HOMOTOPICALLY HOMOGENEOUS SPACES AND MANIFOLDS[1]

BY
PETER RICE

Curtis [1] originally defined a homotopically homogeneous (h.h.) space. X is h.h. if the triple $(\hat{X}, X, p)$ is a (Hurewicz) fiber space [6] where $\hat{X}$ is the deleted product $X \times X$-diagonal, and $p: \hat{X} \to X$ is the projection on the first coordinate. The following remark, which is easily proved, motivates this definition.

REMARK. If X is a compact, n-dimensional polyhedron, and if $(\hat{X}, X, p)$ is a locally trivial fiber space, then X is an n-manifold.

§I of this paper states some basic lemmas used later. §II presents the main theorems, which include the results (1) that a space with a covering of acyclic h.h. open sets with certain weak local properties is a singular homology manifold. and (2) that such a space with stronger local properties is a homotopy manifold. §III deals with some examples, and §IV contains a related theorem on the product of generalized manifolds.

The following notations will be used throughout. w.s.c., l.s.c., p.c. and l.p.c. denote respectively weakly simply connected, locally simply connected, path connected and locally path connected, all of which are defined in [5]. An n-gm will be as defined by Wilder [11], a singular homology n-manifold over K will be as defined by Raymond [9], and a homotopy n-manifold will be as defined by Griffiths [4]. All homology is singular homology and all maps are continuous.

I. Preliminary lemmas.

LEMMA I.1. *If X is path connected and h.h, and if $x, y \in X$, then $X - x$ and $X - y$ have the same homotopy type.*

This is a particularization of the well-known general result for path connected fiber spaces. If α is a path from x to y, then the constructive proof of the general result gives a map

$$X - x \to X - y$$

which will be called $\boldsymbol{\alpha}$. If α^{-1} denotes the reverse of α, then $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^{-1}$ are homotopy inverses.

Received by the editors August 6, 1964.

[1] This paper contains the substance of the author's dissertation, prepared under M. L. Curtis at Florida State University. §§ 1, II, III were presented to the Society, April 20, 1963 under the title *Homotopically homogeneous spaces* and § IV was presented to the Society on January 23, 1964, under the title *The products of polyhedral homotopy manifolds is a homotopy manifold.*

LEMMA I.2. *An l.s.c., l.p.c. space has a basis of p.c., w.s.c. open sets.* (*See* [5] *for a proof.*)

LEMMA I.3. *Let X be a connected, l.p.c., l.s.c., h.h. metric space.*

(1) *If α^{yz} is any path from y to z in x, then α^{yz} canonically induces an isomorphism*

$$\alpha_*^{yz}: H_*(X - y) \to H_*(X - z).$$

(2) *If U is p.c. and w.s.c. and α^{yz}, β^{yz} are two paths from y to z in U, then $\alpha_*^{yz} = \beta_*^{yz}$.*

Proof. Part one. In the triple $(\hat{X}, X, p)$, the fiber over a point x is canonically homeomorphic to $X - x$. By Lemma I.1, α^{yz} induces a homotopy equivalence

$$\alpha^{yz}: X - y \to X - z.$$

Define $\alpha_*^{yz} = \boldsymbol{\alpha}_*^{yz}$. Since the covering homotopy is unique up to the choice of the lifting function [2], α_*^{yz} is canonically induced.

Part two. $\alpha = \alpha^{yz} \cdot (\beta^{yz})^{-1}$ is a loop on y in $\mathscr{U}$, so there is a homotopy

$$G': I^2 \to X$$

shrinking α to a point. It may be assumed that G' contracts α to y rel y. Define $G_0: p^{-1}(\alpha(I)) \to \hat{X}$ to be the inclusion. This covering homotopy situation yields

$$G: p^{-1}(\alpha(I)) \times I \to \hat{X}$$

such that $pG = G'$. Let $G_1 = G \,|\, p^{-1}(\alpha(I)) \times l$. Since X is metric, $(\hat{X}, X, p)$ may be assumed regular [1], so $G_1 \,|\, p^{-1}(\alpha(0)) = id$. α induces

$$\phi: (X - y) \times I \to p^{-1}(\alpha(I)).$$

$$G_1\phi: (X - y) \times I \to (X - y)$$

is a homotopy between the identity and $\boldsymbol{\alpha}$, so $\alpha_* = \boldsymbol{\alpha}_* = id$. It is easy to see that

$$(\alpha^{xy} \cdot \alpha^{yz})_* = \alpha_*^{yz} \alpha_*^{xy}$$

is always the case. Therefore $\alpha_*^{yz} = \beta_*^{yz}$, which concludes the proof.

