

ASYMPTOTIC DISTRIBUTION OF EIGENVALUES AND EIGENFUNCTIONS FOR GENERAL LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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The asymptotic distribution of eigenvalues and eigenfunctions of elliptic operators has been studied extensively by Weyl, Courant, Carleman, Pleijel and others. During the last decade, Gårding [9] and Browder solved the problem for an elliptic operator with infinitely differentiable coefficients and null Dirichlet boundary conditions. It is the purpose of this paper to consider the problem for a general class of elliptic boundary value problems investigated during the last few years by Agmon-Douglis-Nirenberg [2], Browder [3], [6] and Schechter [14].

Let (A, γ) with $\gamma = \{B_1, \dots, B_m\}$ be a uniformly regularly elliptic boundary value problem on S . It is assumed that A is positively strongly elliptic and (A, γ) is formally positive. If A_γ is the realization of A as an operator on $L^2(S)$ with null boundary conditions γ , then the following results are obtained:

(1) When A_γ is self-adjoint:

$$N(t) = \sum_{\lambda_j \leq t} 1 \sim (2\pi)^{-n} t^{n/2m} \int_S \int_{a(x, \xi) < 1} d\xi dx \quad \text{as } t \rightarrow +\infty.$$

Let $e(x, y, t)$ be the spectral function of A_γ , then:

$$t^{-(n+|\alpha|+|\beta|)/2m} D_x^\alpha D_y^\beta e(x, y, t) = t^{-(n+|\alpha|+|\beta|)/2m} \sum_{\lambda_j \leq t} D^\alpha \phi_j(x) D^\beta \phi_j(y) \rightarrow 0$$

as $t \rightarrow +\infty$; x, y in S and $x \neq y$.

$$D^{\alpha+\beta} e(x, x, t) \sim (2\pi)^{-n} t^{-(n+|\alpha|+|\beta|)/2m} \frac{\Gamma(2p)}{\Gamma\left(1 + \frac{n}{2m}\right) \Gamma\left(2p - \frac{n+|\alpha|+|\beta|}{2m}\right)} \\ \cdot \int_{E^n} \xi^{\alpha+\beta} [a(x, \xi) + 1]^{-2p} d\xi$$

for x in S as $t \rightarrow +\infty$ ($4mp > n + |\alpha| + |\beta|$).

Received by the editors May 6, 1965.

(1) This is essentially a Doctoral Dissertation written under the guidance of Professor Felix Browder and submitted to the Department of Mathematics, Massachusetts Institute of Technology in June 1964. I would like to express my deep gratitude to Professor Browder for suggesting the problem, for his generous advices and encouragements.

λ_j, ϕ_j are the eigenvalues and eigenfunctions of A_γ .

(2) When A_γ is nonself-adjoint, then:

$$N(t) = \sum_{\operatorname{Re} \lambda_j \leq t} 1 \sim (2\pi)^{-n} t^{n/2m} \int_S \int_{a(x, \xi) < 1} d\xi dx \quad \text{as } t \rightarrow +\infty.$$

In §1, we give the notations and definitions; in §2, we state some known results; in §3 the Green's function of (A, γ) corresponding to the case of a half-space and constant coefficients is constructed. In §4, using the parametrix method, we construct the Green's function associated with (A, γ) when A and B_j are defined on S, Γ and have infinitely differentiable coefficients. Results for the self-adjoint case are given in §5 and for the nonself-adjoint case in §6.

1. Notations and definitions. Let E^n be the n -dimensional Euclidean space, S an open subset of E^n . The points of E^n will be denoted by $x = (x_1, \dots, x_n)$ and integration with respect to dx over a subset of E^n denotes integration with respect to Lebesgue n -measure.

For $1 \leq j \leq n$, $D_j = i^{-1} \partial / \partial x_j$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is any n -tuple of nonnegative integers, we set:

$$|\alpha| = \sum_{j=1}^n \alpha_j, \quad D^\alpha = \prod_{j=1}^n D_j^{\alpha_j}.$$

The linear partial differential operator $A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$ with coefficients defined on S is said to be uniformly elliptic on S if there exists a constant $c > 0$ such that:

$$|a(x, \xi)| \geq c |\xi|^{2m} \quad \text{for every point } \xi \in R^n.$$

$a(x, \xi)$ is the characteristic form of A and is given by:

$$a(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha, \quad \xi \text{ in } R^n.$$

We shall assume throughout for the sake of simplicity that S is bounded and that its boundary is locally a C^∞ , $(n-1)$ -dimensional manifold with C^∞ imbedding in E^n . In particular, S will satisfy the uniform regularity conditions (Browder [5]).

DEFINITION 1.1. Let j be a nonnegative integer:

$$W^{j,2}(S) = \{u: u \in L^2(S); D^\alpha u \in L^2(S), |\alpha| \leq j\},$$

where $D^\alpha u$ denotes the derivative in the distribution sense.

$W^{j,2}(S)$ is a Hilbert space with respect to the inner product:

$$(u, v)_j = \sum_{|\alpha| \leq j} (D^\alpha u, D^\alpha v), \quad u, v \text{ in } W^{j,2}(S),$$

$$(u, v) = \int_S u(x) \overline{v(x)} dx$$

is the usual inner product in $L^2(S)$.

