QUASI-MARTINGALES

BY
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Introduction. The basic ideas contained in this paper were first introduced by Professor Herman Rubin in an invited address at the Institute of Mathematical Statistics meetings at the University of Washington in 1956([1]).

In this paper we investigate necessary and sufficient conditions for a stochastic process X_T to have a decomposition into the sum of a martingale process and a process having almost every sample function of bounded variation on T. Such a process is called a quasi-martingale.

Necessary and sufficient conditions for such a decomposition have already been obtained by P. Meyer [3] when the process is a sub-martingale. Johnson and Helms [4] have given conditions equivalent to Meyer's when the sub-martingale is sample continuous.

Our main result, Theorem 3.3, gives necessary and sufficient conditions for a sample continuous process X_T to have the above decomposition, where both the processes in the decomposition are sample continuous and the process of sample bounded variation has finite expected variation. When the process is a sample continuous sub-martingale, the conditions reduce to those given in [4].

It is further proved that the decomposition of Theorem 3.3 is unique. The uniqueness follows from Lemma 3.3.1 where we have proved that a martingale which is sample continuous, and of sample bounded variation has constant sample functions. This property, known true for Brownian motion, is seen to be true for all sample continuous martingales.

The dominating technique used throughout the paper is random stopping times defined in terms of the sample functions of the process. The major result involving stopping times is Theorem 2.2 which allows us to approximate a sample continuous process by a sequence of sample equicontinuous and uniformly bounded processes.

1. Notation, definitions and examples.

Let (Ω,F,P) be a probability space on which is defined a family of random variables (r.v.'s) $\{X(t);\ t \in T\}$ where T is a subset of the real line. Let $\{F(t);\ t \in T\}$ be a family of sub σ-fields of F with

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$F(s) \subset F(t)$ for every $s, t \in T$ with $s \leqq t$. The family of r.v.'s is said to be well adapted to the family of σ-fields if $X(t)$ is $F(t)$ measurable for every $t \in T$, and we will write $\{X(t), F(t); t \in T\}$ to indicate this relation. Whenever we speak of a stochastic process X, we will be referring to a family $\{X(t), F(t); t \in T\}$ as defined above. In many cases $F(t)$ is the minimal σ-field with respect to (w.r.t.) which the family of r.v.'s $\{X(s); s \leqq t\}$ is measurable. Such σ-fields are denoted by $\beta(X(s); s \leqq t)$.

We will assume T is the closed unit interval $[0, 1]$.

DEFINITION 1.1. A process $\{X(t), F(t); t \in T\}$ is almost surely sample continuous if there exists $\Lambda \in F$ with $P(\Lambda) = 0$ such that for every $t \in T$

$$\lim_{s \to t} X(s, \omega) = X(t, \omega) \quad \text{if } \omega \notin \Lambda.$$

We will assume the separability properties of stochastic processes are well known [1, p. 50]. We note here that if X is almost surely (a.s.) sample continuous, then X is a separable process w.r.t. the class of closed sets; and if T_0 is any denumerable dense subset of T, then it is a separating set.

DEFINITION 1.2. A process $\{X(t), F(t); t \in T\}$ is called a martingale process if $E(X(t))$ exists for every $t \in T$ and if for every $s, t \in T$ with $s \leqq t$, $E(X(t) \mid F(s)) = X(s)$ a.s., and is caleld a sub-martingale, super-martingale, if respectively

$$E(X(t) \mid F(s)) \geqq X(s), \quad E(X(t) \mid F(s)) \leqq X(s) \text{ a.s.}$$

DEFINITION 1.3. A process $\{X(t), F(t); t \in T\}$ is a.s. of bounded variation (b.v.) if there exists a set $\Lambda \in F$ with $P(\Lambda) = 0$ such that each sample function $X(\cdot, \omega)$ with $\omega \notin \Lambda$ is of b.v. on T.

DEFINITION 1.4. The process $\{X(t), F(t); t \in T\}$ will be called a quasi-martingale if there exists a martingale process $\{X_1(t), F(t); t \in T\}$ and a process

$$\{X_2(t), F(t); t \in T\}$$

with a.e. sample function of b.v. on T such that

$$P([X(t) = X_1(t) + X_2(t); t \in T]) = 1,$$

where $[\cdots]$ denotes the subset of Ω for which "$\cdots$" is true.

If X is a quasi-martingale, we will let $[X]_1$ and $[X]_2$ denote respectively the martingale process and process of b.v. in the decomposition of X.

We now give two simple examples of quasi-martingales.

Let $\{X(t), F(t); t \in T\}$ be a process with independent increments where $F(t) = \beta(X(s); s \leqq t)$, and let $E(X(t)) = m(t)$ exist for all $t \in T$. Assuming $m(0) = 0$, if $s \leqq t$, $E(X(t) \mid F(s)) = X(s) + m(t) - m(s)$. Define $X_2(t) = m(t)$ a.s. and $X_1(t) = X(t) - X_2(t)$ for every $t \in T$. The X-process is a quasi-martingale if $m(t)$ is of b.v. on T.

Let $\{Z(t), F(t); t \in T\}$ be the Brownian motion process; i.e. the process has independent, normally distributed increments with $E(Z(t) - Z(s)) = 0$ and

$E(|X(t) - X(s)|^2) = \sigma^2|t - s|$ where $\sigma > 0$ is fixed. We assume $Z(0) = 0$ a.s. so that the process is a martingale. We further assume $F(t) = \beta(X(s); s \leqq t)$.

Define $X(t) = \exp[Z(t)x]$ for every $t \in T$, where x is an arbitrary positive real number. If $u > 0$, $t + u \leqq 1$,

$$E(X(t + u) \mid F(t)) = E(\exp[(Z(t) + Z(t + u) - Z(t))x] \mid F(t))$$
$$= \exp[Z(t)x]E(\exp[(Z(t + u) - Z(t))x]) = X(t) \exp[u(\sigma x)^2 / 2].$$

If we let

$$X_2(t) = \int_0^t \frac{(\sigma x)^2}{2} X(s)ds \quad \text{for every } t \in T,$$

then the process $\{X_2(t), F(t); t \in T\}$ has a.e. sample function of b.v. on T. It is easily verified that the process $\{X_1(t) = X(t) - X_2(t), F(t); t \in T\}$ is a martingale.

Work on the decomposition of super-martingales has been done by P. Meyer [3] and Johnson and Helms [4]. We will obtain the necessary and sufficient conditions given by Johnson and Helms for the decomposition of an a.s. sample continuous super-martingale as a corollary to our decomposition theorem for quasi-martingales.

2. **Stopping times.** We consider briefly random stopping of a process [2, pp. 530–535] and prove a basic theorem to be used throughout the paper.

Let $\{X(t), F(t); t \in T\}$ be a process defined on the probability space (Ω, F, P) and let $\tau(\omega)$ be a r.v. defined on the same probability space with range T. If for each $t \in T$, $[\tau(\omega) \leqq t] \in F(t)$ the r.v. $\tau(\omega)$ is called a stopping time of the X-process. If we define

$$X_\tau(t, \omega) = X(t, \omega) \qquad t \leqq \tau(\omega)$$
$$(2.1) \qquad\qquad = X(\tau(\omega), \omega) \quad t > \tau(\omega)$$

then $X_\tau(t, \cdot\,)$ is $F(t)$ measurable for every $t \in T$ and the process $\{X_\tau(t), F(t); t \in T\}$ is called the X-process stopped at τ.

The following is a standard theorem which we state here for reference [2, p. 533].

THEOREM 2.1. *If $\{X(t), F(t); t \in T\}$ is an a.s. sample right continuous sub-martingale (martingale) and if τ is a stopping time of the process, then the stopped process $\{X_\tau(t), F(t); t \in T\}$ is also a sub-martingale (martingale).*

DEFINITION 2.2. A process $\{X(t), F(t); t \in T\}$ is a.s. sample equicontinuous if for each $t \in T$, there is a set $\Lambda_t \in F$ with $P(\Lambda_t) = 0$ such that if $\varepsilon > 0$ is given there exists a $\delta > 0$ such that $|X(t, \omega) - X(s, \omega)| < \varepsilon$ whenever $|t - s| < \delta$ for every $\omega \notin \Lambda_t$. If T is compact, then we can find $\Lambda \in F$ independent of $t \in T$ with $P(\Lambda) = 0$.

