUEUEING PROCESSES. (Cf. V. E. Beneš [4] and L. Takács [19].) Suppose that in the time interval $(0, \infty)$ customers arrive at a counter in accordance with a random process. The customers are served by a single server. The server is idle if and only if there is no customer in the system. Denote by $\chi(t)$ the total service time of all those customers who arrive in the interval $(0, t]$. We suppose that $\{\chi(t), 0 \leq t < \infty\}$ is a separable stochastic process with nonnegative, stationary independent increments. Denote by $\eta(t)$ the virtual waiting time at time t, i.e., the time that a customer would have to wait if he arrived at time t and if service is in order of arrival. Let $\alpha(t)$ denote the total idle time of the server in the interval $(0, t)$. If $\eta(0) = 0$, then it can easily be seen that

$$(41) \qquad P\{\eta(t) \leq x\} = P\left\{ \sup_{0 \leq u \leq t} [\chi(u) - u] \leq x \right\}$$

and

$$(42) \qquad P\{\alpha(t) \leq x\} = P\left\{ \sup_{0 \leq u \leq t} [u - \chi(u)] \leq x \right\}.$$

(i) *Poisson input.* (Cf. R. Pyke [15] and L. Takács [17].) Suppose that customers arrive at a counter in accordance with a Poisson process of density λ, and their service time is constant α. Then $\chi(t) = \alpha v(t)$, where $\{v(t), 0 \leq t < \infty\}$ is a Poisson process of density λ. In this case

$$(43) \qquad P\{v(t) = j\} = e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

for all $t \geq 0$ and $j = 0, 1, \cdots$; $\rho = \lambda \alpha$ and $\Phi(s) = \lambda[1 - e^{-s\alpha}]$. By (10) we get

$$(44) \quad \begin{aligned} P\{\eta(t) \leq x\} \\ = P\{\alpha v(t) \leq t + x\} - \sum_{x \leq j\alpha \leq k\alpha \leq t+x} \left(\frac{t + x - k\alpha}{t + x - j\alpha} \right) P\{v(j\alpha - x) = j\} \\ \times P\{v(t + x - j\alpha) = k-j\}. \end{aligned}$$

If $\lambda \alpha < 1$, then by inverting (24) we get

$$(45) \qquad \lim_{t \to \infty} P\{\eta(t) \leq x\} = (1 - \lambda \alpha) \sum_{j=0}^{[x/\alpha]} (-1)^j e^{\lambda(x - j\alpha)} \frac{[\lambda(x - j\alpha)]^j}{j!}.$$

Further by (30)

$$(46) \qquad P\{\alpha(t) \leq x\} = 1 - \sum_{j=0}^{[(t-x)/\alpha]} \left(\frac{x}{x + j\alpha} \right) P\{v(x + j\alpha) = j\}.$$

If $\lambda\alpha > 1$, then by (32)

$$(47) \qquad \lim_{t \to \infty} P\{\alpha(t) \le x\} = 1 - e^{-\omega x}$$

where ω is the largest real root of $\omega = \lambda(1 - e^{-\alpha\omega})$ which is given by

$$(48) \qquad \omega = \lambda\left(1 - \sum_{j=1}^{\infty} e^{-\lambda\alpha j}\frac{(\lambda\alpha j)^{j-1}}{j!}\right).$$

(ii) *Compound Poisson input.* (Cf. L. Takács [17], [19].) Suppose that customers arrive at a counter in accordance with a Poisson process of density λ and the service times are mutually independent, identically distributed random variables with distribution function $H(x)$ and independent of the arrival times. Denote by α the average service time and by $\psi(s)$ the Laplace-Stieltjes transform of $H(x)$. In this case

$$(49) \qquad P\{\chi(t) \le x\} = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^{n}}{n!}H_n(x)$$

where $H_n(x)$ denotes the nth iterated convolution of $H(x)$ with itself; $H_0(x) = 1$ if $x \ge 0$ and $H_0(x) = 0$ if $x < 0$; $\rho = \lambda\alpha$ and $\Phi(s) = \lambda[1 - \psi(s)]$. In this case the distribution and the limiting distribution of $\eta(t)$ and $\alpha(t)$ are given by (10), (13) or (25), (30) and (32) where

$$(50) \qquad H^*(x) = \frac{1}{\alpha}\int_0^x [1 - H(y)]dy$$

if $x \ge 0$ and

$$(51) \qquad \omega = \lambda\left[1 - \sum_{j=1}^{\infty}\int_0^{\infty} e^{-\lambda x}\frac{(\lambda x)^{j-1}}{j!}dH_j(x)\right]$$

if $\lambda\alpha > 1$.