The norm in $W^{j,2}(S)$ is given by:

$$\|u\|_{j,2} = \left(\sum_{|\alpha| \leq j} \|D^\alpha u\|_{L^2(S)}^2 \right)^{1/2}.$$

Let $\gamma = (B_1, \dots, B_m)$ be a family of m differential operators with coefficients defined on S . We assume that the order r_j of each B_j is less than $2m$ and we write:

$$B_j = \sum_{|\beta| \leq r_j} b_{j\beta}(x) D^\beta.$$

We also assume that the coefficients $b_{j\beta}$ lie in $C^\infty(\bar{S})$. For each j , the characteristic form is defined for $\zeta \in C^n$ by

$$b_j(x, \zeta) = \sum_{|\beta| = r_j} b_{j\beta}(x) \zeta^\beta.$$

DEFINITION 1.2. *A is said to be regularly elliptic on S if it is uniformly elliptic on S and if for each x of Γ , the polynomial $a(x, \lambda N_x + T)$ in λ ($\lambda \in C$; N_x is the unit exterior normal vector to Γ at x, T any unit tangent vector to Γ at x) has exactly m roots (counting multiplicities) in the λ upper half plane.*

DEFINITION 1.3. *The boundary value problem (A, γ) is said to be uniformly regular if:*

- (1) *A is uniformly regularly elliptic.*
- (2) *For each unit tangent vector T to Γ at x , let C_T be a closed, Jordan rectifiable curve in the λ half plane which contains in its interior all the zeros of $a(x, \lambda N_x + T)$ with positive imaginary parts.*

For $1 \leq j, k \leq m$; set:

$$c_{jk}(x, T) = \int_{C_T} \lambda^{j-1} b_k(x, \lambda N_x + T) [a(x, \lambda N_x + T)]^{-1} d\lambda.$$

Then there exists $c > 0$ such that:

$$|\text{Det}(c_{jk}(x, T))| \geq c > 0$$

for all $x \in \Gamma$ and all unit tangent vectors to Γ at x .

2. DEFINITION 2.1. *Let x_0 be a point of Γ ; A_0 the homogeneous differential operator of order $2m$ with constant coefficients*

$$A_0 = \sum_{|\alpha| = 2m} a_\alpha(x_0) D^\alpha.$$

Let $\gamma_0 = (B_{10}, \dots, B_{m0})$ where:

$$B_{j0} = \sum_{|\beta| = r_j} b_{j\beta}(x_0) D^\beta.$$

Then (A, γ) is said to be formally positive if the boundary value problem:

$$\begin{aligned}(A_0 + tI)u &= f \text{ on } S, & t > 0, \\ B_j u &= 0 \text{ on } \Gamma, & j = 1, \dots, m,\end{aligned}$$

has a unique solution u in $C^\infty(S) \cap L^2(S)$ for every f in $C_c^\infty(E^n)|_S$ and $t > 0$, such that:

$$\|u\|_{L^2(S)} \leq c \|f\|_{L^2(S)}$$

with the constant c independent of u, f .

We shall take the above definition as our basic assumption on (A, γ) . Alternative assumptions which are equivalent to Definition 2.1 may be given in a more computational form; e.g. as given by Agmon [1]: the polynomial in the complex variable λ , $a(x_0, \lambda N_{x_0} + T) + t$ has no real roots; it has m roots with positive imaginary parts and:

$$\left| \text{Det} \left(\int_{C_{t,T}} \lambda^{j-1} b_r(x_0, \lambda N_{x_0} + T) [a(x_0, \lambda N_{x_0} + T) + t]^{-1} d\lambda \right) \right| > 0,$$

$$r, j = 1, \dots, m.$$

$C_{t,T}$ is a closed Jordan rectifiable curve surrounding all the roots with positive imaginary parts of $a(x_0, \lambda N_{x_0} + T) + t$ considered as a polynomial in λ .

DEFINITION 2.2. (1) Let A_γ be the operator on $L^2(S)$ defined as follows:

$$\begin{aligned}D(A_\gamma) &= \{u: u \text{ in } W^{2m,2}(S); B_j u = 0 \text{ on } \Gamma; j = 1, \dots, m\}, \\ A_\gamma u &= Au \text{ if } u \in D(A_\gamma).\end{aligned}$$

(2) (A, γ) is said to be formally self-adjoint if A_γ is a symmetric operator in $L^2(S)$; i.e.

$$(A_\gamma u, v) = (u, A_\gamma v)$$

for all u, v in $D(A_\gamma)$.

THEOREM 2.1. Let (A, γ) be a uniformly regularly elliptic boundary value problem as above, such that A is uniformly strongly elliptic and (A, γ) is formally positive.

Then:

(1) If (A, γ) is formally self-adjoint; A_γ is then a self-adjoint operator on $L^2(S)$.

(2) If $t \geq t_0 > 0$; $(A_\gamma + tI)^{-1}$ exists on all of $L^2(S)$ and:

$$\|(A_\gamma + tI)^{-1}\| \leq ct^{-1}.$$

(3) If p is a positive integer such that $2mp > n$; $(A_\gamma + tI)^{-p}$ is an operator of Hilbert-Schmidt type,

$$(A_\gamma + tI)^{-p}f(x) = \int_S \mathcal{G}_{(p)}(x, y, t)f(y)dy \quad f \text{ in } L^2(S)$$

and $\mathcal{G}_{(p)}(x, y, t) \in L^2(S) \times L^2(S)$.

Proof. It has been proved by Agmon [1] that when (A, γ) is formally positive then $\|(A_\gamma + tI)^{-1}\| \leq ct^{-1}$ and moreover if A_γ is formally self-adjoint, then A_γ is self-adjoint.