Let $(\mathscr{F}, X) = \bigcup_{x \in X} H_*(X - x)$ with X as in Lemma I.3 and let $\mathscr{U}$ be a basis of X consisting of p.c. and w.s.c. open sets (Lemma I.2). Let $x \in X$ and $U \in \mathscr{U}$ such that $x \in U$, and let $s \in H_*(X - x)$. If $y \in U$, there is a canonical isomorphism

$$H_*(X - x) \to H_*(X - y)$$

which takes s to s_y by Lemma I.3. Define

$$W(U, s) = \{s_y \,|\, y \in U\}.$$

Let $\mathscr{W} = \{W(U,s) \mid U \in \mathscr{U}, s \in H_*(X-x) \text{ where } x \in U\}$.

LEMMA I.4. *$\mathscr{W}$ is a basis for a topology on $(\mathscr{F},X)$, and with this topology, $(\mathscr{F},X)$ is a locally constant sheaf.*

Proof. This is an easy consequence of the preceding lemma and the fact that a sheaf over $U \subset X$ is constant if and only if two sections are equal or have disjoint images and every section over a connected open set may be extended to U.

DEFINITION. Let $(\mathscr{K},X)$ be the sheaf generated by the presheaf $V \to H_*(X-V)$, let $(\mathscr{L},X)$ be generated by $V \to H_*(X,X-V)$, and let $(\mathscr{H},X)$ be the constant sheaf with stalk $H_*(X)$. $(\mathscr{K},X)^n$, $(\mathscr{L},X)^n$ and $(\mathscr{H},X)^n$ denote these sheaves restricted to the nth homology. Note that $(\mathscr{K},X)^n(x) = H_n(X-x)$ and $(\mathscr{L},X)^n(x) = H_n(X,X-x)$.

LEMMA I. 5. *$(\mathscr{F},X)$ is isomorphic to $(\mathscr{K},X)$. In particular, if X is a locally compact connected, l.p.c., l.s.c., h.h. metric space, then $(\mathscr{K},X)$ with coefficients in a module is locally constant.*

Proof. $(\mathscr{F},X)$ and $(\mathscr{K},X)$ are the same point sets and the identity point set function is an isomorphism on each stalk. Because of the definition of the topology of a sheaf, every continuous sheaf homomorphism is an open map. Thus it is sufficient to establish the continuity of the identity map $(\mathscr{K},X) \to (\mathscr{F},X)$. Let $S \in H_*(X-x)$ and let $W(U,s)$ be a basic neighborhood of s in $(\mathscr{F},X)$ (chosen so that there is an element $s_U \in H_*(X-U)$ which is sent onto s by $H_*(X-U) \to H_*(X-x)$). Then $W(U,s)$ is open in $(\mathscr{K},X)$ if the following diagram is commutative for all $y \in U$, where j_x, j_y are induced by the inclusion:

$$\begin{array}{ccc} & H_n(X-U) & \\ {\scriptstyle j_x}\swarrow & & \searrow{\scriptstyle j_y} \\ H_n(X-x) & \xrightarrow{\alpha_*} & H_n(X-y). \end{array}$$

To see this, let $U \in \mathscr{U}$, x, $y \in U$, and let α be a path in U from x to y. Let

$$H:(X-U) \times I \to \hat{X}$$

be the homotopy with the properties:

$$\begin{aligned} pH(z,t) &= \alpha(1-t), \\ H(z,0) &= (y,\boldsymbol{\alpha}(z)), \\ H(z,1) &= (x,\boldsymbol{\alpha}^{-1}\boldsymbol{\alpha}(z)). \end{aligned}$$

Let

$$K:(X-U) \times I \to \{x\} \times (X-x)$$

be the homotopy such that

$$K(z,0) = (x,\boldsymbol{\alpha}^{-1}\boldsymbol{\alpha}(z)),$$
$$K(z,1) = (x,z).$$

