ON CHARACTERISTIC FUNCTIONS AND RENEWAL THEORY(1)

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1. Introduction and statement of results. In this paper we will present a self-contained treatment of renewal theory for a distribution function having finite moment of order m and first moment $\mu > 0$. For $m=1$ Blackwell's theorem is proven and for $m \geqq 2$ new estimates of the remainder term are obtained.

Let m denote any positive integer (see, however, the remark at the end of this section) and let F denote a one-dimensional right-continuous probability distribution function such that

$$\int_{-\infty}^{\infty} |x|^m dF(x) < \infty \quad \text{and} \quad \mu = \int_{-\infty}^{\infty} x\, dF(x) > 0.$$

Set

$$\mu_k = \int_{-\infty}^{\infty} x^k\, dF(x), \qquad k \leqq m,$$

let f denote the characteristic function of F defined by

$$f(\theta) = \int_{-\infty}^{\infty} e^{i\theta x}\, dF(x),$$

and let $F^{(n)}$ denote the n-fold convolution of F with itself.

We call F a lattice distribution function with lattice constant $d > 0$ if the measure corresponding to F is concentrated on the set $\{jd \mid -\infty < j < \infty\}$, but not on the set $\{jd' \mid -\infty < j < \infty\}$ for any $d' > d$. Without loss of generality we can assume $d = 1$. F is lattice with lattice constant $d = 1$ if and only if f is periodic with period 2π and $f(\theta) \neq 1$ for $0 < |\theta| \leqq \pi$. In this case, let $P_n(k)$ denote the jump of $F^{(n)}$ at k and set $p_k = P_1(k)$, $q_k = \sum_{j>k} p_j$ for $k \geqq 0$ and $q_k = \sum_{j \leqq k} p_j$ for $k < 0$, $r_k = \sum_{j>k} q_j$ for $k \geqq 0$ and $r_k = \sum_{j \leqq k} q_j$ for $k < 0$, $s_k = \sum_{j \geqq k} r_j$ for $k \geqq 0$ and $s_k = \sum_{j \leqq k} r_j$ for $k < 0$. Two quantities of interest in renewal theory are

$$u_k = \sum_{n=0}^{\infty} P_n(k) \qquad h_k = \sum_{j=-\infty}^{k} u_j .$$

THEOREM 1. *Let F be a lattice distribution function with lattice constant $d = 1$ having finite moment of order $m \geqq 1$ and first moment $\mu > 0$. Then u_k is finite and*

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(1) $$\lim_{k\to\infty}\left(u_k-\frac{1}{\mu}\right)=\lim_{k\to-\infty}u_k=0.$$

Suppose additionally that $m \geqq 2$. *Then*

(2) $$\lim_{k\to\infty}k^m\left(u_k-\frac{1}{\mu}-\frac{r_k}{\mu^2}\right)=\lim_{k\to-\infty}k^m\left(u_k-\frac{r_k}{\mu^2}\right)=0,$$

h_k *is finite, and*

(3) $$\lim_{k\to\infty}k^{m-1}\left(h_k-\frac{k}{\mu}-\frac{\mu_2+\mu}{2\mu^2}+\frac{s_k}{\mu^2}\right)=\lim_{k\to-\infty}k^{m-1}\left(h_k-\frac{s_k}{\mu^2}\right)=0.$$

Equation (1) is the well-known main renewal theorem ([3], [4], [5], [11]). (2) and (3) were strongly suggested by a recent result of A. O. Gelfond [9] that under the additional condition that $p_k=0$ for $k<0$,

$$u_k=\frac{1}{\mu}+\frac{r_k}{\mu^2}+O\left(\frac{\log k}{k^m}\right)\qquad \text{as } k\to\infty.$$

(3) follows immediately from (2) and the known result ([7], [10], [12], [15]) that

(4) $$\lim_{k\to\infty}\left(h_k-\frac{k}{\mu}-\frac{\mu_2+\mu}{2\mu^2}\right)=0.$$

Note that $k^{m-1}r_k\to 0$ as $|k|\to\infty$ and if $m\geqq 2$, then $k^{m-2}s_k\to 0$ as $|k|\to\infty$ and $\sum_{|k|\geqq 1}|k|^{m-2}r_k<\infty$. Thus we obtain immediately from Theorem 1 the following known results ([7], [10]).