Since (A, γ) is a regular elliptic boundaryvalue problem; we have the a priori estimate:

$$\begin{aligned} \|u\|_{W^{2mp, 2}} &\leq C\{\|u\|_{L^2} + \|(A_\gamma + tI)u\|_{W^{2m(p-1), 2}}\}, \\ \|u\|_{W^{2mp, 2}} &\leq C\|(A_\gamma + tI)^p u\|_{L^2}. \end{aligned}$$

Therefore $(A_\gamma + tI)^{-p}$ is a continuous mapping of $L^2(S)$ into $W^{2mp, 2}(S)$. From the Sobolev imbedding theorem, the injection mapping:

$$W^{2mp, 2}(S) \rightarrow L^\infty(S)$$

is continuous when $2mp > n$. Hence $(A_\gamma + tI)^{-p}$ considered as a mapping of $L^2(S)$ to $L^\infty(S)$ is continuous. By the Dunford-Pettis theorem, it follows that $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type:

$$(A_\gamma + tI)^{-p}f(x) = \int_S \mathcal{G}_{(p)}(x, y, t)f(y)dy, \quad f \text{ in } L^2(S).$$

3. Let A be a homogeneous linear elliptic differential operator on E^n with constant coefficients, of order $2m$.

Let $\gamma = \{B_j, j = 1, \dots, m\}$ be a family of homogeneous linear differential operators defined on E^n with constant coefficients and of order $r_j < 2m - 1$. Let t be a positive parameter such that (A, γ) is formally positive in the sense of Definition 2.1. In this section, we construct: (1) the Green's function $G(x, y, t)$ associated with $(A + tI, \gamma)$; (2) the iterates of G . Finally we study the asymptotic behavior of $G_{(p)}$ as $t \rightarrow +\infty$.

LEMMA 3.1. *Let A be a homogeneous, linear elliptic operator of order $2m$, on E^n with constant coefficients. Let t be a positive parameter such that $A + tI$ is positively strongly elliptic. Then $A + tI$ has a fundamental solution $E(x, y, t)$ given by:*

$$E(x, y, t) = (2\pi)^{-n} \int_{E^n} \exp(i\langle x - y, \xi \rangle) [a(\xi) + t]^{-1} d\xi$$

where the integral is taken as the Fourier transform of a tempered distribution if $2m < n$.

$E(x, y, t)$ is infinitely differentiable for $x \neq y$ and if $t = \tau^{2m}$ then:

$$D^\alpha E(x, y, t) = O(1)\tau^{-\varepsilon}|x - y|^{-n+2m-|\alpha|-\varepsilon}(1 + |\tau(x - y)|^N)^{-1}$$

if $-n + 2m - |\alpha| \leq 0$ and

$$D^\alpha E(x, y, t) = O(1)\tau^{-\varepsilon}(1 + |\tau(x - y)|^N)^{-1}$$

if $-n + 2m - |\alpha| > 0$, where $0 < \varepsilon < 1$; N is an arbitrary positive number (Gårding [9]).

THEOREM 3.1. Let v be the solution of the boundary value problem:

$$(A + tI)v = 0 \quad \text{on } E_+^n,$$

$$B_j v = B_j E \quad \text{on } E^{n-1}, \quad j = 1, \dots, m,$$

where (A, γ) is a regular elliptic boundary value problem; A and B_j are homogeneous and have constant coefficients. Then:

(1) $v(x, y, t)$ is infinitely differentiable for $x \neq y$ and is given by:

$$v(x, y, t) = \sum_{r,j=1}^m (2\pi)^{-n} \int_{E^{n-1}} \exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) V_{rj}(\hat{\xi}, x_1, y_1, t) d\hat{\xi}$$

with:

$$V_{rj}(\hat{\xi}, x_1, y_1, t) = Q_{rj}(\hat{\xi}, t) H_j(\hat{\xi}, y_1, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1$$

where $C_{\hat{\xi}, t}$ is a closed, Jordan rectifiable curve in the ζ_1 upper half plane and surrounding all the m roots (counting multiplicities) of $a(\zeta_1, \hat{\xi}) + t = 0$ for fixed $\hat{\xi}, t$.

$$H_j(\hat{\xi}, y_1, t) = \int_{E^1} \exp(-iy_1 \xi_1) b_j(\xi_1, \hat{\xi}) [a(\xi_1, \hat{\xi}) + t]^{-1} d\xi_1.$$

Finally Q_{rj} are the elements of the transpose of the inverse of the matrix $(c_{rj}(\hat{\xi}, t))$ with:

$$c_{rj}(\hat{\xi}, t) = \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1, \quad r, j = 1, \dots, m.$$

(2) Let $t = \tau^{2m}$ then $D^\alpha v(x, y, t) = O(1)\tau^{-\varepsilon}|\hat{x} - \hat{y}|^{-n+2m-|\alpha|-\varepsilon}(1 + |\hat{x} - \hat{y}|^N)^{-1}$ if $-n + 2m - |\alpha| \leq 0$ and $D^\alpha v = O(1)\tau^{-\varepsilon}(1 + |\hat{x} - \hat{y}|^N)^{-1}$ if $-n + 2m - |\alpha| > 0$; $0 < \varepsilon < 1$; N is an arbitrary positive number.