Let

$$L:(X-U)\times I \to \hat{X}$$

be the homotopy such that

$$L(z,t) = (\alpha(t),z).$$

α can be shrunk to y leaving y fixed. If

$$G:p^{-1}(\alpha(I)) \to (X-y)$$

is the final map in the homotopy covering this situation, then $G(y,z)=(y,z)$ by regularity. Define

$$J:(X-U)\times I \to (X-y)$$

by

$$J(z,t)=\begin{cases} GH(z,3t) & 0 \leqq t \leqq 1/3, \\ GK(z,3t-1) & 1/3 \leqq t \leqq 2/3, \\ GL(z,3t-2) & 2/3 \leqq t \leqq 1. \end{cases}$$

J proves that $\boldsymbol{\alpha}:(X-U)\to(X-y)$ is homotopic to the inclusion map. This establishes the commutativity which proves the lemma.

LEMMA I.6. *If X is acyclic, then $(\mathscr{H},X)$ is isomorphic to $(\mathscr{L},X)$.*

Proof. This follows from the exactness of the sequence

$$(\mathscr{H},X)^n \to (\mathscr{L},X)^n \to (\mathscr{H},X)^{n-1} \to (\mathscr{H},X)^{n-1}.$$

LEMMA I.7. *If $W \subset V \subset \bar{V} \subset U$, then the excision isomorphism*

$$H_*(U,U-W) \to H_*(X,X-W)$$

induces a sheaf isomorphism

$$(\mathscr{L},U)\,|\,V \to (\mathscr{L},X)\,|\,V.$$

Proof. This is a simple consequence of the fact that the excision isomorphism is induced by the inclusion map.

II. Main theorems.

THEOREM II.1. *If X is a locally compact, connected, l.p.c. metric space with covering of acyclic, l.s.c., h.h. open sets, then $(\mathscr{L},X)$ with coefficients in a module is locally constant.*

Proof. Let U be an acyclic, l.s.c., h.h. open set in X. Then U is also l.p.c. and by Lemma I.5, $(\mathscr{H}, U)$ is locally constant. By Lemma I.6, $(\mathscr{L}, U)$ is locally constant, and by Lemma I.7, $(\mathscr{L}, X) \mid V$ is locally constant, where $V \subset \bar{V} \subset U$. Since X may be covered by sets such as V, $(\mathscr{L}, X)$ is locally constant.

THEOREM II.2. *If X is a locally compact, connected, l.p.c., metric space with a covering of acyclic, l.s.c., h.h. open sets and $\dim_K X = n$, where K is a field, then X is a singular homology manifold over K.*

Proof. Lee [7] proved that whenever X has dimension n over a field K and the sheaf $(\mathscr{L}, X)$ is locally constant, then $H_i(X, X - x; K)$ is K if $i = n$ and is zero otherwise. (Lee did not state this result specifically, but it is an easy corollary to the proof of his Theorem 1.) Thus this theorem follows from Theorem II.1.

COROLLARY II.1. *If X is an absolute retract, h.h., and $\dim_K X = n$ where K is a field, then X is a singular homology n-manifold over K*(2).

COROLLARY II.2. *If X is HLC over the principal ideal domain K, satisfies the conditions of Theorem* II.2 *over K, and $H_p(X, X - x)$ is finitely generated for each p, then X is a singular homology manifold over K.*

Proof. This is an application of Lee's Theorem 2.

COROLLARY II.3. *If X is as in Corollary* II.2, *X is an n-gm.*

DEFINITION II.1. $\mathscr{U}_x$ is a *contractible basis* for a space X at a point x if $\mathscr{U}_x$ is a basis of open sets such that for each $U \in \mathscr{U}_x$, U is contractible and $U - x$ is 0-connected and 1-connected (in the homotopy sense).

DEFINITION II.2. X is a *strong homology n-manifold* or *n-shm* if, for $n \geqq 3$, X is a singular n-manifold and X has a contractible basis at each point, or if, for $n < 3$, X is an n-manifold. Note that it is equivalent to require that X be a separable, locally contractible, singular n-manifold for $n < 3$, since X would then be an n-gm.

THEOREM II.3. *An n-shm over Z (the integers) is a homotopy n-manifold.*

Proof. This is obvious for $n \leqq 2$. Let X be an n-shm, $n \geqq 3$, and let $x \in X$, $U \in \mathscr{U}_x$ such that the singular homology sheaf is constant over U. From the homotopy exact sequence of the pair $(U, U - x)$

$$\pi_{m+1}(U, U - x) = \pi_m(U - x).$$

By excision

$$H_{m+1}(U, U - x) = H_{m+1}(X, X - x) = \begin{cases} Z \text{ for } m + 1 = n, \\ 0 \text{ for } m + 1 \neq n. \end{cases}$$

(2) I am indebted to the referee for suggesting this corollary.