THEOREM 2.2. *Let $\{X(t), F(t); t \in T\}$ be an a.s. sample continuous process. There is a sequence of processes $\{X_\nu(t), F(t); t \in T\}$ $\nu \geqq 1$ with the following properties:*

(i) *For each* $\nu \geqq 1$ *the* X_ν*-process is*

(a) *a.s. sample equicontinuous, and*

(b) *uniformly bounded by* ν.

(ii) *There is a set* $\Lambda \in F$ *with* $P(\Lambda) = 0$ *such that if* $\omega \notin \Lambda$, *then there exists* $\nu(\omega)$ *such that* $X(t,\omega) = X_\nu(t,\omega)$ *for every* $t \in T$ *if* $\nu \geqq \nu(\omega)$.

(iii) *If* $\lim_{r\to\infty} rP([\sup_t |X(t)| \geqq r]) = 0$, *then*

$$\lim_{\nu\to\infty} E(|X_\nu(t) - X(t)|) = 0 \quad \textit{for every } t \in T.$$

The proof of this theorem is easily obtained from the following lemmas.

LEMMA 2.2.1. *If the X-process is as defined in Theorem* 2.2, *then there exists a sequence of processes satisfying conditions* (ia) *and* (ii).

Proof. We show there exists a sequence of processes X_ν, $\nu \geqq 1$ such that

$$P([X_\nu \neq X]) < 2^{-\nu}.$$

It then follows from the Borel-Cantelli lemma that property (ii) is satisfied.

By the a.s. sample continuity of the X-process, for each $n \geqq 1$ we can find $\delta_{n\nu} > 0$ such that

$$P\left(\left[\sup_{|t-s|\leqq\delta_{n\nu}} |X(t,\omega) - X(s,\omega)| \geqq 1/n\right]\right) \leqq 2^{-(n+\nu)}$$

for every $\nu \geqq 1$. We can assume that for each n, $\delta_{n1} > \delta_{n2} > \cdots$. Let $\tau_{n\nu}(\omega)$ be the first t such that

$$\sup_{|s-s'|\leqq\delta_{n\nu};\,s,s'\leqq t} |X(s,\omega) - X(s',\omega)| \geqq 1/n.$$

If no such t exists define $\tau_{n\nu}(\omega) = 1$. Then $0 < \tau_{n\nu} \leqq 1$ a.s., and $\tau_{n\nu}$ is a stopping time of the X-process since for any $t \in T$,

$$[\tau_{n\nu}(\omega) > t] = \left[\sup_{|s-s'|\leqq\delta_{n\nu};\,s,s'\leqq t} |X(s',\omega) - X(s,\omega)| < 1/n\right].$$

Define $\tau_\nu(\omega) = \inf_n \tau_{n\nu}(\omega)$. $0 \leqq \tau_\nu(\omega) \leqq 1$ a.s. for all $\nu \geqq 1$. $\tau_\nu(\omega)$ will be a stopping time for the X-process if $[\tau_\nu(\omega) \leqq t]$ differs from an $F(t)$ set by a set of measure zero [1, p. 365].

Let $\Lambda_\nu(t) = [\tau_{n\nu}(\omega) > t$ for every $n]$, $\nu \geqq 1$, $t \in [0,1)$. Then

$$[\tau_\nu(\omega) \leqq t] = \{[\tau_\nu(\omega) \leqq t] \cap \Lambda_\nu(t)\} \cup \{[\tau_\nu(\omega) \leqq t] \cap \sim\Lambda_\nu(t)\}.$$

If $\omega \in \{[\tau_\nu(\omega) \leqq t] \cap \Lambda_\nu(t)\}$, then for every $\varepsilon > 0$ such that $t + \varepsilon < 1$,

$$t < \tau_{n\nu}(\omega) < t + \varepsilon$$

for infinitely many n. Consequently

$$\omega \in [\tau_{n\nu}(\omega) < 1 \text{ for infinitely many } n].$$

Thus for every $\nu \geqq 1$, $t \in [0,1)$,

$$\{[\tau_{n\nu}(\omega) \leqq t] \cap \Lambda_\nu(t)\} \subset \limsup_n \, [\tau_{n\nu}(\omega) < 1].$$

But

$$P(\limsup [\tau_{n\nu}(\omega) < 1]) = P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} [\tau_{n\nu}(\omega) < 1]\right)$$

$$\leqq \lim_{m\to\infty} \sum_{n=m}^{\infty} P([\tau_{n\nu}(\omega) < 1]) \leqq \lim_{m\to\infty} 2^{-(m+\nu-1)} = 0.$$

Now, $\{[\tau_{n\nu}(\omega) \leqq t] \cap \sim\Lambda_\nu(t)\} = \bigcup_{n=1}^{\infty}[\tau_{n\nu}(\omega) \leqq t]$ and thus

$$\left\{[\tau_\nu(\omega) \leqq t] - \left(\bigcup_{n=1}^{\infty} [\tau_{n\nu}(\omega) \leqq t]\right)\right\} \subset \Lambda_\nu(t) \in F(t)$$

and $P(\Lambda_\nu(t)) = 0$. Hence, $[\tau_\nu \leqq t]$ differs from a $F(t)$ set by a set of measure zero. Defining

$$\begin{aligned} X_\nu(t,\omega) &= X(t,\omega) \quad \text{if } t \leqq \tau_\nu(\omega) \\ &= X(\tau_\nu(\omega),\omega) \quad \text{if } t > \tau_\nu(\omega) \end{aligned}$$

for each $\nu \geqq 1$, the lemma is now evident.

LEMMA 2.2.2. *If the X-process is as defined in Theorem 2.2, then there exists a sequence of processes satisfying conditions* (ib) and (ii).

Proof. For every $\nu \geqq 1$, define $\tau_\nu(\omega)$ to be the first t such that $\sup_{s \geqq t} |X(s,\omega)| \geqq \nu$. If no such t exists, let $\tau_\nu(\omega) = 1$. Clearly $\tau_\nu(\omega)$ defines a stopping time for the X-process. Define X_ν, $\nu \geqq 1$, as usual. Then X_ν is uniformly bounded by ν. Since a.e. sample function has an absolute maximum on T, property (ii) of Theorem 2.2 is obviously satisfied.

The proof of Theorem 2.2 is now trivial. For let $\tau_\nu'(\omega)$ and $\tau_\nu''(\omega)$, $\nu \geqq 1$, be the stopping times of the X-process defined respectively in Lemmas 2.2.1 and 2.2.2. If $\tau_\nu(\omega) = \min[\tau_\nu'(\omega), \tau_\nu''(\omega)]$, then τ_ν is a stopping time of the X-process, and the stopped X-processes X_ν, $\nu \geqq 1$, have properties (i) and (ii).

We now show property (iii). Let $\Lambda_\nu = [\tau_\nu(\omega) < 1]$,

$$\begin{aligned} E(|X_\nu(t) - X(t)|) &= \int_{\Lambda_\nu} |X_\nu(t) - X(t)| dP \\ &\leqq \nu P(\Lambda_\nu) + \int_{\Lambda_\nu} |X(t)| dP. \end{aligned}$$

The second term clearly goes to zero as $\nu \to \infty$. The first term is bounded by

$$\nu P([\tau_\nu' < 1]) + \nu P([\tau_\nu'' < 1]) \leqq \nu 2^{-\nu} + \nu P([\sup |X(t)| \geqq \nu]) \to 0 \quad \text{as } \nu \to \infty.$$

Some additional notation and theorems will be needed to prove the general decomposition theorem. Although the theorems are somewhat specialized, they are not in the trend of the argument leading to the general decomposition theorem and are therefore stated and proved at this time.