COROLLARY 1. *Suppose that the conditions of Theorem* 1 *hold with* $m\geqq 2$. *Then*

(5) $$\lim_{k\to\infty}k^{m-1}\left(u_k-\frac{1}{\mu}\right)=\lim_{k\to-\infty}k^{m-1}u_k=0,$$

(6) $$\sum_{k=-\infty}^{-1}|k|^{m-2}u_k+\sum_{k=1}^{\infty}k^{m-2}\left|u_k-\frac{1}{\mu}\right|<\infty,$$

and

(7) $$\lim_{k\to\infty}k^{m-2}\left(h_k-\frac{k}{\mu}-\frac{\mu_2+\mu}{2\mu^2}\right)=\lim_{k\to-\infty}k^{m-2}h_k=0.$$

F is nonlattice if and only if $f(\theta)\neq 1$ for $\theta\neq 0$. As a special case, we call F strongly nonlattice if

$$\liminf_{|\theta|\to\infty}|1-f(\theta)|>0.$$

(This is easily shown to be equivalent to Cramer's condition C:

$$\limsup_{|\theta|\to\infty}|f(\theta)|<1.)$$

Set

$$R(x) = \begin{cases} \int_x^\infty (1 - F(y))\, dy, & x \geqq 0, \\ \int_{-\infty}^x F(y)\, dy, & x < 0, \end{cases}$$

and

$$S(x) = \begin{cases} \int_x^\infty R(y)\, dy, & x \geqq 0, \\ \int_{-\infty}^x R(y)\, dy, & x < 0. \end{cases}$$

We are interested in the functions

$$U(x,h) = \sum_{n=0}^{\infty} (F^{(n)}(x + h/2) - F^{(n)}(x - h/2)), \qquad h > 0,$$

and

$$H(x) = \sum_{n=0}^{\infty} F^{(n)}(x).$$

THEOREM 2. *Let F be a nonlattice distribution function having finite moment of order $m \geqq 1$ and first moment $\mu > 0$. Then $U(x,h)$ is finite and*

(8) $$\lim_{x\to\infty}\left(U(x,h) - \frac{h}{\mu}\right) = \lim_{x\to-\infty} U(x,h) = 0, \qquad h > 0.$$

Suppose additionally that $m \geqq 2$. Then for fixed $h > 0$

(9) $$U(x,h) - \frac{hR(x)}{\mu^2} \leqq o(R(x) + |x|^{-m}) \qquad \text{as } x \to -\infty;$$

also $H(x)$ is finite,

(10) $$\lim_{x\to\infty}\left(H(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2}\right) = 0,$$

and

(11) $$H(x) - \frac{S(x)}{\mu^2} \leqq o(S(x) + |x|^{1-m}) \qquad \text{as } x \to -\infty.$$

Equation (8) is Blackwell's theorem ([1], [2]) and (10) is also known (Smith [13]). As in the case of Theorem 1 we have immediately

COROLLARY 2. *Suppose that the conditions of Theorem 1 hold with $m \geqq 2$. Then*

(12) $$\lim_{x\to-\infty} x^{m-1} U(x,h) = 0, \qquad h > 0,$$

$$\int_{-\infty}^{0} |x|^{m-2} U(x,h)\, dx < \infty, \qquad h > 0, \tag{13}$$

and

$$\lim_{k \to -\infty} k^{m-2} H(x) = 0. \tag{14}$$

In order to obtain analogies of the other results of Theorem 1 and Corollary 1, we need to assume that F is strongly nonlattice.

THEOREM 3. *Let F be a strongly nonlattice distribution function having finite moment of order $m \geqq 2$ and first moment $\mu > 0$. Then*

$$\begin{aligned} &\lim_{x \to \infty} \frac{x^m}{\log x}\left(U(x,h) - \frac{h}{\mu} - \frac{hR(x)}{\mu^2}\right) \\ &\quad = \lim_{x \to -\infty} \frac{x^m}{\log |x|}\left(U(x,h) - \frac{hR(x)}{\mu^2}\right) = 0, \qquad h > 0, \end{aligned} \tag{15}$$

both limits being uniform for h in bounded sets, and

$$\begin{aligned} &\lim_{x \to \infty} \frac{x^{m-1}}{\log x}\left(H(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} + \frac{S(x)}{\mu^2}\right) \\ &\quad = \lim_{x \to -\infty} \frac{x^{m-1}}{\log |x|}\left(H(x) - \frac{S(x)}{\mu^2}\right) = 0. \end{aligned} \tag{16}$$

COROLLARY 3. *Suppose the conditions of Theorem 3 hold. Then*

$$\lim_{x \to \infty} x^{m-1}\left(U(x,h) - \frac{h}{\mu}\right) = 0, \qquad h > 0, \tag{17}$$

the limit being uniform for h in bounded sets,

$$\int_{0}^{\infty} x^{m-2} \left| U(x,h) - \frac{h}{\mu}\right| dx < \infty, \qquad h > 0, \tag{18}$$

and

$$\lim_{x \to \infty} x^{m-2}\left(H(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2}\right) = 0. \tag{19}$$

REMARK. It is immediately apparent that Corollaries 1–3 hold for $m \geqq 2$ but not necessarily an integer. For we have only to replace m by $[m]$ in the corresponding theorem.