Proof. We write:

$$A = \sum_{q=0}^{2m} A_q(\hat{D}) D_1^q, \quad B_j = \sum_{q=0}^{r_j} B_{jq}(\hat{D}) D_1^q,$$

where $\hat{D}$ denotes differential operators involving only $D_2, \dots, D_n$. Taking the

Fourier transform with respect to the tangential variables $\hat{x} = (x_2, \dots, x_n)$ we are reduced to the initial value problem:

$$\left(\sum_{q=0}^{2m} a_q(\hat{\xi}) D_1^q + tI \right) V(x_1, \hat{\xi}) = 0, \quad x_1 > 0,$$

$$\sum_{q=0}^{r_j} b_{jq}(\hat{\xi}) D_1^q V(x_1, \hat{\xi}) = H_j(\hat{\xi}, t); \quad x_1 = 0; j = 1, \dots, m,$$

where:

$$V(x_1, \hat{\xi}) = (2\pi)^{-(n-1)/2} \int_{E^{n-1}} \exp(-i\langle \hat{x}, \hat{\xi} \rangle) v(\hat{x}_1, \hat{x}) d\hat{x}.$$

$H_j(\hat{\xi}, t)$ is similarly defined ($r_j < 2m - 1$).

We consider a solution of the form

$$V(x_1, \hat{\xi}) = (2\pi)^{-1} \int_{C_{\hat{\xi}, t}} \exp(ix_1 \zeta_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} p(\zeta_1) d\zeta_1,$$

where $p(\zeta_1)$ is a polynomial of degree less than or equal to $2m$ and $C_{\hat{\xi}, t}$ is a closed, Jordan rectifiable curve in the upper half plane surrounding all the m roots (counting multiplicities) of $a(\zeta_1, \hat{\xi}) + t = 0$ for fixed $\hat{\xi}, t$.

By Cauchy's theorem, we may assume that $p(\zeta_1)$ is a polynomial of degree less than m :

$$p(\zeta_1) = \sum_{r=1}^m p_r \zeta_1^{r-1}.$$

We obtain a system of m equations with m unknowns.

$$(2\pi)^{-1} \sum_{r=1}^m p_r(\hat{\xi}) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 = H_j(\hat{\xi}, t); \quad j = 1, \dots, m.$$

We may solve it in a unique fashion. Indeed, we have:

$$\text{Det} \left(\int_{\hat{\xi}, t} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}) [a(\zeta_1, \hat{\xi}') + t]^{-1} d\zeta_1 \right)$$

$$= |\hat{\xi}|^{\sum_j r_j + (m-m^2)/2} \text{Det} \left(\int_{C_{\hat{\xi}, t}, |\hat{\xi}'|^{-m_2}} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}') [a(\zeta_1, \hat{\xi}') + t |\hat{\xi}'|^{-2m}]^{-1} d\zeta_1 \right)$$

and the last expression is nonnull by hypothesis for $|\hat{\xi}| \neq 0$. We get:

$$p_r(\hat{\xi}) = \sum_{j=1}^m Q_{rj}(\hat{\xi}, t) H_j(\hat{\xi}, t),$$

$$V(x_1, \hat{\xi}, t) = \sum_{j=1}^m Q_{rj}(\hat{\xi}, t) H_j(\hat{\xi}, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1.$$

We want to take the inverse Fourier transform of V . To prove the summability of $V(x_1, \hat{\xi}, t)$ with respect to $\hat{\xi}$ we establish the following lemmas.

LEMMA 3.2. *The following estimates hold uniformly for $t \geq t_0 > 0$:*

$$\begin{aligned} f(x_1, \hat{\xi}, t) &= \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 \\ &= O((1 + |\hat{\xi}|^{2m-r})^{-1}) \exp(-d|\hat{\xi}|x_1) \end{aligned}$$

where $x_1 > 0$, $d > 0$ independent of t and $C_{\hat{\xi}, t}$ is a closed, Jordan rectifiable curve in the ζ_1 upper half plane surrounding all the m roots of $a(\zeta_1, \hat{\xi}) + t$ for fixed $\hat{\xi}, t$. $a(\zeta_1, \hat{\xi})$ is a homogeneous polynomial of degree $2m$.

Proof. Making the change of variables $\hat{\xi} = t^{1/2m} \hat{\xi}'$, $\zeta_1 = t^{1/2m} \zeta_1'$, we get:

$$\begin{aligned} f(x_1, \hat{\xi}, t) &= t^{r/2m-1} \int_{C_{\hat{\xi}', 1}} \zeta_1'^{r-1} \exp(i\zeta_1' x_1 t^{1/2m}) [a(\zeta_1', \hat{\xi}') + 1]^{-1} d\zeta_1' \\ &= t^{r/2m-1} f(x_1 t^{1/2m}, \hat{\xi}' t^{-1/2m}, 1). \end{aligned}$$

We consider:

$$f(x_1, \hat{\xi}, 1) = \int_{C_{\hat{\xi}}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) (a(\zeta_1, \hat{\xi}) + 1)^{-1} d\zeta_1.$$

(1) the equation $a(\zeta_1, \hat{\xi}) + 1 = 0$ may be written as $c\zeta_1^{2m} + P(\zeta_1, \hat{\xi}) = 0$ where $P(\zeta_1, \hat{\xi})$ is a polynomial of degree less than or equal to $2m-1$ in the ζ_1 variable and such that $P(\zeta_1, 0) = 1$; c is a constant. The roots of $a(\zeta_1, \hat{\xi}) + 1 = 0$ depend continuously on the parameter $\hat{\xi}$ and as $\hat{\xi}$ goes to zero they go to the roots of $c\zeta_1^{2m} + 1 = 0$. Therefore, there is a closed curve C containing all the roots of $a(\zeta_1, \hat{\xi}) + 1 = 0$ for fixed and small $\hat{\xi}$.