(2.2) We let $\{T_n; n \geqq 1\}$ be a sequence of partitions of $T = [0,1]$ with the following properties:

(i) T_{n+1} is a refinement of T_n for every n, and

(ii) $N(T_n) \to 0$ as $n \to \infty$, where $N(T_n)$ denotes the norm of the partition. The points of the partition T_n will be denoted as follows:

$$0 = t_{n,0} < t_{n,1} < \cdots < t_{n,N_n} = 1.$$

If $m > n$, we let

$$t_{n,j} = t_{nmj,0} < t_{nmj,1} < \cdots < t_{nmj,k_{nmj}} = t_{n,j+1}$$

denote the points of T_m contained in the jth subinterval of the T_n partition.

Let $\{X(t), F(t) : t \in T\}$ be a real-valued process with $E(X(t))$ existing for every $t \in T$. The following notation will be used in an effort to avoid, as much as possible, the cumbersome notation of sums and multiple subscripts.

(2.3) For any partition T_n we write

$$\begin{aligned} \Delta_{n,j}(X) &= X(t_{n,j+1}) - X(t_{n,j}), \\ F_{n,j} &= F(t_{n,j}), \\ C_{n,j}(X) &= E(\Delta_{n,j}(X) \mid F_{n,j}), \\ T_n(X) &= \sum_j C_{n,j}(X). \end{aligned}$$

If $n < m$ so that T_m is a refinement of T_n, we can write

$$T_{nmj}(X) = \sum_{k \leqq k_{nmj} - 1} C_{nmj,k}(X)$$

and then $T_m(X) = \sum_j T_{nmj}(X)$.

(2.4) We define

$$\sum_t C_{n,j}(X) = \begin{cases} 0, & 0 = t_{n,0} \leqq t < t_{n,1}, \\ \sum_{j=0}^{k-1} C_{n,j}(X), & t_{n,k} \leqq t < t_{n,k+1}, 1 \leqq k \leqq N_n. \end{cases}$$

We note that $\sum_0 C_{n,j}(X) \equiv 0$, but that if $t \in T$ and $t > 0$, then for all n sufficiently large we will have $t \geqq t_{n,1}$.

THEOREM 2.3. *Let $\{X(t), F(t); t \in T\}$ be a second-order process. Let $[\alpha, \beta]$ be a closed subinterval of T and let $\alpha = a_0 < a_1 < \cdots < a_{n+1} = \beta$ be a partition of $[\alpha, \beta]$.*

Let $\varepsilon > 0$ be given. If

$$\varepsilon_{\alpha,\beta} = \operatorname*{ess.sup.}_{\omega} \max_k \left| X(\beta, \omega) - X(a_k, \omega) \right| < \varepsilon,$$

then

$$P\left(\left[\max_m \left| \sum_{k=0}^{m} C_k(X) \right| \geqq \varepsilon \right]\right) \leqq E(\left| \sum C_k(X) \right|^2) / (\varepsilon - \varepsilon_{\alpha,\beta})^2,$$

where $C_k(X) = E(X(a_{k+1}) - X(a_k) \mid F(a_k))$.

Proof. The argument is the following: Assume $\Lambda \in F$, $|A| \geqq \varepsilon$ and A is F measurable. Assume further $|E(B \mid F)| \leqq \delta < \varepsilon$. Then

$$\begin{aligned} \int_\Lambda (A^2 + 2AB + B^2) dP &= \int_\Lambda (A^2 + 2AE(B \mid F) + E(B^2 \mid F)) dP \\ &\geqq \int_\Lambda (A^2 + 2AE(B \mid F) + [E(B \mid F)]^2) dP \\ &\geqq \int_\Lambda (A^2 + \delta^2 - 2|A|\delta) dP \\ &= \int_\Lambda (|A| - \delta)^2 dP \geqq (\varepsilon - \delta)^2 P(\Lambda). \end{aligned}$$

Now we let

$$\Lambda_m = \left[\left| \sum_{k=0}^{v} C_k(X) \right| < \varepsilon \text{ for } v < m \text{ and } \left| \sum_{k=0}^{m} C_k(X) \right| \geqq \varepsilon \right]$$

for $0 \leqq m \leqq n$. Then $\Lambda_m \in F_m$, since $C_k(X)$ is measurable w.r.t. F_m for $0 \leqq k \leqq m$. We also have $\Lambda_m \cap \Lambda_j = \varnothing$ for $m \neq j$, and

$$\left[\max_m \left| \sum_{k=0}^{m} C_k(X) \right| \geqq \varepsilon \right] = \bigcup_{m=0}^{n} \Lambda_m.$$

Now

$$E\left(\left| \sum C_k(X) \right|^2 \right) = \sum_{m=0}^{n} \int_{\Lambda_m} \left| \sum_{k=0}^{m} C_k(X) + \sum_{k=m+1}^{n} C_k(X) \right|^2 dP.$$

Letting $A_m = \sum_{k=0}^{m} C_k(X)$, $B_m = \sum_{k=m+1}^{n} C_k(X)$, and replacing Λ and F with Λ_m and F_m respectively and δ with $\varepsilon_{\alpha,\beta}$, the above argument gives

$$E(\left| \sum C_k(X) \right|^2) \geqq (\varepsilon - \varepsilon_{\alpha,\beta})^2 P\left(\bigcup_{m=0}^{n} \Lambda_m \right).$$

COROLLARY 2.3.1. *Let $\{X(t), F(t); t \in T\}$ be a second order process which is a.s. sample equicontinuous. Let $\{T_n; n \geqq 1\}$ be a sequence of partitions as defined in* (2.2). *Given any $\varepsilon > 0$, there exists $n(\varepsilon)$ such that if $m > n \geqq n(\varepsilon)$ then*

$$P\left(\left[\max_j \max_k \left| \sum_{i=0}^{k} C_{nmj,i}(X) \right| \geqq \varepsilon\right]\right) \leqq E\left(\sum_j \left| T_{nm,j}(X)\right|^2\right) \Big/ (\varepsilon - \varepsilon_n)^2,$$

where

$$\varepsilon_n = \operatorname{ess.sup.max}_{\omega \quad j} \sup_{t_{n,j} \leqq s,t \leqq t_{n,j+1}} \left| X(t,\omega) - X(s,\omega)\right| < \varepsilon.$$

Proof. Because of the a.s. uniform sample equicontinuity of the X-process, given any $\varepsilon > 0$ there exists $\delta(\varepsilon)$ and Λ with $P(\Lambda) = 0$ such that

$$\sup_{\omega \notin \Lambda} \sup_{|t-s| \leqq \delta(\varepsilon)} \left| X(t,\omega) - X(s,\omega)\right| < \varepsilon.$$

Let $n(\varepsilon)$ be such that $N(T_n) < \delta(\varepsilon)$ if $n \geqq n(\varepsilon)$. Then $\varepsilon_n < \varepsilon$, and for all $m > n$,

$$\sup_{\omega \notin \Lambda} \max_j \max_{0 \leqq k,k' \leqq k_{nmj}} \left| X(t_{nmj,k}) - X(t_{nmj,k'})\right| \leqq \varepsilon_n < \varepsilon.$$

Using Theorem 2.3, we now have

$$P\left(\left[\max_j \max_k \left| \sum_{i=0}^{k} C_{nmj,i}(X) \right| \geqq \varepsilon\right]\right) \leqq \sum_j P\left(\left[\max_k \left| \sum_{i=0}^{k} C_{nmj,i}(X)\right| \geqq \varepsilon\right]\right)$$

$$\leqq E\left(\sum_j \left| T_{nmj}(X)\right|^2\right) \Big/ (\varepsilon - \varepsilon_n)^2.$$

THEOREM 2.4. *Assume $\{X_n(t); t \in T\}$, $n \geqq 1$, is a sequence of processes with the following properties*:

(i) *There is a countable dense subset T_0 of T, containing the points* 0 *and* 1, *such that $P \lim_{n \to \infty} X_n(t)$ exists for each $t \in T_0$.*

(ii) *Given $\varepsilon, \gamma > 0$, there exists $n = n(\varepsilon, \gamma)$ and $\delta = \delta(\varepsilon, \gamma)$ such that if $m \geqq n$,*

$$P\left(\left[\sup_{|t-s| \leqq \delta} \left| X_m(t) - X_m(s)\right| > \varepsilon\right]\right) < \gamma.$$

Then there exists a subsequence of processes $\{X_{n_k}(t); t \in T\}$, $k \geqq 1$, and a process $\{X(t); t \in T\}$ such that

(i)′ $P([\lim_k X_{n_k} = X]) = 1$ *and*

(ii)′ *the X-process is a.s. sample continuous. Furthermore,*

(iii)′ $P \lim X_n(t) = X(t)$ *for every $t \in T$.*

Proof. We first show that conditions (i) and (ii) imply that

$$\limsup_{m,n \to \infty} P\left(\left[\sup_t \left| X_n(t) - X_m(t)\right| > \varepsilon\right]\right) = 0$$

for every $\varepsilon > 0$.