2. **Preliminaries.** Let M, $0 < M < \infty$, be such that $f(\theta) \neq 1$ for $0 < |\theta| \leqq M$. Then *on* $|\theta| \leqq M$, $\Re(1/(1 - f(\theta)))$ *is integrable and* (Feller and Orey [8]) *as* $r \uparrow 1$ *the measure with density* $\Re(1/(1 - rf(\theta)))$ *converges weakly to the measure with density* $\Re(1/(1 - f(\theta)))$ *for* $\theta \neq 0$ *and a point measure of value* π/μ *at* $\theta = 0$.

For completeness we give a proof of these results. Since

$$\Re\left(\frac{1}{1-f(\theta)}\right) = \frac{\Re(1-f(\theta))}{|1-f(\theta)|^2},$$

in order to verify the integrability of $\Re(1/(1-f(\theta)))$ it suffices to verify that of

$$\begin{aligned}\theta^{-2}\Re(f(\theta)-1) &= \theta^{-2}\int_{-\infty}^{\infty}\Re(e^{i\theta x}-1-i\theta x)\,dF(x)\\ &= \theta^{-2}\int_{|\theta x|\le 1}\Re(e^{i\theta x}-1-i\theta x)\,dF(x)\\ &= \theta^{-2}\int_{|\theta x|>1}\Re(e^{-i\theta x}-1)\,dF(x).\end{aligned}$$

But this follows easily by an interchange of the order of integration. Moreover

$$\begin{aligned}&\lim_{r\uparrow 1}\int_{-M}^{M}\Re\left(\frac{1}{1-rf(\theta)}-\frac{1}{1-f(\theta)}\right)d\theta\\ &=\lim_{r\uparrow 1}\int_{-M}^{M}\Re\left(\frac{(r-1)f(\theta)}{(1-rf(\theta))(1-f(\theta))}\right)d\theta\\ &=\lim_{r\uparrow 1}\int_{-M}^{M}(1-r)\Re(f(\theta))\,\Im\left(\frac{1}{1-rf(\theta)}\right)\Im\left(\frac{1}{1-f(\theta)}\right)d\theta\\ &=\lim_{r\uparrow 1}\int_{-M}^{M}\frac{1-r}{|1-rf(\theta)|^2}\,d\theta\\ &=\lim_{r\uparrow 1}\int_{-M}^{M}\frac{1-r}{(1-r)^2+\mu^2\theta^2}\,d\theta=\frac{\pi}{\mu},\end{aligned}$$

as desired.

3. **Proof of Theorem 1.** We assume the conditions of Theorem 1 hold. Then

$$\begin{aligned}u_k &= \sum_{n=0}^{\infty}P_n(k)=\lim_{r\uparrow 1}\sum_{n=0}^{\infty}r^nP_n(k)\\ &=\lim_{r\uparrow 1}\sum_{n=0}^{\infty}r^n\frac{1}{2\pi}\int_{-\pi}^{\pi}\Re(e^{-ik\theta}f^n(\theta))\,d\theta\\ &=\lim_{r\uparrow 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Re\left(e^{-ik\theta}\left(\frac{1}{1-rf(\theta)}\right)\right)d\theta\\ &=\frac{1}{2\mu}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\Re\left(e^{-ik\theta}\left(\frac{1}{1-f(\theta)}\right)\right)d\theta<\infty.\end{aligned}$$

In particular, for $f(\theta) = e^{i\theta}$ we have

$$1 = \frac{1}{2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(e^{-ik\theta}\left(\frac{1}{1-e^{i\theta}}\right)\right) d\theta, \qquad k \geqq 0,$$

and

$$0 = \frac{1}{2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(e^{-ik\theta}\left(\frac{1}{1-e^{i\theta}}\right)\right) d\theta, \qquad k < 0.$$

Thus

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(e^{-ik\theta}\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right)\right) d\theta \tag{20}$$
$$= \begin{cases} u_k - \dfrac{1}{\mu} & k \geqq 0, \\ u_k & k < 0. \end{cases}$$

We first prove (1). Since $\Re\,(1/(1-f(\theta)) - 1/\mu(1-e^{i\theta}))$ is absolutely integrable on $-\pi \leqq \theta \leqq \pi$, it follows from the Riemann-Lebesgue lemma that

$$\lim_{|k| \to \infty} \int_{-\pi}^{\pi} \cos k\theta \Re\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right) d\theta = 0.$$

Also, for $k \neq 0$

$$\int_{-\pi}^{\pi} \sin k\theta \Im\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right) d\theta$$
$$= \int_{|\theta| \leqq \pi/2|k|} \sin k\theta \Im\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right) d\theta$$
$$+ \int_{\pi/2|k| < |\theta| \leqq \pi} \frac{\cos k\theta}{k} \Im\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right)' d\theta$$
$$= o_k(1) \qquad \text{as } |k| \to \infty,$$

and (1) now follows from (20).