If, in particular, $H(x) = 1 - e^{-\mu x}$ for $x \ge 0$, then $\alpha = 1/\mu$, $\psi(s) = \mu/(\mu + s)$, $H^*(x) = H(x)$ and $\omega = \mu/\lambda$ if $\lambda > \mu$. In this particular case

$$(52) \qquad P\{\chi(t) \le x\} = e^{-\lambda t}\left[1 + \sqrt{(\lambda\mu t)}\int_0^x e^{-\mu y}y^{-1/2}I_1(2\sqrt{(\lambda\mu t y)})\,dy\right],$$

where $I_1(x)$ denotes the modified Bessel function of the first kind. If $\lambda < \mu$, then

$$(53) \qquad \lim_{t \to \infty} P\{\eta(t) \le x\} = 1 - \frac{\lambda}{\mu}e^{-(\mu-\lambda)x}$$

for $x \ge 0$, and if $\lambda > \mu$, then

$$(54) \qquad \lim_{t \to \infty} P\{\alpha(t) \le x\} = 1 - e^{-(\lambda-\mu)x}$$

for $x \ge 0$.

2. DAM PROCESSES. (Cf. J. Gani and R. Pyke [9], and J. Gani and N. U. Prabhu [8].) Consider a dam (reservoir) with infinite capacity and suppose that water is flowing into the dam in accordance with a random process. Denote by $\chi(t)$ the total quantity of water flowing into the dam in the time interval $(0, t]$. Suppose that $\{\chi(t), 0 \leq t < \infty\}$ is a separable stochastic process with nonnegative, stationary independent increments. Suppose that the release is continuous at a constant unit rate when the dam is not empty. Denote by $\eta(t)$ the content of the dam at time t. Let $\alpha(t)$ denote the total time in the interval $(0, t)$ during which the dam is empty. If $\eta(0) = 0$, then the distributions of $\eta(t)$ and $\alpha(t)$ are given by (41) and (42) respectively.

(i) *Gamma input.* (Cf. P. A. P. Moran [13] and D. G. Kendall [11].) Suppose that

$$(55) \qquad P\{\chi(t) \leq x\} = \frac{1}{\Gamma(t)} \int_0^{\mu x} e^{-y} y^{t-1} \, dy$$

where μ is a positive constant. Then $\rho = 1/\mu$, $\Phi(s) = \log(1 + s/\mu)$, and

$$(56) \qquad \frac{dH^*(x)}{dx} = \mu \int_{\mu x}^{\infty} \frac{e^{-y}}{y} \, dy$$

for $x > 0$.

Now the distribution function $P\{\eta(t) \leq x\}$ is given by (10) and if $\mu > 1$, then by (13)

$$(57) \qquad \lim_{t \to \infty} P\{\eta(t) \leq x\} = 1 - (\mu - 1)e^{-\mu x} \int_0^{\infty} e^{-\mu y} \frac{[\mu(x + y)]^{y-1}}{\Gamma(y)} \, dy$$

for $x \geq 0$. By (30)

$$(58) \qquad P\{\alpha(t) \leq x\} = 1 - \mu x \int_0^{t-x} e^{-\mu y} \frac{(uy)^{x+y-1}}{\Gamma(x + y + 1)} \, dy$$

for $0 \leq x \leq t$, and if $\mu < 1$, then by (32)

$$(59) \qquad \lim_{t \to \infty} P\{\alpha(t) \leq x\} = 1 - e^{-\omega x}$$

for $x \geq 0$ where

$$(60) \qquad \omega = \mu \left(\sum_{j=1}^{\infty} e^{-j\mu} \frac{(j\mu)^{j-1}}{j!} - 1 \right).$$

(ii) *Stable input.* (Cf. H. Pollard [14] and D. A. Darling [7].) Suppose that for $0 < c < 1$

$$(61) \qquad P\{\chi(t) \leq t^{1/c} x\} = \int_0^x h_c(y) \, dy$$

where

$$(62) \qquad h_c(x) = \frac{1}{\pi x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(nc+1)\sin nc\pi}{n! x^{nc}}$$

for $x > 0$. Then $\rho = \infty$ and $\Phi(s) = |s|^c$. If, in particular, $c = 1/2$, then

$$(63) \qquad h_{1/2}(x) = \frac{1}{\sqrt{(2\pi x^3)}} e^{-1/2x}$$

for $x > 0$. Now the distributions of $\eta(t)$ and $\alpha(t)$ are given by (10) and (30) respectively.