(The homology here is augmented singular homology with coefficients in Z.) Since

$$\pi_0(U-x) = \pi_1(U-x) = 0,$$

the Hurewicz isomorphism theorem applies and

$$H_m(U, U-x) = \pi_m(U, U-x), \qquad m \leqq n,$$

so

$$\pi_{n-1}(U-x) = Z.$$

Thus, there is a map $f: S^{n-1} \to U - x$ such that

$$f_{\#}: \pi_{n-1}(S^{n-1}) \to \pi_{n-1}(U-x)$$

is an isomorphism.

$$f_*: H_m(S^{n-1}) \to H_m(U-x)$$

is an isomorphism for $m \leqq n-1$ by the preceding, using the functorial property of the Hurewicz map, an isomorphism for $m > n$ because of dimension, and an isomorphism for $m = n$ because of the dimension of S^{n-1} and the exact homology sequence of $(U, U-x)$.

It follows from Whitehead's result [10] that

$$f_{\#}: \pi_m(S^{n-1}) \to \pi_m(U-x)$$

is an isomorphism for all m.

Let $U_1, U_2 \in \mathscr{U}_x$, $U_1 \subset U_2$, and let

$$i_{\#}: \pi_m(U_1 - x) \to \pi_m(U_2 - x)$$

be induced by the inclusion map. The inclusion

$$(U_1, U_1 - x) \to (U_2, U_2 - x)$$

induces an isomorphism of the exact homology sequences (by the "five lemma"), so

$$i_*: H_m(U_1 - x) \to H_m(U_2 - x)$$

is an isomorphism. By Whitehead's theorem, $i_{\#}$ is an isomorphism.

Now let $U_1 \in \mathscr{U}_x$ be chosen as above, let $V_1 \subset \bar{V}_1 \subset U_1$, V_1 open, let $U_2 \in \mathscr{U}_x$ such that $U_2 \subset V_1$, let $V_2 \subset \bar{V}_2 \subset U_2$, V_2 open, and let $U_3 \in \mathscr{U}_x$ such that $U_3 \subset V_2$. In the following diagram, $\alpha_2\alpha_1$ and $\alpha_4\alpha_3$ are isomorphisms:

$$\pi_m(U_3 - x) \overset{\alpha_1}{\to} \pi_m(\bar{V}_2 - x) \overset{\alpha_2}{\to} \pi_m(U_2 - x) \overset{\alpha_3}{\to} \pi_m(\bar{V}_1 - x) \overset{\alpha_4}{\to} \pi_m(U_1 - x).$$

Thus α_2 is onto and α_3 is one-to-one. Therefore

$$\operatorname{Im}(\alpha_3\alpha_2) = \operatorname{Im}(\alpha_2) = \pi_m(U_2 - x) = \pi_m(S^{n-1}),$$

so X is a homotopy n-manifold.

COROLLARY II. 4. *If X is a connected metric space with a contractible basis at each point, if each point has a contractible neighborhood which is h.h., if $Hp(X, X - x)$ is finitely generated for each p, and if* $\dim_Z X = n$, *where Z is the ring of integers, then X is a homotopy n-manifold.*

III. Examples. Curtis proved that manifolds are h.h. Since the fibers in a fiber space with path connected base have the same homotopy type, this affords a criterion for determining when a space is not h.h.

LEMMA III. 1. An n-manifold M with nonempty boundary is not h.h.

Proof. Let $x \in M^{\cdot}$, $y \varepsilon M_{\circ}$. A path α from x to y induces an isomorphism $\alpha_*: H_*(M - x) \to H_*(M - y)$ as in Lemma I.3.

$$\begin{array}{ccccccccc} H_n(M-x) & \to & H_n(M) & \to & H_n(M, M-x) & \to & H_{n-1}(M-x) & \to & H_{n-1}(M) \\ \downarrow \alpha_* & & \downarrow id & & & & \downarrow \alpha^* & & \downarrow id \\ H_n(M-y) & \to & H_n(M) & \to & H_n(M, M-y) & \to & H_{n-1}(M-y) & \to & H_{n-1}(M). \end{array}$$

It is easy to prove that this diagram is commutative since the homotopy connecting $\alpha: M - x \to M$ and inc.: $M - x \to M$ is exactly the homotopy covering the path α. If coefficients are taken in a field, the exact sequences split, and it is possible to define an isomorphism

$$\beta_*: H_n(M, M - x) \to H_n(M, M - y).$$

But these two vector spaces are not isomorphic, hence M cannot be h.h.