We suppose next that $m \geqq 2$ and proceed to a proof of (2). Write

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(e^{-ik\theta}\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right)\right) d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta}\left(\frac{1}{1-f(\theta)} - \frac{1}{\mu(1-e^{i\theta})}\right) d\theta$$
$$= \frac{1}{2\pi\mu^2} \int_{-\pi}^{\pi} \frac{e^{-ik\theta}(f(\theta) - 1 + \mu(1-e^{i\theta}))}{(1-e^{i\theta})^2} d\theta \tag{21}$$
$$+ \frac{1}{2\pi\mu^2} \int_{-\pi}^{\pi} \frac{e^{-ik\theta}(f(\theta) - 1 + \mu(1-e^{i\theta}))^2}{(1-e^{i\theta})^2(1-f(\theta))} d\theta.$$

Observe that

$$r_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-ik\theta}\,(f(\theta) - 1 + \mu(1 - e^{i\theta}))}{(1 - e^{i\theta})^2}\, d\theta. \tag{22}$$

For if $k \neq 0, 1$

$$p_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta}\,(f(\theta) - 1 + \mu(1 - e^{i\theta}))\, d\theta;$$

and hence if $k \geqq 0$

$$q_{k+1} = \sum_{j=k+2}^{\infty} p_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(k+2)\theta}(f(\theta) - 1 + \mu(1 - e^{i\theta}))}{1 - e^{-i\theta}}\, d\theta,$$

and another summation yields (22) for $k \geqq 0$. A similar proof works for $k < 0$.

By (20)–(22), in order to prove (2) we must show that

$$k^m \int_{-\pi}^{\pi} e^{-ik\theta}\,\theta \frac{\left(\dfrac{f(\theta) - 1 + \mu(1 - e^{i\theta})}{\theta^2}\right)^2}{\left(\dfrac{1 - e^{i\theta}}{\theta}\right)^2 \left(\dfrac{1 - f(\theta)}{\theta}\right)}\, d\theta = o(1) \quad \text{as } |k| \to \infty. \tag{23}$$

The general idea is to integrate by parts the left side of (23) m times and get a finite number of terms, each of the form $o(1)$.

We now make some observations valid for an arbitrary distribution function F with finite moment of order $m \geqq 2$ and first moment μ $(-\infty < \mu < \infty)$. These observations will also be used in the proofs of Theorems 2 and 3. First

$$f(\theta) = \sum_{j=0}^{m} \frac{(i\theta)^j}{j!}\mu_j + \int_{-\infty}^{\infty} \left(e^{ix\theta} - \sum_{j=0}^{m} \frac{(ix\theta)^j}{j!}\right) dF(x). \tag{24}$$

Thus for $k \leqq m$

$$\begin{aligned} &\left(\frac{1}{\theta}\left(f(\theta) - \sum_{j=0}^{m} \frac{(i\theta)^j}{j!}\mu_j\right)\right)^{(k)} \\ &= \sum_{\nu=0}^{k} \binom{k}{\nu} (-1)^{\nu} \nu!\, \theta^{-\nu-1} \int_{-\infty}^{\infty} (ix)^{k-\nu} \left(e^{ix\theta} - \sum_{j=0}^{m-k+\nu} \frac{(ix\theta)^j}{j!}\right) dF(x) \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\left(\frac{1}{\theta^2}\left(f(\theta) - \sum_{j=0}^{m} \frac{(i\theta)^j}{j!}\mu_j\right)\right)^{(k)} \\ &= \sum_{\nu=0}^{k} \binom{k}{\nu} (-1)^{\nu} (\nu+1)!\, \theta^{-\nu-2} \int_{-\infty}^{\infty} (ix)^{k-\nu} \left(e^{ix\theta} - \sum_{j=0}^{m-k+\nu} \frac{(ix\theta)^j}{j!}\right) dF(x). \end{aligned} \tag{26}$$

Consequently

$$\lim_{\theta\to 0}\left(\frac{f(\theta)-1}{\theta}\right)^{(k)} = \frac{i^{k+1}\mu_{k+1}}{k+1}, \quad k \leqq m-1, \tag{27}$$

$$\lim_{\theta\to 0}\theta\left(\frac{f(\theta)-1}{\theta}\right)^{(m)} = 0, \tag{28}$$

$$\lim_{\theta\to 0}\left(\frac{f(\theta)-1-i\mu\theta}{\theta^2}\right)^{(k)} = \frac{i^{k+2}\mu_{k+2}}{(k+2)(k+1)}, \quad k \leqq m-2, \tag{29}$$

$$\lim_{\theta\to 0}\theta\left(\frac{f(\theta)-1-i\mu\theta}{\theta^2}\right)^{(m-1)} = 0, \tag{30}$$

and

$$\lim_{\theta\to 0}\theta^2\left(\frac{f(\theta)-1-i\mu\theta}{\theta^2}\right)^{(m)} = 0. \tag{31}$$