3. ORDER STATISTICS (Cf. N. V. Smirnov [16].) Let $\xi_1, \xi_2, \cdots, \xi_n$ be mutually independent random variables with a common continuous distribution function $F(u)$. Let $F_n(u)$ denote the empirical distribution function of the sample $(\xi_1, \xi_2, \cdots, \xi_n)$, that is, the number of variables $\leq u$ divided by n. Let

$$(64) \qquad \delta_n^+ = \sup_{-\infty < u < \infty} [F_n(u) - F(u)]$$

and

$$(65) \qquad \delta_n^- = \sup_{-\infty < u < \infty} [F(u) - F_n(u)].$$

They are distribution-free statistics. If $F(u) = u$ for $0 \leq u \leq 1$, then $F_n(u) = \chi(u)$, where the process $\{\chi(u), 0 \leq u \leq 1\}$ is defined as follows: We choose n points independently in the interval $(0, 1)$ such that each point has a uniform distribution over $(0, 1)$. Denote by $\chi(u)$ the ratio of the number of points in the interval $(0, u]$ to n. Then $\{\chi(u)\}$ has nonnegative interchangeable increments, and

$$(66) \qquad P\left\{\chi(u) = \frac{j}{n}\right\} = \binom{n}{j} u^j (1-u)^{n-j}$$

for $0 \leq u \leq 1$ and $j = 0, 1, \cdots, n$.

By (10)

$$(67) \qquad \begin{aligned} P\{\delta_n^+ \leq x\} &= P\left\{\sup_{0 \leq u \leq 1} [\chi(u) - u] \leq x\right\} \\ &= 1 - \sum_{nx \leq j \leq n} \left(\frac{nx}{n(1+x)-j}\right) P\left\{\chi\left(\frac{j}{n} - x\right) = \frac{j}{n}\right\} \end{aligned}$$

and by (30)

$$(68) \qquad \begin{aligned} P\{\delta_n^- \leq x\} &= P\left\{\sup_{0 \leq u \leq 1} [u - \chi(u)] \leq x\right\} \\ &= 1 - \sum_{0 \leq j \leq n(1-x)} \left(\frac{nx}{nx+j}\right) P\left\{\chi\left(\frac{j}{n} + x\right) = \frac{j}{n}\right\} \end{aligned}$$

for $x \geq 0$. Obviously (67) and (68) are identical.

6. REMARKS. We have found two different expressions for the distribution functions of $\sup_{0 \leq u < \infty} [\chi(u) - u]$ and $\sup_{0 \leq u < \infty} [u - \chi(u)]$ when the process $\{\chi(t), 0 \leq t < \infty\}$ has nonnegative stationary independent increments. These results make it possible to discover interesting identities.

For example, if we compare formulas (13) and (25) for $W(x) = P\{\sup_{0 \leq u < \infty} [\chi(u) - u] \leq x\}$, then we get that

$$(69) \qquad \int_x^\infty d_y P\{\chi(y) \leq y - x\} = \frac{1}{1 - \rho}$$

if $x > 0$ and $\rho < 1$. If, in particular, $\{\chi(t), 0 \leq t < \infty\}$ is a Poisson process of density λ, then (69) reduces to

$$(70) \qquad \sum_{j=0}^\infty e^{-\lambda(x+j)} \frac{[\lambda(x+j)]^j}{j!} = \frac{1}{1 - \lambda}$$

provided that $\lambda < 1$ and $x > 0$. This identity was found by J. L. W. V. Jensen [10].

If we compare (30) and (32), then we get for $x > 0$ that

$$(71) \qquad \int_x^\infty \frac{1}{y} d_y P\{\chi(y) \leq y - x\} = \frac{e^{-\omega x}}{x}$$

where ω is the largest real root of $\Phi(\omega) = \omega$. If, in particular, $\{\chi(t), 0 \leq t < \infty\}$ is a Poisson process of density λ, then

$$(72) \qquad \sum_{j=0}^\infty \frac{1}{(x+j)} e^{-\lambda(x+j)} \frac{[\lambda(x+j)]^j}{j!} = \frac{e^{-\omega x}}{x}$$

for $x > 0$ where ω is the largest real root of $\omega = \lambda[1 - e^{-\omega}]$. If $\lambda \leq 1$, then $\omega = 0$.

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