Let $\{T_\nu;\ \nu \geqq 1\}$ be a sequence of partitions of T defined as in (2.2) but with the points of each T_ν a subset of T_0. Let $\varepsilon, \gamma > 0$ be given. First choose $n_1 = n_1(\varepsilon, \gamma)$ and $\delta = \delta(\varepsilon, \gamma)$ such that

$$P\left(\left[\sup_{|t-s|\leqq\delta} |X_n(t) - X_n(s)| > \varepsilon/3\right]\right) < \gamma/3$$

for every $n \geqq n_1$. This can be done by condition (ii). Now choose ν such that $N(T_\nu) < \delta$. Next choose n_ν such that $P([\max_j |X_n(t_{\nu,j}) - X_m(t_{\nu,j})| > \varepsilon/3]) < \gamma/3$ for every $m > n \geqq n_\nu$. This is possible because there are only a finite number of points in T_ν and for each $t_{\nu,j} \in T_\nu$, $P\lim_{n\to\infty} X_n(t_{\nu,j})$ exists. Now let $n = \max(n_1, n_\nu)$ and consider

$$\begin{aligned} P\left(\left[\sup_t |X_n(t) - X_m(t)| > \varepsilon\right]\right) &= P([\max\sup |X_n(t) - X_m(t)| > \varepsilon]) \\ &\leqq P([\max\sup |X_n(t) - X_m(t_{\nu,j})| > \varepsilon/3]) \\ &\quad + P([\max |X_n(t_{\nu,j}) - X_m(t_{\nu,j})| > \varepsilon/3]) \\ &\quad + P([\max\sup |X_m(t_{\nu,j}) - X_m(t)| > \varepsilon/3]) < \gamma, \end{aligned}$$

where the supremun is over t with $t_{n,j} \leqq t \leqq t_{n,j+1}$ and the maximum is over j.

Now $\lim\sup_{n,m\to\infty} P([\sup_t |X_n(t) - X_m(t)| > \varepsilon]) = 0$ for every $\varepsilon > 0$ implies the existence of a subsequence $\{X_{n_k}(t); t \in T\}$, $k \geqq 1$, a process $\{X(t); t \in T\}$ and a set Λ with $P(\Lambda) = 0$ such that if $\omega \notin \Lambda$ then

$$\lim_{k\to\infty}\left(\sup_t |X_{n_k}(t) - X(t)|\right) = 0.$$

We have now established (i)′.

Since the X_n-processes are not necessarily sample continuous, the convergence in the supremum metric does not imply the a.s. sample continuity of the limit process. However, condition (ii) could be called "asymptotic sample continuity" and this is sufficient.

Let $\{X'_k(t),\ k \geqq 1\}$ be the above derived subsequence. By (ii), for every $n \geqq 1$ we can find a k_n and a $\delta(n)$ such that for $k \geqq k_n$

$$P\left(\left[\sup_{|t-s|\leqq\delta(n)} |X'_k(t) - X'_k(s)| > n^{-1}\right]\right) < 2^{-n}.$$

Let $X^*_n(t) = X'_{k_n}(t)$ for every $t \in T$ $n \geqq 1$.

Let $A^*_n = [\sup_{|t-s|\leqq\delta(n)} |X^*_n(t) - X^*_n(s)| > n^{-1}]$.

If $B^*_m = \bigcup_{n=m}^{\infty} A^*_n$, then $A^* = \lim\sup A^*_n = \bigcap_{m=1}^{\infty} B^*_m$ and $P(A^*) = 0$.

If $\omega \notin B^*_m$, then for every $n \geqq m$,

$$\sup_{|t-s| \leqq \delta(n)} \left| X_n^*(t) - X_n^*(s) \right| \leqq m^{-1}.$$

Consider the following inequality true for any $\delta > 0$;

$$\sup_{|t-s| \leqq \delta} \left| X(t) - X(s) \right| \leqq 2 \sup_t \left| X_m^*(t) - X(t) \right| + \sup_{|t-s| \leqq \delta} \left| X_m^*(t) - X_m^*(s) \right|.$$

Now if $\omega \notin \{\Lambda \cup A^*\}$, and $\varepsilon > 0$ is given, we first choose $m_0(\omega)$ such that $2 \sup_t \left| X_m^*(t) - X(t) \right| < \varepsilon/2$ for every $m \geqq m_0(\omega)$. This can be done since $\omega \notin \Lambda$. Next choose $m_1(\omega)$ such that $\omega \notin B_{m_1(\omega)}$ and $m_1(\omega)^{-1} < \varepsilon/2$. If $m(\omega) = \max\{m_0(\omega), m_1(\omega)\}$ and $\delta \leqq \delta(m(\omega))$, we have

$$\sup_{|t-s| \leqq \delta} \left| X(t) - X(s) \right| \leqq \varepsilon.$$

Then for $\omega \notin \{\Lambda \cap A^*\}$, $P(\Lambda \cap A^*) = 0$,

$$\lim_{\delta \to 0} \sup_{|t-s| \leqq \delta} \left| X(t) - X(s) \right| = 0.$$

Therefore the X-process is a.s. sample continuous.

Property (iii)′ is now easily derived using the continuity of the X-process and the conditions of the theorem. For we have,

$$\begin{aligned} P([\left| X_n(t) - X(t) \right| > \varepsilon]) \leqq & P([\left| X_n(t) - X_n(t_0) \right| > \varepsilon/3]) \\ & + P([\left| X_n(t_0) - X(t_0) \right| > \varepsilon/3]) \\ & + P([\left| X(t_0) - X(t) \right| > \varepsilon/3]) \end{aligned}$$

where $t_0 \in T_0$.

3. **Decomposition theorem.** If one is familiar with the decomposition of a sub-martingale sequence into the sum of a martingale sequence and a sequence which is a.s. non-negative and nondecreasing, then it will be seen that this simple decomposition is a motivating force in what follows.

Let $\{X(t), F(t); t \in T\}$ be a process and let $\{T_n; n \geqq 1\}$ be a sequence of partitions of T as defined in (2.2). From the given process we construct a sequence of what can be called "simple" quasi-martingales (in that the sample functions of each component are step functions) as follows:

For each partition T_n, $n \geqq 1$, we define

$$\left.\begin{aligned} X_n(t) &= X(t_{n,j}) \\ F_n(t) &= F(t_{n,j}) \end{aligned}\right\} t_{n,j} \leqq t < t_{n,j+1}; \quad 0 \leqq j \leqq N_n \tag{3.1}$$

$$X_{2n}(t) = \sum_t C_{n,j}(X), \quad t \in T.$$

If we now let $X_{1n}(t) = X_n(t) - X_{2n}(t)$, $t \in T$, then for each

$$n \geqq 1, \quad \{X_{1n}(t), F_n(t); t \in T\}$$

is a martingale process as is easily verified, and clearly the process

$$\{X_{2n}(t), F_n(t); t \in T\}$$

has a.e. sample function of b.v. on T.