Moreover (for $m \geqq 2$)

$$\int_{-M}^{M}\left|\left(\frac{f(\theta)-1-i\mu\theta}{\theta^2}\right)^{(m-1)}\right| d\theta < \infty, \quad 0 < M < \infty. \tag{32}$$

To see this we set $k = m-1$ in (26) and break up the range of integration of the integral on the right side of (26) into two parts, corresponding to $|x\theta| \leqq 1$ and $|x\theta| > 1$. The contribution of the first part is dominated by a constant times the integrable function

$$\int_{|x\theta|\leqq 1} |x|^{m+1}\, dF(x).$$

The contribution of the second part is dominated by a constant times the integrable function

$$\theta^{-2}\int_{|x\theta|>1} |x|^{m-1}\, dF(x).$$

This results from the cancellation of the highest order terms, a consequence of the identity

$$\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}(-1)^\nu = (1-1)^{m-1} = 0, \qquad m \geqq 2.$$

We would like to assert that (for $m \geqq 2$)

$$\int_{-M}^{M}\left|\theta\left(\frac{f(\theta)-1-i\mu\theta}{\theta^2}\right)^{(m)}\right| d\theta < \infty.$$

Set $k = m$ in (26) and break up the range of integration of the integral on the right side of (26) into two parts, corresponding to $|x\theta| \leqq N$ and $|x\theta| > N$. The contribution of the first part is dominated by a constant times the integrable function

$$\int_{|x\theta|\leqq N} |x|^{m+1} dF(x).$$

The contribution of the second part is dominated by a constant times the integrable function

$$\theta^{-2} \int_{|x\theta|>N} |x|^{m-1}\, dF(x)$$

plus the function

$$\theta^{-1} i^m \int_{|x\theta|>N} x^m e^{i\theta x}\, dF(x).$$

This function does not appear to be absolutely integrable. However

$$\begin{aligned}(33)\qquad &\int_{-M}^{M} ie^{-ik\theta}\theta^{-1}\, d\theta \int_{|x\theta|>N} x^m e^{i\theta x} dF(x) \\ &= -\int_{|xM|>N} x^m dF(x) \int_{N<|x\theta|\leqq|Mx|} \frac{\sin(x-k)\theta}{\theta} d\theta = o_N(1)\end{aligned}$$

where for fixed M, $o_N(1) \to 0$ as $N \to \infty$ uniformly in k (k is not necessarily an integer here).

We would also like to assert that

$$\int_{-M}^{M} \left| \theta^{-3} \left(f(\theta) - 1 - i\mu\theta + \frac{\mu_2 \theta^2}{2} \right) \right| d\theta < \infty.$$

From (24) and the methods used above we see that the function appearing under the absolute value signs is the sum of an integrable function and the function

$$\frac{1}{2\theta} \int_{|x\theta|>N} x^2 dF(x).$$

This function is integrable if $m > 2$ but not necessarily if $m = 2$. Even in the latter case, however,

$$\begin{aligned}(34)\qquad &\int_{-M}^{M} ie^{-ik\theta}\theta^{-1}\, d\theta \int_{|x\theta|>N} x^2 dF(x) \int_{|y\theta|>N} y^2 e^{iy\theta} dF(y) \\ &= -\int_{|xM|>N} x^2 dF(x) \int_{|yM|>N} y^2 dF(y) \int_{*} \frac{\sin(y-k)\theta}{\theta} d\theta = o_N(1)\end{aligned}$$

where for fixed M, $o_N(1) \to 0$ as $N \to \infty$ uniformly in k, k not necessarily an integer. (*The range of integration is $\max\{N/|x|, N/|y|\} < |\theta| \leqq \pi$.)

Note finally that if F is a lattice distribution function with lattice constant $d = 1$, then

$$\left(\frac{i^{j+1}(f(\theta) - 1 + \mu(1 - e^{i\theta}))^2}{(1 - e^{i\theta})^2(1 - f(\theta))}\right)^{(j)}\Bigg|_{-\pi}^{\pi} = 0, \qquad j \leqq m. \tag{35}$$

If we set $M = \pi$ in the above discussion and recall the Riemann-Lebesgue lemma, we have just what is needed to prove (23) and hence (2) as well.

To complete the proof of Theorem 1, we show that (4) holds. Let $u_k' = u_k$ for $k<0$ and $u_k' = u_k - \mu^{-1}$ for $k \geqq 0$. Then for $m \geqq 2$, $\sum_{-\infty}^{\infty} |u_k'| < \infty$ and

$$\sum_{-\infty}^{\infty} e^{ik\theta} u_k' = \frac{1}{1 - f(\theta)} - \frac{1}{\mu(1 - e^{i\theta})}.$$

Thus

$$\begin{aligned}\lim_{k\to\infty}\left(h_k - \frac{k+1}{\mu}\right) &= \lim_{k\to\infty} \sum_{-\infty}^{k} u_k' = \sum_{-\infty}^{\infty} u_k' \\ &= \lim_{\theta\to 0}\left(\frac{1}{1 - f(\theta)} - \frac{1}{\mu(1 - e^{i\theta})}\right) = \frac{\mu_2}{2\mu^2} - \frac{1}{2\mu}.\end{aligned}$$

4. **Proof of Theorems 2 and 3.** We assume throughout the rest of the paper that F is nonlattice. Some of the techniques used in this section were suggested by another recent work of the author [14].