Curtis also proved that topological groups are h.h., thus the Cantor set and the dyadic solenoid are h.h. Products of the solenoid with itself provide examples of spaces in all dimensions which are compact, connected, metric, l.s.c. and h.h. but are not l.p.c.

IV. The product of shm's is an shm.

THEOREM IV.1. *A locally orientable homotopy n-manifold with a contractible basis is an n-shm over Z.*

Proof. All locally orientable homotopy n-manifolds are singular n-manifolds, and for $n \leqq 2$, all homotopy n-manifolds are n-manifolds.

LEMMA IV.1. *If U and V are contractible spaces and $x \in U$, $y \in V$, and if $U - x$ is* 0*-connected and* 1*-connected, then $U \times V - \{(x, y)\}$ is* 1*-connected.*

Proof. Write $U \times V - \{(x, y)\} = U \times (V - y) \cup (U - x) \times V$. Van Kampen's theorem applies easily.

THEOREM IV. 2. *If X is an n-shm and Y is an m-shm, then $X \times Y$ is an $(m+n)$-shm.*

Proof. Since X and Y are locally contractible, they are HLC, so they are generalized manifolds. Thus $X \times Y$ is an $(m+n)$-gm. Since the product of contractible spaces is contractible, $X \times Y$ is locally contractible, thus a singular $(m+n)$-manifold [8]. For $m \leqq 2$ and $n \leqq 2$, this theorem is the product theorem for manifolds. Let $m \geqq 3$, and $x \in X$, $y \in Y$. Let U be a member of the contractible basis $\mathscr{U}_x$ at x and V a member of the contractible basis $\mathscr{V}_y$ at y (or an n-cell for $n \leqq 2$). $W = U \times V - \{(x,y)\}$ is 1-connected by Lemma 1, and it is clearly 0-connected. $U \times V$ is contractible, so $\mathscr{W} = \{U \times V \mid U \in \mathscr{U}_x, V \in \mathscr{V}_x\}$ is a contractible basis at (x,y), and $X \times Y$ is an $(m+n)$-shm.

COROLLARY IV. 1 (K. W. KWUN). *The product of polyhedral homotopy manifolds is a polyhedral homotopy manifold.*

Proof. Every polyhedron is locally conical, so by Lemma 2 of [7], it is locally orientable. Then by Theorem IV.1, a polyhedral homotopy manifold is an shm over Z, thus the product of two of them is an shm by Theorem IV.2, and a homotopy manifold by Theorem II.3.

REFERENCES

1. M. L. Curtis, *Homotopically homogeneous polyhedra*, Michigan Math. J. **8** (1961), 55–60.
2. ———, *The covering homotopy theorem*, Proc. Amer. Math. Soc. **7** (1956), 682–684.
3. Roger Godement, *Théorie des faisceaux*, Hermann, Paris, 1958.
4. H. B. Griffiths, *A contribution to the theory of manifolds*, Michigan Math. J. **2** (1953–54), 61–89.
5. P. J. Hilton and S. Wylie, *Homology theory*: Cambridge Univ. Press, New York, 1960.
6. Witold Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961.
7. C. N. Lee, *Kosinski's r-spaces and homology manifolds*, Michigan Math. J. **10** (1963), 289–293.
8. Frank Raymond, *Some remarks on the coefficients used in the theory of homology manifolds* (to appear).
9. ———, *A note on the local "C" groups of Griffiths*, Michigan Math. J. **7** (1960), 1–5.
10. J. H. C. Whitehead, *On the homotopy type of ANR's*, Bull. Amer. Math. Soc. **54** (1948), 1133–1145.
11. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloq. Publ. Vol. 32, Amer. Math. Soc., Providence, R. I., 1949.

FLORIDA STATE UNIVERSITY,
TALLAHASSEE, FLORIDA
UNIVERSITY OF GEORGIA,
ATHENS, GEORGIA