If the X-process has any continuity properties, then it is the limit of a sequence of "simple" quasi-martingales. Thus we might expect that under certain continuity, and possibly other conditions, the X-process will itself be a quasi-martingale.

We now prove three lemmas which will give us rather strong sufficient conditions for the X-process to be a quasi-martingale.

LEMMA 3.1.1. *Let $\{X(t), F(t); t \in T\}$ be continuous in the mean, and let $\{X_{2n}(t); t \in T\}$ be defined as in* (3.1). *Assume there is a process $\{X_2(t), F(t); t \in T\}$ such that $E(|X_{2n}(t) - X_2(t)|) \to 0$, $t \in T$. Then the process*

$$\{X_1(t) = X(t) - X_2(t), F(t); t \in T\}$$

is a martingale.

Proof. We need to show $\int_\Lambda X_1(t)dP = \int_\Lambda X_1(s)dP$ for all $\Lambda \in F(s)$. Assume $s \leqq t$, and let $t_{n,k'} \leqq s < t_{n,k'+1}$, $t_{n,k} \leqq t < t_{n,k+1}$, where of course the k and k' will change with n. By the mean continuity of X and the mean convergence of X_{2n}, we have

$$\begin{aligned}
\int_\Lambda X_1(t)dP &= \int_\Lambda (X(t) - X_2(t))dP \\
&= \lim_{n\to\infty} \int_\Lambda (X_n(t) - X_{2n}(t))dP \\
&= \lim_{n\to\infty} \int_\Lambda \left(X(t_{n,k}) - \sum_{j=0}^{k'-1} C_{n,j}(X) - \sum_{j=k'}^{k-1} C_{n,j}(X)\right)dP \\
&= \lim_{n\to\infty} \int_\Lambda \left(X(t_{n,k}) - \sum_{j=0}^{k'-1} C_{n,j}(X) - [X(t_{n,k}) - X(t_{n,k'})]\right)dP \\
&= \lim_{n\to\infty} \int_\Lambda (X_n(s) - X_{2n}(s))dP = \int_\Lambda X_1(s)dP.
\end{aligned}$$

Our problem now is to obtain conditions to insure the existence of the X_2-process alluded to in Lemma 3.1.1 and to insure this process has a.e. sample function or b.v. on T.

It is trivial to show that $E(\sum |C_{n,j}(X)|)$ is monotone nondecreasing in n. We are then led naturally to processes satisfying the following condition.

(3.2) The process $\{X(t), F(t); t \in T\}$ is such that there exists a sequence of partitions $\{T_n; n \geqq 1\}$, of T, as defined in (2.2) such that

$$\lim E(\sum | C_{n,j}(x) |) \leqq K_x < \infty.$$

Note. If X is a quasi-martingale with $[X]_2 = X_2$ the process of b.v., and if $E(V(\omega)) < \infty$ where $V(\omega)$ is the total variation of $X_2(\cdot, \omega)$ over T, then condition (3.2) is satisfied for any sequence of partitions. For we have

$$E(\sum | C_{n,j}(x) |) = E(\sum | C_{n,j}(X_2) |) \leqq E(\sum | \Delta_{n,j}(X_2) |) \leqq E(V(\omega)) < \infty.$$

We also note that if the X_2 process is a.s. sample continuous,

$$V(\omega) = \lim \sum | \Delta_{n,j}(X_2) | \text{ a.s.}$$

LEMMA 3.1.2. *Let the process* $\{X(t), F(t); t \in T\}$ *satisfy* (3.2) *and let* $\{X_{2n}(t), F_n(t); t \in T\}$, $n \geqq 1$, *be as defined in* (3.1). *If there exists a process* $\{X_2(t), F(t); t \in T\}$ *such that* $P([\lim X_{2n}(t) = X_2(t); t \in T]) = 1$, *then the* X_2 *process has a.e. sample function of b.v. on T. Furthermore, if the* X_2 *process is a.s. sample continuous, the total variation,* $V(\omega)$, *of* $X_2(\cdot, \omega)$ *over T is a r.v. with*

$$E(V(\omega)) \leqq \lim E(\sum | C_{n,j}(X) |) \leqq K_x.$$

Proof. Let $K(\omega) = \liminf \sum | C_{n,j}(X) |$, then condition (3.2) and Fatou's lemma imply $K(\omega)$ is a.s. finite and integrable. Let $0 = a_1 < a_2 < \cdots < a_{m+1} = 1$ be any partition of T. We write

$$\sum_{\Delta a_i} C_{n,j}(X) = \sum_{a_{i+1}} C_{n,j}(X) - \sum_{a_i} C_{n,j}(X).$$

Then

$$\begin{aligned} \sum_i | \Delta_i(X_2) | &= \sum_i \left| \lim_n \Delta_i(X_{2n}) \right| = \sum_i \left| \lim_n \sum_{\Delta a_i} C_{n,j}(X) \right| \\ &= \sum_i \lim_n \left| \sum_{\Delta a_i} C_{n,j}(X) \right| \leqq \sum_i \liminf \sum_{\Delta a_i} | C_{n,j}(X) | \\ &\leqq \liminf \sum_i \sum_{\Delta a_i} | C_{n,j}(X) | = K(\omega) \text{ a.s.} \end{aligned}$$

The assertions are now evident.

LEMMA 3.1.3. *Let* $\{X(t), F(t); t \in T\}$ *be a uniformly bounded, a.s. sample equicontinuous process satisfying condition* (3.2). *If the sequence of processes* $\{X_{2n}(t), F_n(t); t \in T\}$, $n \geqq 1$, *are defined as in* (3.1), *then*

(a) *for each* $t \in T$, *the sequence* $\{X_{2n}(t); n \geqq 1\}$ *is uniformly integrable;*

(b) *the sequence of processes* $\{X_{2n}; n \geqq 1\}$ *satisfy the hypotheses of Theorem* 2.4 *with* $T_0 = \bigcup_{n=1}^{\infty} T_n$.

Proof. We show (a) is satisfied by showing $E(|X_{2n}(t)|^2) \leqq K < \infty$ for every $n \geqq 1$ and $t \in T$. Let $M_x = \sup_{t,\omega}|X(t,\omega)|$ and let K_x be as defined in (3.2). We have

$$\begin{aligned} E(|X_{2n}(t)|^2) &= E(|\sum_t C_{n,j}(X)|^2) \\ &= E\left(\sum_t |C_{n,j}(X)|^2 + 2\sum_t C_{n,j}(X)\left[\sum_{k>j} C_{n,k}(X)\right]\right) \\ &= E\left(\sum_t |C_{n,j}(X)|^2\right) + 2E\left(\sum_t C_{n,j}(X)E\left[\sum_{k>j} C_{n,k}(X)|F_{n,j}\right]\right) \\ &\leqq E\left(\sum_t |C_{n,j}(X)|\,|\Delta_{n,j}(X)|\right) \\ &\quad + 2E\left(\sum_t |C_{n,j}(X)|\,|X(t_{n,j(t)}) - X(t_{n,j+1})|\right) \\ &\leqq 6M_xK_x, \end{aligned}$$

where $j(t)$ denotes the last j such that $t_{n,j} \leqq t$. We now prove condition (i) of Theorem 2.4 by showing $E(|X_{2n}(t) - X_{2m}(t)|^2) \to 0$ for every $t \in T_0$. If $t \in T_0$, there exists n_t such that $t \in T_n$ for every $n \geqq n_t$. We assume now $m > n > n_t$. Then we can write

$$X_{2n}(t) = \sum_t E(T_{nm,j}(X)|F_{n,j})$$

$$X_{2m}(t) = \sum_t T_{nm,j}(X),$$

where $T_{nm,j}(X)$ is as defined in (2.4). Now using orthogonality (Parseval's Identity),

$$\begin{aligned} E(|X_{2m}(t) - X_{2n}(t)|^2) &= E\left(\left|\sum_t T_{nm,j}(X) - E(T_{nm,j}(X)|F_{n,j})\right|^2\right) \\ &= E\left(\sum_t |T_{nm,j}(X) - E(T_{nm,j}(X)|F_{n,j})|^2\right) \\ &= E\left(\sum_t |T_{nm,j}(X)|^2\right) - E\left(\sum_t |E(T_{nm,j}(X)|F_{n,j}|^2\right) \\ &\leqq E(\sum |T_{nm,j}(X)|^2) = E(\sum\sum |C_{nmj,k}(X)|^2) \\ &\quad + 2E\left(\sum\sum C_{nmj,k}(X)E\left(\sum_{i>k} C_{nmj,i}(X)|F_{nmj,k}\right)\right) \\ &\leqq E(\max|\Delta_{nmj,k}(X)|\sum\sum|C_{nmj,k}(X)|) \\ &\quad + 2E(\max|X(t_{n,j+1}) - X(t_{nmj,k})|\sum\sum|C_{nmj,k}(X)|). \end{aligned}$$