For $|\hat{\xi}| < 1$, we have:

$$f(x_1, \hat{\xi}, 1) = \int_C \zeta_1^{r-1} \exp(ix_1 \zeta_1) [a(\zeta_1, \hat{\xi}) + 1]^{-1} d\zeta_1$$

so:

$$\begin{aligned} |f(x_1, \hat{\xi}, 1)| &\leq M, \\ |f(x_1, \hat{\xi}, t)| &\leq M t_0^{r/2m-1}. \end{aligned}$$

(2) To study the behavior of $f(x_1, \hat{\xi}, 1)$ as $|\hat{\xi}|$ goes to infinity, we make the change of variables $\hat{\xi} = \tau \hat{\xi}'$; $\zeta_1 = \tau \zeta_1'$ with $\tau = |\hat{\xi}|$. Then

$$f(x_1, \hat{\xi}, 1) = |\hat{\xi}|^{r-2m} \int_{C_{\hat{\xi}', t^{-2m}}} \exp(i\zeta_1' \tau x_1) [a(\zeta_1', \hat{\xi}') + \tau^{-2m}]^{-1} d\zeta_1'.$$

Consider the equation $a(\zeta_1, \hat{\xi}') + \tau^{-2m} = 0$ for large τ . The roots of the equation considered as a polynomial in ζ_1 depend continuously on τ^{-1} and so as τ goes to infinity they go to the roots of $a(\zeta_1, \hat{\xi}') = 0$; $|\hat{\xi}'| = 1$. Therefore there exists

a closed curve $C_{\xi'}$ in the upper half plane surrounding all the roots of $a(\zeta_1, \xi') + \tau^{-2m} = 0$ for large τ

$$f(x_1, \xi, 1) = \tau^{-2m} \int_{C_{\xi'}} \zeta_1^{r-1} \exp(i\zeta_1 \tau x_1) [a(\zeta_1, \xi') + \tau^{-2m}]^{-1} d\zeta_1.$$

On the curve $C_{\xi'}$, we have, $|a(\zeta_1, \xi)| \geq c > 0$. On the other hand $|a(\zeta_1, \xi') + \tau^{-2m}| \geq |a(\zeta_1, \xi')| - \tau^{-2m}$. For sufficiently large τ , $|a(\zeta_1, \xi') + \tau^{-2m}| \geq c_1 > 0$. Therefore for large ξ

$$|f(x_1, \xi, 1)| \leq M \exp(-d|\xi| x_1) |\xi|^{r-2m}$$

where $d = \inf_{\zeta_1 \in C_{\xi'}} (\text{Im } \zeta_1) > 0$, M is a constant and

$$|f(x_1, \xi, t)| \leq M \exp(-d|\xi| x_1) |\xi|^{r-2m}, \quad |\xi| > 1.$$

M is independent of x_1, t .

Combining (1), (2) we get the lemma.

LEMMA 3.3. 1. The following estimates hold uniformly for $t \geq t_0 > 0$

$$c_{r,j}(\xi, t) = \int_{C_{\xi,t}} \zeta_1^{r-1} b_j(\zeta_1, \xi) [a(\zeta_1, \xi) + t]^{-1} d\zeta_1 = O((1 + |\xi|^{r+r_j})(1 + |\xi|^{2m})^{-1})$$

where $r = 1, \dots, m$; $r_j < 2m$ and $C_{\xi,t}$ is a closed Jordan rectifiable curve in the ζ_1 upper half plane surrounding all the m roots of $a(\zeta_1, \xi) + t = 0$ for fixed ξ, t . $b_j(\zeta_1, \xi)$; $a(\zeta_1, \xi)$ are homogeneous polynomials of degree $r_j, 2m$ respectively.

2. Let $Q_{r,j}(\xi, t)$ be the elements of the transpose of the inverse of the matrix $(c_{r,j}(\xi, t))$ then:

$$Q_{r,j}(\xi, t) = O((1 + |\xi|^{2m})(1 + |\xi|^{r+r_j})^{-1}).$$

Proof. The proof is similar to that of Lemma 3.2. We will not repeat it.

We return to the proof of Theorem 3.1. From Lemmas 3.2, 3.3 and noting that $H_j(\xi, t) = O(1 + |\xi|^{r_j})(1 + |\xi|^{2m})^{-1}$ we obtain:

$$V(x_1, \xi, t) = O((1 + |\xi|^{2m})^{-1}) \exp(-d|\xi| x_1)$$

where $d > 0$; $x_1 > 0$; $t \geq t_0 > 0$.