Choose a probability density function $K(x)$, $-\infty < x < \infty$, with characteristic function $k(\theta)$, $-\infty < \theta < \infty$, such that $K(x)$ is symmetric about $x = 0$ and has finite moment of order $m + 2$ and $k(\theta) = 0$ for $|\theta| \geqq 1$. (Examples of such functions can be constructed easily along the lines suggested by Esseen [6, pp. 30–36].)

For fixed $n \geqq 0$ and $h > 0$, the function $F^{(n)}(x + h/2) - F^{(n)}(x - h/2)$, $-\infty < x < \infty$, is absolutely integrable and

$$\int_{-\infty}^{\infty} e^{ix\theta}(F^{(n)}(x + h/2) - F^{(n)}(x - h/2))\, dx = \frac{\sin h\theta/2}{h\theta/2} f^n(\theta).$$

Fourier inversion yields that for $a > 0$

$$\begin{aligned}&\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} r^n(F^{(n)}(x + h/2 - y) - F^{(n)}(x - h/2 - y))a^{-1}K(a^{-1}y)\, dy \\ &\quad = \frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} \Re\left(e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right)k(a\theta)\left(\frac{1}{1 - rf(\theta)}\right)\right) d\theta.\end{aligned}$$

Set

$$V(x, h, a) = \int_{-\infty}^{\infty} a^{-1}K(a^{-1}y)U(x - y, h)\, dy.$$

From the monotone convergence theorem we obtain

(36) $$V(x,h,a) = \frac{h}{2\mu} + \frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} \Re\left(e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right)k(a\theta)\left(\frac{1}{1-f(\theta)}\right)\right)d\theta.$$

It is easily seen that

$$\lim_{x\to\pm\infty} \frac{1}{2\pi}\int_{-a^{-1}}^{a^{-1}} \Re\left(e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right)k(a\theta)\frac{i}{\mu\theta}\right)d\theta = \pm\frac{h}{2\mu}.$$

By the same method as used in proving (1) we get

(37) $$\lim_{x\to\infty}\left(V(x,h,a)-\frac{h}{\mu}\right) = \lim_{x\to-\infty} V(x,h,a)=0.$$

Thus for each a and h, $V(x,h,a)$ is bounded as a function of x. From the obvious inequality

$$U(x-y,2h) \geqq U(x,h), \qquad |y| \leqq \frac{h}{2},$$

we see that $U(x,h)$ is finite and, for fixed $h>0$, is bounded in x.

We now complete the proof of (8). Let $h>0$ be fixed and let $N<\infty$ be an upper bound to $U(x,h)$, $-\infty<x<\infty$. Choose ε $(0<\varepsilon\leqq 1/2)$ and a_0 such that

$$\int_{|y|\geqq\varepsilon h} a_0^{-1}K(a_0^{-1}y)\,dy \leqq \varepsilon.$$

Then from

$$U(x-y,(1-2\varepsilon)h) \leqq U(x,h) \leqq U(x-y,(1+2\varepsilon)h), \qquad |y|\leqq\varepsilon h,$$

it follows that

$$V(x,(1-2\varepsilon)h,a_0) - N\varepsilon \leqq U(x,h) \leqq (1-\varepsilon)^{-1}V(x,(1+\varepsilon)h,a_0).$$

Since ε is arbitrary, (8) now follows from (37).

We show next that

(38) $$\frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} \Re\left(e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right)k(a\theta)\frac{i}{\theta}\right)d\theta \mp \frac{h}{2} = \frac{o_x(1)}{x^m},$$

where $o_x(1)\to 0$ as $|x|\to\infty$ uniformly for a and h in bounded sets. Indeed, (38) follows from the inversion formula

(39 $$\frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} \Re\left(e^{-ix\theta}\frac{\sin h\theta/2}{h\theta/2}k(a\theta)\frac{i}{\theta}\right)d\theta = \int_{-h/2}^{h/2}dz\int_{-\infty}^{x+z}K_a(y)\,dy - \frac{h}{2}.$$