Letting

$$\varepsilon_n = \operatorname*{ess.\,sup.}_{\omega} \max_{j} \sup_{t_{n,j} \leqq s,t \leqq t_{n,j+1}} \left| X(t,\omega) - X(s,\omega) \right|,$$

we have

$$E(\left| X_{2m}(t) - X_{2n}(t) \right|^2) \leqq 3\varepsilon_n K_x$$

and $\varepsilon_n \to 0$ as $n \to \infty$ by the a.s. sample equicontinuity of the X-process.

To prove condition (ii) of Theorem 2.4, let $\varepsilon, \gamma > 0$ be given. We can choose $n = n(\varepsilon, \gamma)$ such that

$$\varepsilon_n = \operatorname*{ess.\,sup}_{\omega} \max_{j} \sup_{t_{n,j} \leqq s,t \leqq t_{n,j+1}} \left| X(t,\omega) - X(s,\omega) \right| < \varepsilon$$

and

$$3\varepsilon_n K_x / (\varepsilon - \varepsilon_n)^2 < \gamma.$$

If $\delta < \min \left| t_{n,j+1} - t_{n,j} \right|$, then for every $m \geqq n$, we have by Corollary 2.3.1

$$\begin{aligned} P&\left(\left[\sup_{|t-s| \leqq \delta} \left| X_{2m}(t) - X_{2m}(s) \right| > 3\varepsilon\right]\right) \\ &\leqq P\left(\left[\max_j \max_k \left| \sum_{i=0}^{k} C_{nmj,i}(X) \right| > \varepsilon\right]\right) \\ &\leqq E(\left| T_{nm,j}(X) \right|^2) / (\varepsilon - \varepsilon_n)^2 \leqq 3\varepsilon_n K_x / (\varepsilon - \varepsilon_n)^2 < \gamma. \end{aligned}$$

THEOREM 3.1. *If $\{X(t), F(t); t \in T\}$ is a uniformly bounded, a.s. sample equicontinuous process satisfying condition* (3.2), *then the process is a quasi-martingale. If $[X]_2 = X_2$, then the X_2 process satisfies the following conditions:*

(i) *If the sequence of processes $\{X_{2n}; n \geqq 1\}$ are as defined in* (3.1), *then P $\lim X_{2n}(t) = X_2(t)$, $t \in T$.*

(ii) *The X_2 process is a.s. sample continuous*; *and*

(iii) *if $V(\omega)$ denotes the variation of $X_2(\cdot, \omega)$ over T, then $E(V(\omega)) \leqq K_x < \infty$.*

Proof. The proof is now a matter of applying the three lemmas.

By Lemma 3.1.3 the sequence of processes $\{X_{2n}; n \geqq 1\}$ as defined in (3.1) satisfies the conditions of Theorem 2.4 and hence there is an a.s. sample continuous process $\{X_2(t), F(t); t \in T\}$ such that $P \lim X_{2n}(t) = X_2(t)$, $t \in T$, and there is a subsequence of processes $\{X_{2n_k}; k \geqq 1\}$ such that $P([\lim_k X_{2n_k}(t) = X_2(t); t \in T]) = 1$. Also according to Lemma 3.1.3, for each $t \in T$, the sequence $\{X_{2n}(t); n \geqq 1\}$ is uniformly integrable.

The X-process is obviously continuous in the mean so that from $P \lim X_{2n}(t) = X_2(t), t \in T$, and the uniform integrability of the sequence $\{X_{2n}(t); n \geqq 1\}; t \in T$, we can conclude by Lemma 3.1.1 that the process $\{X_1(t) = X(t) - X_2(t), F(t); t \in T\}$ is a martingale.

That the X-process satisfies condition (3.2) and $P([\lim X_{2n_k}(t) = X_2(t); t \in T]) = 1$ allows us to conclude by Lemma 3.1.2 that the X_2-process is a.s. of b.v.

Thus the X-process is a quasi-martingale, with the stated properties.

With a little more work we will be in a position to prove our main decomposition theorem.

LEMMA 3.2.1. *Let* $\{Y(t),F(t);t\in T\}$ *be a martingale process having a.e. sample function continuous and of b.v. on* T. *Then* $P([Y(t)=Y(0);t\in T])=1$.

Proof. Since the Y-process has a.e. sample function continuous and of b.v. on T, $V(t,\omega)$, the variation of $Y(\cdot,\omega)$ on $[0,t]$ is sample continuous and monotone nondecreasing on T. Further, $V(\cdot,\omega)$ is measurable w.r.t. $F(t)$, $t\in T$.

Thus we can apply Theorem 2.2 to both the V and Y processes. Let τ_ν' and τ_ν'', $\nu\geqq 1$, be the stopping times defined in Theorem 2.2 for the V and Y processes respectively. If $\tau_\nu=\min\{\tau_\nu',\tau_\nu''\}$, then τ_ν is a stopping time for both V and Y. Let V_ν and Y_ν be the stopped processes. By Theorem 2.1, each Y_ν is again a martingale process. It is clear that $V_\nu(t,\omega)$ is the variation of $Y_\nu(\cdot,\omega)$ over $[0,t]$.

Let $\{T_n;n\geqq 1\}$ be a sequence of partitions as defined in (2.1) and let $T_0=\bigcup_{n=1}^\infty T_n$. Also let

$$\varepsilon_{n\nu} = \operatorname*{ess.sup.}_{\omega}\ \max_j\left|\Delta_{n,j}(Y_\nu)\right|.$$

Then because of the equicontinuity of the Y_ν-process, $\lim_n \varepsilon_{n\nu}=0$ for each $\nu\geqq 1$.

If $t\in T_0$, then there exists n_t such that $t\in T_n$ for $n\geqq n_t$. Assume now $n\geqq n_t$. Since each Y_ν-process is a uniformly bounded martingale, it has orthogonal increments. Thus we have

$$\begin{aligned} E(|Y_\nu(t)-Y_\nu(0)|^2) &= E\left(\left|\sum_t \Delta_{n,j}(Y_\nu)\right|^2\right)\\ &= E\left(\sum_t |\Delta_{n,j}(Y_\nu)|^2\right) \leqq E\left(\varepsilon_{n\nu}\sum_t |\Delta_{n,j}(Y_\nu)|\right)\\ &\leqq \varepsilon_{n\nu}E(V_\nu(1,\omega)) \leqq \varepsilon_{n\nu}\nu\to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Thus, $P([Y_\nu(t)=Y_\nu(0);t\in T_0])=1$. But T_0 is dense in T, and since each Y_ν is a.s. sample continuous we have $P([Y_\nu(t)=Y_\nu(0);t\in T])=1$ for every $\nu\geqq 1$. By Theorem 2.2, for a.e. ω, when ν is sufficiently large $Y_\nu(t,\omega)=Y(t,\omega)$ for every $t\in T$. Hence $P([Y(t)=Y(0);\ t\in T])=1$.

THEOREM 3.2. *If* $\{X(t),F(t);t\in T\}$ *is a quasi-martingale with the following decompositions*

$$P([X=X_1^*+X_2^*])=P([X=X_1+X_2])=1$$

where X_i *and* X_i^* $(i=1,2)$ *are a.s. sample continuous processes, then*

$$P([X_1(t)=X_1^*(t)+(X_1(0)-X_1^*(0));t\in T])=1.$$

Proof. Let $Y_1=X_1-X_1^*$, $Y_2=X_2^*-X_2$. Then

$$P([Y_1 = Y_2]) = 1.$$

The conclusion now follows from Lemma 3.2.1.