For $x_1 > 0$, $V(x_1, \xi, t)$ is integrable with respect to ξ we have:

$$(2\pi)^n v(x, t) =$$

$$\sum_{r,j=1}^m \int_{E^{n-1}} \exp(i\langle x, \xi \rangle) Q_{r,j}(\xi, t) H_j(\xi, t) \int_{C_{\xi,t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \xi) + t]^{-1} d\zeta_1 d\xi.$$

We study the regularity of $v(x, t)$. The results are stated in the following lemma:

LEMMA 3.4. Let $v(x, t)$ be given by the expression:

$$v(x, t) =$$

$$\sum_{r,j=1}^m (2\pi)^{-n} \int_{E^{n-1}} \exp(i\langle \hat{x}, \hat{\xi} \rangle) Q_{r,j}(\hat{\xi}, t) H_j(\hat{\xi}, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 d\hat{\xi}$$

where $Q_{r,j}; H_j$ are defined in Theorem 3.1, then:

- (1) $v(x, t)$ is infinitely differentiable for $\hat{x} \neq 0$.
- (2) $D^\alpha v(0, \hat{x}, t) = O(1)t^{-\varepsilon/2m} |\hat{x}|^{-n-|\alpha|+2m-\varepsilon} (1 + |\hat{x}|^N)^{-1}$ if $-n - |\alpha| + 2m \leq 0$ and $D^\alpha v(0, \hat{x}, t) = O(1)t^{-\varepsilon/2m} (1 + |\hat{x}|^N)^{-1}$ otherwise. $0 < \varepsilon < 1$ and N is an arbitrary positive integer.

Proof. (1) First we consider the case when $x_1 > 0$. We may take the differentiation under the integral sign. Indeed:

$$K_{r,j}(\hat{\xi}, x_1, t, \alpha + \beta) = \hat{\xi}^\alpha Q_{r,j}(\hat{\xi}, t) H_j(\hat{\xi}, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1+\beta} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1,$$

$$K_{r,j}(\hat{\xi}, x_1, t, \alpha + \beta) = O(|\hat{\xi}|^{\alpha+\beta} (1 + |\hat{\xi}|^{2m-1})^{-1} \exp(-d|\hat{\xi}|x_1)),$$

hence for $x_1 > 0$; $v(x, t)$ is infinitely differentiable.

(2) We now study the regularity of $v(x, t)$ as x_1 goes to zero.

For $x_1 > 0$, we have:

$$\hat{x}^p D^{\alpha+\beta} v(x_1, \hat{x}, t) = (2\pi)^{-n} \sum_{r,j=1}^m \int_{E^{n-1}} \exp(i\langle \hat{x}, \hat{\xi} \rangle) D_{\hat{\xi}}^p K_{r,j}(\hat{\xi}, x_1, \alpha + \beta, t) d\hat{\xi}.$$

Making the change of variables $\hat{\xi} = t^{1/2m} \hat{\xi}'$, $\zeta_1 = t^{1/2m} \zeta'_1$, we get:

$$\begin{aligned} (2\pi)^n \hat{x}^p D^{\alpha+\beta} v(x_1, \hat{x}, t) \\ = \sum_{r,j=1}^m t^{(\alpha+\beta+n-p)/2m-1} \int_{E^{n-1}} \exp(i\langle \hat{x} t^{1/2m}, \hat{\xi}' \rangle) D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m}, \alpha + \beta) d\hat{\xi}'. \end{aligned}$$

Consider the expression $D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m}, \alpha + \beta)$. We have:

$$|D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m}, \alpha + \beta)| \leq M \quad \text{for } |\hat{\xi}'| \leq 1.$$

M is a constant independent of x_1, t .

Making the change of variables $\hat{\xi} = \tau \hat{\xi}'$; $\zeta_1 = \tau \zeta'_1$ with $\tau = |\hat{\xi}|$; we get:

$$D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m}, \alpha + \beta) = \tau^{\alpha+\beta+p-2m+1} D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m} \tau, \alpha + \beta)$$

so:

$$|D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m}, \alpha + \beta)| \leq M |\hat{\xi}|^{|\alpha|+|\beta|-2m+1-p} \quad \text{for } |\hat{\xi}| \geq 1.$$

If $|\alpha| + |\beta| + n + 1 - 2m \geq 0$, we take $p = n + 1 + |\alpha| + |\beta| - 2m$ and:

$$|D_{\hat{\xi}'}^p K_{r,j}(\hat{\xi}', x_1 t^{1/2m}, \alpha + \beta)| \leq M (1 + |\hat{\xi}|^N)^{-1} \quad \text{for all } \hat{\xi}.$$

Since $\hat{x}^p D^{\alpha+\beta} v(x_1, \hat{x}, t) = 0$ for $\hat{x} = 0$ and $p = n + |\alpha| + |\beta| + 1 - 2m \geq 0$, we obtain:

$$D^{\alpha+\beta}v(x_1, \hat{x}, t) = O(1)t^{-\varepsilon/2m} |\hat{x}|^{-n-|\alpha|-|\beta|+2m-\varepsilon}(1 + |\hat{x}|^N)^{-1}; \quad x_1 \geq 0.$$

If $|\alpha| + |\beta| + 1 + n - 2m < 0$, we take $p = 0$, then:

$$|D_{\xi}^p K_{rj}(\hat{\xi}, x_1 t^{1/2m}, \alpha + \beta)| \leq M |\hat{\xi}|^{-n} \quad \text{for } |\hat{\xi}| \geq 1.$$

Therefore:

$$|D_{\xi}^p K_{rj}(\hat{\xi}, x_1 t^{1/2m}, \alpha + \beta)| \leq M(1 + |\hat{\xi}|^N)^{-1} \quad \text{for all } \hat{\xi}.$$

We get:

$$D^{\alpha+\beta}v(x_1, \hat{x}, t) = O(1)t^{-\varepsilon/2m}(1 + |\hat{x}|^N)^{-1} \quad \text{for } x_1 \geq 0.$$

The lemma is proved.