We assume from now on that $m \geqq 2$. Write

$$\frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\left(\frac{1}{1-f(\theta)}-\frac{i}{\mu\theta}\right) d\theta$$

$$= \frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\left(\frac{f(\theta)-1-i\mu\theta}{-\mu^2\theta^2}\right) d\theta$$

$$-\frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\left(\frac{(f(\theta)-1-i\mu\theta)^2}{\mu^2\theta^2(1-f(\theta))}\right) d\theta$$

$$= I_1 + I_2.$$

Then for fixed a and h, $x^m I_2 = o_x(1)$ as $|x| \to \infty$. If F is strongly nonlattice, then $x^m I_2 = (1+|\log a|)o_x(1)$, where $o_x(1) \to 0$ as $|x| \to \infty$ uniformly for a and h in bounded sets. The proof of these two results is patterned after the proof of (23). We give the proof of the second result, that of the first being similar and easier. We must justify

$$\text{(40)}\qquad \frac{-h}{2\pi\mu^2}\int_{-a^{-1}}^{a^{-1}} i^{-m}e^{-ix\theta}\left[\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\theta\,\frac{\left(\dfrac{f(\theta)-1-i\mu\theta}{\theta^2}\right)^2}{\left(\dfrac{1-f(\theta)}{\theta}\right)}\right]^{(m)} d\theta$$

$$= (1+|\log a|)o_x(1).$$

New difficulties arise only for a small, so we can assume $0 < a < 1$. The contribution of the integral over $|\theta| \leqq 1$ is of the form $o_x(1)$, just as in the proof of (23). The only term of the contribution of the integral over $1 < |\theta| \leqq a^{-1}$ which is not clearly of the form $o_x(1)$ is

$$\frac{1}{2\pi}\int_{1<|\theta|\leqq a^{-1}} i^{-m}e^{-ix\theta}(2\sin h\theta/2)\frac{k(a\theta)f^{(m)}(\theta)}{\theta(1-f(\theta))^2}\,d\theta.$$

Since the family of functions

$$(\sin h\theta/2)k(a\theta)f^{(m)}(\theta), \qquad 1 < |\theta| < \infty$$

(where a and h range over bounded sets), is uniformly bounded and equicontinuous, it follows that this last integral is of the form $(1+|\log a|)o_x(1)$, as desired.

Next we consider the term I_1. It is easily shown that

$$\frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)(1-i\mu\theta)\,d\theta = x^{-m-2}o_x(1).$$

Thus

$$\int_{-\infty}^{\infty} a^{-1}K(a^{-1}y)(F(x+h/2-y)-F(x-h/2-y))\,dy$$

$$=\frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\ (f(\theta)-1-i\mu\theta)\,d\theta + x^{-m-2}o_x(1).$$

Set

$$Q(x,h)=\begin{cases}\displaystyle\int_x^{\infty}(F(y+h/2)-F(y-h/2))\,dy, & x\geqq 0,\\ \displaystyle\int_{-\infty}^{x}(F(y+h/2)-F(y-h/2))\,dy, & x<0,\end{cases}$$

and

$$R(x,h)=\begin{cases}\displaystyle\int_x^{\infty}Q(y,h)\,dy, & x\geqq 0,\\ \displaystyle\int_{-\infty}^{x}Q(y,h)\,dy, & x<0.\end{cases}$$

Then

$$\int_{-\infty}^{\infty} a^{-1}K(a^{-1}y)Q(x-y)\,dy$$

$$=\pm\frac{h}{2\mu}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\frac{f(\theta)-1-i\mu\theta}{i\theta}\,d\theta + x^{-m-1}\,o_x(1)$$

as $x\to\pm\infty$. Further

$$\int_{-\infty}^{\infty} a^{-1}K(a^{-1}y)R(x-y,h)\,dy$$

$$=\frac{h}{2\mu}\int_{-a^{-1}}^{a^{-1}} e^{-ix\theta}\left(\frac{\sin h\theta/2}{h\theta/2}\right) k(a\theta)\,\frac{(f(\theta)-1-i\mu\theta)}{-\theta^2}\,d\theta + x^{-m}o_x(1).$$

It is easy to show that

$$\begin{aligned}\int_{-\infty}^{\infty} a^{-1}K(a^{-1}y)R(x-y,h)\,dy &= R(x,h)+x^{-m}o_x(1)\\ &= hR(x)+x^{-m}o_x(1).\end{aligned}$$

Thus we have that

$$I_1=\frac{hR(x)}{\mu^2}+x^{-m}o_x(1), \tag{41}$$

where $o_x(1)\to 0$ as $|x|\to\infty$ uniformly for a and h in bounded sets.

We can now prove (9). We see from the above that for fixed a and h

$$\int_{-\infty}^{\infty} a^{-1}K(a^{-1}y)U(x-y,h)\,dy = \frac{hR(x)}{\mu^2} + o(x^{-m}) \quad \text{as } x \to -\infty.$$

Let $h > 0$ be fixed. Choose $\varepsilon > 0$. Let $a_0 > 0$ be such that

$$\int_{|y| \geqq \varepsilon h} a_0^{-1}K(a_0^{-1}y)\,dy \leqq \varepsilon.$$

Clearly

$$U(x-y,(1+2\varepsilon)h) \geqq U(x,h), \qquad |y| \leqq \varepsilon h,$$

and hence

$$U(x,h) \leqq (1-\varepsilon)^{-1}\frac{(1+2\varepsilon)R(x)}{\mu^2} + o(x^{-m}).$$

Since ε is arbitrary, we have (9) as desired.