LEMMA 3.3.1. *Let* $\{X(t), F(t); t \in T\}$ *be a.s. sample continuous. Assume*
(i) $\lim_{r\to\infty} rP([\sup_t |X(t)| \geqq r]) = 0$ *and*
(ii) *the* X*-process satisfies condition* (3.2).
If the sequence of processes $\{X_v(t), F(t); t \in T\}$ *are as defined in Theorem* 2.2, *then each* X_v *process also satisfies condition* (3.2) *and the bound* K *is independent of* v.

Proof. We wish to prove the existence of a $K > 0$ such that

$$\lim_{n\to\infty} E(\Sigma \,|\, C_{n,j}(X_v) \,|\,) \leqq K$$

for every $v \geqq 1$. Let $A(v; n,j) = [\tau_v(\omega) \geqq t_{n,j}]$,

$$Q(v; n,j) = [t_{n,j} \leqq \tau_v(\omega) < t_{n,j+1}]$$

$$(1) \quad \Sigma \int_{A(v;n,j)} (|C_{n,j}(X_v)| - |C_{n,j}(X)|) dP$$

$$\leqq \Sigma \int_{A(v;n,j)} |C_{n,j}(X_v) - C_{n,j}(X)| dP$$

$$\leqq \Sigma \int_{A(v;n,j)} |X_v(t_{n,j+1}) - X(t_{n,j+1})| dP$$

$$\leqq \Sigma \int_{Q(v;n,j)} |X(t_{n,j+1})| \, dP + vP([\tau_v(\omega) < 1]),$$

$$(2) \quad \Sigma \int_{\sim A(v;n,j)} |C_{n,j}(X)| dP = \sum_j \sum_{k<j} \int_{Q(v;n,k)} |C_{n,j}(X)| dP$$

$$= \sum_k \int_{Q(v;n,k)} \left(\sum_{j>k} |C_{n,j}(X)| \right) dP$$

$$\geqq \sum_k \int_{Q(v;n,k)} \left(\sum_{j>k} E(-\operatorname{sgn} X(t_{n,k+1}) \Delta_{n,j}(X) | F_{n,j}) \right) dP$$

$$= \sum_k \int_{Q(v;n,k)} \left(\sum_{j>k} -\operatorname{sgn} X(t_{n,k+1}) \Delta_{n,j}(X) \right) dP$$

$$\geqq \sum_k \int_{Q(v;n,k)} (|X(t_{n,k+1})| - |X(1)|) dP$$

$$= \Sigma \int_{Q(v;n,k)} |X(t_{n,k+1})| dP - \int_{[\tau_v(\omega) < 1]} |X(1)| \, dP.$$

Now by (1) and (2)

$$\begin{aligned} E(\textstyle\sum|C_{n,j}(X_\nu)|) - E(\textstyle\sum|C_{n,j}(X)|) &= \sum \int_{A(\nu;n,j)} (|C_{n,j}(X_\nu)| - |C_{n,j}(X)|)dP \\ &\quad - \sum \int_{\sim A(\nu;n,j)} |C_{n,j}(X)|dP \\ &\leqq \nu P([\tau_\nu(\omega) < 1]) + \int_{[\tau_\nu(\omega) > 1]} |X(1)|\,dP. \end{aligned}$$

The theorem is now evident.

THEOREM 3.3. *In order that the a.s. sample continuous first order process* $\{X(t), F(t); t \in T\}$ *have a decomposition into the sum of two processes,*

$$P([X = X_1 + X_2]) = 1,$$

where

(i) $\{X_1(t), F(t); t \in T\}$ *is an a.s. sample continuous martingale,*

(ii) $\{X_2(t), F(t); t \in T\}$ *has a.e. sample function continuous, of b.v. on T, and having finite expected variation, and*

(iii) $X_2(t) = P\lim_{n\to\infty} \sum_t C_{n,j}(X)$, $t \in T$

it is nessary and sufficient that

(i)′ $\lim rP([\sup_t|X(t,\omega)| \geqq r]) = 0$, *and*

(ii)′ *For any sequence of partitions* $\{T_n; n \geqq 1\}$ *of T with* $\lim N(T_n) = 0$ *and* T_{n+1} *a refinement of* T_n *for every n,* $\lim E|\sum(C_{n,j}(X)_|) \leqq K_0 < \infty$.

Proof. (1) *Necessity.* If the X-process is a quasi-martingale with the stated decomposition, we have already indicated (ii)′ is true. We need to prove (i)′. Consider

$$\begin{aligned} rP\left(\left[\sup_t|X(t)| \geqq r\right]\right) &\leqq rP\left(\left[\sup_t|X(t) - X(0)| \geqq r/2\right]\right) \\ &\quad + rP([|X(0)| \geqq r/2]) \end{aligned}$$

the second term goes to zero as $r \to \infty$. Now

$$\begin{aligned} rP\left(\left[\sup_t|X(t) - X(0)| \geqq r/2\right]\right) &\leqq rP\left(\left[\sup_t|X_1(t) - X_1(0)| \geqq r/4\right]\right) \\ &\quad + rP\left(\left[\sup_t|X_2(t) - X_2(0)| \geqq r/4\right]\right). \end{aligned}$$

The process $\{(X_1(t) - X_1(0)), F(t); t \in T\}$ is a martingale and hence [1, Theorem 3.2, §II, Chapter VII] we have the first term bounded by

$$4\int_{[\sup_t|X_1(t)-X_1(0)|\geqq r/4]} |X_1(1)-X_1(0)|\,dP \to 0 \quad \text{as} \quad r\to\infty.$$

We also have the second term bounded by

$$rP\left(\left[\sup_t V(t,\omega) \geqq r/4\right]\right) \leqq rP([V(1,\omega)\geqq r/4]) \to 0$$

as $r\to\infty$, where $V(t,\omega)$ denotes the variation of $X_2(\cdot,\omega)$ over the interval $[0,t]$.

(2) *Sufficiency*. Let $\tau_\nu(\omega)$, $\nu \geqq 1$ be the stopping times defined in Theorem 2.2 and let $\{X_\nu(t); F(t); t\in T\}$ be the corresponding sequence of stopped processes.

(a) By Lemma 3.3.1, each X_ν-process satisfies condition (3.2) and hence by Theorem 3.1, each X_ν is a quasi-martingale. Let $\sim\Lambda_\nu = [X_\nu = X_{1\nu} + X_{2\nu}]$, then $P(\Lambda_\nu) = 0, \nu\geqq 1$.

(b) By Theorem 2.2, there is a set Λ with $P(\Lambda)=0$ such that if $\omega\notin\Lambda$, then there exists $\nu(\omega)$ such that:

$X_\nu(t,\omega) = X_{\nu^*}(t,\omega) = X(t,\omega)$ for all $t\in T$ if $\nu,\nu^*\geqq \nu(\omega)$.

Since $X_{2\nu}(0)=0$ for every ω, by Theorem 3.2 we have:

$X_{i\nu}(t,\omega) = X_{i\nu^*}(t,\omega)$ for all $t\in T$ if $\nu\geqq\nu^*\geqq\nu(\omega)$ and $\omega\notin(\bigcup_{\nu=1}^\infty \Lambda_\nu)\cup\Lambda = \Lambda_0$.
We can therefore define

$$X_i(t,\omega) = \lim_{\nu\to\infty} X_{i\nu}(t,\omega) \quad \text{for all } t\in T \text{ and } \omega\notin\Lambda_0.$$

We then have

$$P([X = X_1 + X_2]) = P(\sim\Lambda_0) = 1.$$

It thus remains to show that the given decomposition has the stated properties.