THEOREM 3.2. Let A be a positively strongly elliptic operator on E^n , homogeneous, of order $2m$, with constant coefficients. Let $B_1, \dots, B_m$ be m homogeneous, differential operators on E^n of order $r_j < 2m - 1$ with constant coefficients. Let t be a positive parameter such that (A, γ) is formally positive. Then $G(x, y, t)$ is given by the expression:

$$G(x, y, t) = (2\pi)^{-n} \int_{E^n} \exp(i\langle x - y, \xi \rangle) [a(\xi) + t]^{-1} d\xi + v(x, y, t)$$

where the integral is taken as the Fourier transform of a tempered distribution if $2m < n$; $v(x, y, t)$ is given by Theorem 3.1.

(1) $G(x, y, t)$ is infinitely differentiable for $x \neq y$,

(2) $D^\alpha G(x, y, t) = O(1)t^{-\varepsilon/2m} |x - y|^{n+2m-|\alpha|-\varepsilon}(1 + |x - y|^N)^{-1}$ if $2m - n \leq |\alpha|$ and

$$D^\alpha G(x, y, t) = O(1)t^{-\varepsilon/2m}(1 + |x - y|^N)^{-1}, \quad \text{if } 0 \leq |\alpha| < 2m - n,$$

where $0 < \varepsilon < 1$ and N is an arbitrary positive number.

Proof. The theorem follows immediately from Lemma 3.1 and Theorem 3.1.

THEOREM 3.3. Let $G(x, y, t)$ be the Green's function defined in Theorem 3.2 and $2m > n$; then:

(1) $x \neq y$, $t^{1-n/2m}G(x, y, t) \rightarrow 0$ as $t \rightarrow +\infty$,

(2) $t^{1-n/2m}G(x, y, t) \sim (2\pi)^{-n} \int_{E^n} [a(\xi) + 1]^{-1} d\xi + O(1)t^{-\varepsilon/2m}$ as $t \rightarrow +\infty$.

Proof. Let $t = \tau^{2m}$ and make the change of variables $\hat{\xi} = \tau \hat{\xi}'$; $\zeta_1 = \tau \zeta'_1$. We obtain:

$$G(x, y, t) = \tau^{n-2m}E(\tau x, \tau y, 1) + v(x, y, t).$$

We consider the expression $v(x, y, t)$. From Lemma 3.4, we have:

$$v(x, y, t) = \sum_{r,j=1}^m \tau^{n-1-r_j} \int_{E^{n-1}} \exp(i\langle \tau \hat{x} - \tau \hat{y}, \hat{\xi} \rangle) Q_{rj}(\hat{\xi}, 1) H_j(\hat{\xi}, y_1, t) \\ \cdot \int_{C_{\hat{\xi}}} \zeta_1^{r-1} \exp(i\zeta_1 \tau x_1) [a(\zeta_1, \hat{\xi}) + 1]^{-1} d\zeta_1 d\hat{\xi}.$$

On the other hand; $H_j(\hat{\xi}, y_1, t) = O(1)\tau^{r_j+1-2m-\varepsilon}(1 + |\hat{\xi}|^{2m-r_j})^{-1}$. When $x \neq y$ then by the Riemann-Lebesgue theorem, we have:

$$\tau^{2m-n} G(x, y, t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

THEOREM 3.4. Let $G_{(p)}(x, y, t)$ be the p th iterate of the Green's function G defined by Theorem 3.1. Let p be such that $2mp > n$; then:

- (1) If $x \neq y$, $t^{p-n/2m} G_{(p)}(x, y, t) \rightarrow 0$ as $t \rightarrow +\infty$,
- (2) $t^{p-n/2m} G_{(p)}(x, x, t) = (2\pi)^{-n} \int_{E^n} [a(\xi) + 1]^{-p} d\xi + O(1)t^{-\varepsilon/2m}$, $0 < \varepsilon < 1$, $t \rightarrow +\infty$.

Proof. We construct the iterates of G . We know that:

$$G_{(2)}(x, y, t) = \int_{E^n} G(x, z, t) G(z, y, t) dz, \quad x \neq y.$$

The integral is well defined. We also have:

$$F_{\hat{x}} G(x, z, t) = (2\pi)^{-(n-1)/2} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) \int_{E'} \exp(i(x_1 - z_1)\xi_1) [a(\xi_1, \hat{\xi}) + t]^{-1} d\xi_1 \\ + \exp(-i\langle \hat{z}, \hat{\xi} \rangle) V(x_1, z_1, \hat{\xi}, t)$$

with:

$$V(x_1, z_1, \hat{\xi}, t) = \sum_{r,j=1}^m Q_{rj}(\hat{\xi}, t) H_j(\hat{\xi}, z_1, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1.$$

Consider the Fourier transform of $G_{(2)}(x, y, t)$ with respect to the tangential variables $\hat{x}$.

$$F_{\hat{x}} G_{(2)}(x, y, t) = (2\pi)^{(1-n)/2} \int_{E^{n-1}} \exp(-i\langle \hat{x}, \hat{\xi} \rangle) G_{(2)}(x, y, t) d\hat{x}, \quad x \neq y, \\ = (2\pi)^{(1-n)/2} \int_{E^{n-1}} \int_{E^n} \exp(-i\langle \hat{x}, \hat{\xi} \rangle) G(x, z, t) G(y, z, t) dz d\hat{x}.$$