We suppose now that F is strongly nonlattice and proceed to a proof of (15). From the above we have that

$$\text{(42)} \qquad V(x,h,a) = \begin{cases} \dfrac{h}{\mu} + \dfrac{hR(x)}{\mu^2} + (1+\log|a|)x^{-m}o_x(1), & x \to \infty, \\[2ex] \dfrac{hR(x)}{\mu^2} + (1+\log|a|)x^{-m}o_x(1), & x \to -\infty, \end{cases}$$

where $o_x(1) \to 0$ as $x \to \pm\infty$ uniformly for a and h in bounded sets. By a proof similar to that of (9) we see that (15) holds for $h \leqq 2|x|^{-m-1}$. Choose $M > 0$. Let N be such that $U(x,h) \leqq N$ for $h \leqq M$ and $-\infty < x < \infty$. Choose $\varepsilon > 0$. There is an x_0 such that for $|x| \geqq x_0$

$$\int_{|y| \geqq |x|} K(y)\,dy \leqq \frac{\varepsilon}{|x|^m} < 1.$$

Then for $h \geqq 2|x|^{-m-1}$

$$U\left(x-y,\, h - \frac{2}{|x|^{m+1}}\right) \leqq U(x,h)$$

$$\leqq U\left(x-y,\, h + \frac{2}{|x|^{m+1}}\right), \quad |y| \leqq \frac{1}{|x|^{m+1}}.$$

Consequently, for $2|x|^{-m-1} \leqq h \leqq M$ and $|x| \geqq x_0$

$$V\left(x,\, h - \frac{2}{|x|^{m+1}},\, \frac{1}{|x|^{m+2}}\right) - \frac{N\varepsilon}{|x|^m} \leqq U(x,h)$$

$$\leqq \left(1 - \frac{\varepsilon}{|x|^m}\right)^{-1} V\left(x, h + \frac{2}{|x|^{m+1}},\, \frac{1}{|x|^{m+2}}\right).$$

Since ε is arbitrary (42) now yields (15), as desired.

We now give a proof of (10), from which (16) follows easily. We first prove that for each a and h

$$\lim_{x\to\infty} \int_{-\infty}^{x} V(y,h,a)dy - h\left(\frac{x}{\mu}+\frac{\mu_2}{2\mu^2}\right) = 0. \tag{43}$$

From (36) and (39) we have that

$$V(y,h,a) - \frac{1}{\mu}\int_{-h/2}^{h/2} dz \int_{-\infty}^{y+z} K_a(w)\,dw$$
$$= \frac{h}{2\pi}\int_{-a^{-1}}^{a^{-1}} e^{-iy\theta}\frac{\sin h\theta/2}{h\theta/2}k(a\theta)\left(\frac{1}{1-f(\theta)}-\frac{i}{\mu\theta}\right)d\theta$$

and hence

$$\lim_{x\to\infty}\int_{-\infty}^{x} V(y,h,a)dx - \frac{hx}{\mu}$$
$$= \int_{-\infty}^{\infty}\left(V(y,h,a) - \frac{1}{\mu}\int_{-h/2}^{h/2} dz \int_{-\infty}^{y+z} K_a(w)dw\right)dy$$
$$= \lim_{\theta\to 0}\left(\frac{1}{1-f(\theta)}-\frac{i}{\mu\theta}\right) = \frac{h\mu_2}{2\mu^2},$$

i.e. (43) holds.

Since the probability density function $a^{-1}K(a^{-1}y)$, $-\infty < y < \infty$, has zero first moment

$$\int_{-\infty}^{x} V(y,h,a)\,dy$$
$$= \int_{-\infty}^{\infty} a^{-1}K(a^{-1}z)\,dz\left(\int_{-\infty}^{x} U(y,h)\,dy - \int_{x-z}^{x}\left(U(y,h)-\frac{h}{\mu}\right)dy\right).$$

From (8), the boundedness of $U(x,h)$, and the dominated convergence theorem

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} a^{-1}K(a^{-1}z)dz\int_{x-z}^{x}\left(U(y,h)-\frac{h}{\mu}\right)dy = 0.$$

Thus (43) implies that

$$\lim_{x\to\infty}\int_{-\infty}^{x} U(y,h)dy - h\left(\frac{x}{\mu}+\frac{\mu_2}{2\mu^2}\right) = 0.$$

Since

$$\int_{-\infty}^{x} U(y,h)dy = \int_{-\infty}^{x}(H(y+h/2)-H(y-h/2))dy = \int_{x-h/2}^{x+h/2} H(y)\,dy,$$

(10) now follows from the monotonicity of H.

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