By definition of X_1 and X_2, they are a.s. sample continuous since each $X_{1\nu}$ and $X_{2\nu}$ are a.s. sample continuous according to Theorem 3.1. Also by the definition of X_2, it is a.s. of sample b.v. since each $X_{2\nu}$ is a.s. of sample b.v.

We will now show that $E(V(1,\omega))<\infty$ and that $E(|X_{2\nu}(t)-X_2(t)|)\to 0$ as $\nu\to\infty$. In view of the fact that condition (i)′ implies, according to Theorem 2.2, $E(|X_\nu(t)-X(t)|)\to 0$ as $\nu\to\infty$, it follows that $E(|X_{1\nu}(t)-X_1(t)|)\to 0$ as $\nu\to\infty$, and hence, X_1 being the limit in the mean of a sequence of martingale processes is itself a martingale process.

Letting $V(t,\omega)$ and $V_\nu(t,\omega)$ denote respectively the variation of $X_2(\cdot,\omega)$ and $X_{2\nu}(\cdot,\omega)$ over $[0,t]$, it is clear that $V_\nu(t,\omega)$ is $V(t,\omega)$ stopped at $\tau_\nu(\omega)$, and $P([\lim V_\nu(t,\omega) = V(t,\omega)]) = 1$.

Since $\tau_\nu(\omega)$ is a.s. nondecreasing in ν, $V_\nu(1,\omega) = V(\tau_\nu(\omega),\omega)$ is a.s. monotone nondecreasing in ν. By Lemma 3.1.2 and Lemma 3.3.1 $\lim E(V_\nu(1,\omega))\leqq K<\infty$ and hence $E(|V_\nu(1,\omega)-V(1,\omega)|)\to 0$ as $\nu\to\infty$ by the monotone convergence theorem. Now

$$\begin{aligned}\sup_t \left|X_2(t,\omega) - X_{2\nu}(t,\omega)\right| &= \sup_{t>\tau_\nu(\omega)} \left|X_2(t,\omega) - X_2(\tau_\nu(\omega),\omega)\right| \\ &\leqq \sup_{t>\tau_\nu(\omega)} \left|V(t,\omega) - V(\tau_\nu(\omega),\omega)\right| \\ &\leqq V(1,\omega) - V_\nu(1,\omega),\end{aligned}$$

and hence $E(\sup_t|X_2(t) - X_{2\nu}(t)|) \leqq E(V(1,\omega) - V_\nu(1,\omega)) \to 0$ as $\nu \to \infty$.

We now prove property (iii). Let

$$\begin{aligned}{}_n\Lambda_\nu^\varepsilon &= [\sup_t |\textstyle\sum_t(C_{n,j}(X) - C_{n,j}(X_\nu))| \geqq \varepsilon], \text{ and} \\ A(\nu; n,j) &= [\tau_\nu(\omega) \geqq t_{n,j}], \\ Q(\nu; n,j) &= [t_{n,j} \leqq \tau_\nu(\omega) < t_{n,j+1}],\end{aligned}$$

then

$$P({}_n\Lambda_\nu^\varepsilon) = P({}_n\Lambda_\nu^\varepsilon \cap [\tau_\nu(\omega) < 1]) + P({}_n\Lambda_\nu^\varepsilon \cap [\tau_\nu(\omega) = 1]).$$

The first term is bounded by $P([\tau_\nu(\omega) < 1])$ and goes to zero as $\nu \to \infty$. The second term is bounded by

$$\begin{aligned}&(1/\varepsilon) \sum \int_{A(\nu;n,j)} |C_{n,j}(X_2) - C_{n,j}(X_{2\nu})|dP \\ &\leqq (1/\varepsilon) \sum \int_{A(\nu,n,j)} |X_2(t_{n,j+1}) - X_{2\nu}(t_{n,j+1})|dP \\ &= (1/\varepsilon) \sum \int_{Q(\nu;n,j)} |X_2(t_{n,j+1}) - X_{2\nu}(t_{n,j+1})|dP \\ &\leqq (1/\varepsilon) \sum \int_{Q(\nu;n,j)} 2V(1,\omega)dP = (1/\varepsilon) \int_{[\tau_\nu(\omega)<1]} 2V(1,\omega)dP.\end{aligned}$$

Hence, since this last term also goes to zero, $P({}_n\Lambda_\nu^\varepsilon) \to 0$ as $\nu \to \infty$ uniformly in n. The conclusion is now apparent.

We make the following observation: If the process $\{X(t),F(t); 0 \leqq t < \infty\}$ satisfies the conditions of Theorem 3.3 for every interval $[0,b]$, then the process is a quasi-martingale. This follows trivially from the uniqueness of the decomposition on every finite interval.

It is also clear that the process of bounded variation in the decomposition need not have finite expected variation. If, however, the conditions of Theorem 3.3 are met and if

$$\lim_{n\to\infty} E(\sum |C_{n,j}(X)|) < \infty,$$

where we have a sequence of partitions becoming dense in $[0,\infty)$ and such that

T_{n+1} is a refinement of T_n, then the process of bounded variation in the decomposition will have finite expected variation.

An immediate corollary to Theorem 3.3 is

COROLLARY 3.3.1. *If* $\{X(t), F(t); t \in T\}$ *is an a.s. sample continuous submartingale, then it has the decomposition stated in Theorem* 3.3 *if and only if*

(i) $\lim_{r\to\infty} rP([\sup_t |X(t,\omega)| \geqq r]) = 0$.

If, in particular, the X-process has a.e. sample function non-negative, then (i) *is always satisfied.*

Proof. We need only show condition (ii) is satisfied. But

$$\begin{aligned} E(\textstyle\sum |C_{n,j}(X)|) &= E(\textstyle\sum |E(\Delta_{n,j}(X) \mid F_{n,j})|) \\ &= E(\textstyle\sum |E(X(t_{n,j+1}) \mid F_{n,j}) - X(t_{n,j})|) \\ &= E(\textstyle\sum (E(X(t_{n,j+1}) \mid F_{n,j}) - X(t_{n,j}))) = E(X(1)) - E(X(0)). \end{aligned}$$

If the X-process has a.e. sample function non-negative, then by a fundamental sub-martingale inequality [2, p. 524]

$$\begin{aligned} rP\left(\left[\sup_t |X(t)| \geqq r\right]\right) &= rP\left(\left[\sup_t X(t,\omega) \geqq r\right]\right) \\ &\leqq \int_{[\sup_t X(t) \geqq r]} X(1)dP \end{aligned}$$

which goes to zero as $r \to \infty$.

We saw previously that if Z is the Brownian motion process, then

$$X(t) = \exp[xZ(t)], \qquad x > 0$$

is a quasi-martingale with

$$X_2(t) = \int_0^t \frac{(x\sigma)^2}{2} \exp[xZ(s)]ds, \qquad t \in T.$$

Consider now

$$\begin{aligned} &\sum_t E(\Delta_{n,j}(X) \mid F_{n,j}) \\ &\quad = \sum_t E(\exp[xZ(t_{n,j+1})] - \exp[xZ(t_{n,j})] \mid F_{n,j}) \\ &\quad = \sum_t \exp[xZ(t_{n,j})]E(\exp[x(Z(t_{n,j+1}) - Z(t_{n,j}))] - 1 \mid F_{n,j}) \\ &\quad = \sum_t \exp[xZ(t_{n,j})]\left[\exp\left[\frac{(x\sigma)^2}{2}(t_{n,j+1} - t_{n,j})\right] - 1\right] \\ &\quad = \sum_t \exp[xZ(t_{n,j})]\left[\frac{(x\sigma)^2}{2}(t_{n,j+1} - t_{n,j}) + \sum_{k=2}^{\infty} \frac{\alpha^k}{k'}\right], \end{aligned}$$

where

$$\alpha = \frac{(x\sigma)^2}{2} (t_{n,j+1} - t_{n,j}).$$

Hence

$$\text{a.s.} \lim \sum_t E(\Delta_{n,j}(X) \mid F_{n,j}) = \int_0^t \frac{(x\sigma)^2}{2} \exp[xZ(s)]ds$$

as was to be expected.

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