By Fubini's theorem we may interchange the order of integration. We obtain:

$$F_{\hat{x}} G_{(2)}(x, y, t) = \int_{E^n} F_{\hat{x}} G(x, z, t) G(y, z, t) dz, \\ = (2\pi)^{(1-n)/2} \int_{E^n} \int_{E^1} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) \exp(i\xi_1(x_1 - z_1)) [a(\xi_1, \hat{\xi}) + t]^{-1} G(y, z, t) d\xi_1 dz \\ + (2\pi)^{(1-n)/2} \int_{E^n} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) V(\hat{\xi}, x_1, z_1, t) G(y, z, t) dz.$$

From Theorem 3.2; Lemmas 3.2; 3.3; it follows that all the integrals are well defined. The first integral of the above expression may be written as follows:

$$(2\pi)^{(1-n)/2} \int_{E^n} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) G(y, z, t) \int_{E^1} \exp[i\xi_1(x_1 - z_1)] [a(\xi_1, \hat{\xi}) + t]^{-1} d\xi_1 dz.$$

Denote by $k(z_1, x_1, t, \hat{\xi}) = \int_{E^1} \exp[i\xi_1(x_1 - z_1)] [a(\xi_1, \hat{\xi}) + t]^{-1} d\xi_1$. Then:

$$\begin{aligned} & (2\pi)^{(1-n)/2} \int_{E^n} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) G(y, z, t) k(z_1, x_1, \hat{\xi}, t) dz \\ &= (2\pi)^{(1-n)/2} \int_{E^n} k(z_1, x_1, \hat{\xi}, t) \int_{E^{n-1}} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) G(y, z, t) d\hat{z} dz_1 \\ &= \int_{E^1} k(z_1, x_1, \hat{\xi}, t) F_2 G(x, y, t) dz_1 \\ &= (2\pi)^{(1-n)/2} \int_{E^1} k(z_1 - x_1, \hat{\xi}, t) \exp(-i\langle \hat{y}, \hat{\xi} \rangle) \int_{E^1} \exp[i\eta_1(z_1 - y_1)] [a(\eta_1, \hat{\xi}) + t]^{-1} d\eta_1 dz_1 \\ &\quad + (2\pi)^{(1-n)/2} \int_{E^1} k(z_1 - x_1, \hat{\xi}, t) \exp(-i\langle \hat{y}, \hat{\xi} \rangle) V(\hat{\xi}, y_1, z_1, t) dz_1. \end{aligned}$$

Consider the first term. It is easy to see that $k(u, \hat{\xi}, t)$ is integrable with respect to u . Applying Fubini's theorem, we get:

$$(2\pi)^{-n/2} \exp(-i\langle \hat{y}, \hat{\xi} \rangle) \int_{E^1} \exp[i\xi_1(x_1 - y_1)] [a(\xi_1, \hat{\xi}) + t]^{-2} d\xi_1$$

since

$$\int_{E^1} \exp(i\eta_1 z_1) k(z_1 - x_1, \hat{\xi}, t) dz_1 = (2\pi)^{-1/2} \exp(i\eta_1 x_1) [a(\eta_1, \hat{\xi}) + t]^{-1}.$$

We consider:

$$(2\pi)^{(1-n)/2} \int_{E^n} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) V(\hat{\xi}, x_1, z_1, t) G(y, z, t) dz.$$

As before we may write it as

$$\int_{E^1} V(\hat{\xi}, x_1, z_1, t) \left\{ (2\pi)^{(1-n)/2} \int_{E^{n-1}} \exp(-i\langle \hat{z}, \hat{\xi} \rangle) G(y, z, t) d\hat{z} \right\} dz_1.$$

The integral in the bracket is the Fourier transform of G with respect to the tangential variable $\hat{z}$. We obtain:

$$\begin{aligned} & \int_{E^1} \int_{E^1} \exp(-i\langle \hat{y}, \hat{\xi} \rangle) \exp[i\xi_1(z_1 - y_1)] V(\hat{\xi}, y_1, z_1, t) [a(\xi_1, \hat{\xi}) + t]^{-1} d\xi_1 dz_1 \\ &+ \int_{E^1} \exp(-i\langle \hat{y}, \hat{\xi} \rangle) V(\hat{\xi}, t, y_1, z_1) V(\hat{\xi}, t, y_1, x_1) dz_1. \end{aligned}$$

Denote the first integral by $h_1(\xi, x_1, y_1, t)$ and the second one by $h_2(\xi, x_1, y_1, t)$; then:

$$F_{\hat{x}}G_{(2)}(x, y, t) = (2\pi)^{(1-n)/2} \exp(-i\langle \hat{y}, \hat{\xi} \rangle) \int_{E^1} \exp[i\xi_1(x_1 - y_1)] [a(\xi_1, \hat{\xi}) + t]^{-2} d\xi_1 \\ + h_1(\xi, x_1, y_1, t) + h_2(\xi, x_1, y_1, t) + g(\xi, x_1, y_1, t)$$

with

$$g(\xi, x_1, y_1, t) = \int_{E^1} \exp(-i\langle \hat{y}, \hat{\xi} \rangle) k(z_1 - x_1, \xi, t) V(\xi, z_1, y_1, t) dz_1.$$

If $4m > n$, we want to take the inverse Fourier transform of $F_{\hat{x}}G_{(2)}$. First we establish the following lemma:

LEMMA 3.5. Let $g(\xi, x_1, y_1, t)$; $h_1(\xi, x_1, y_1, t)$, $h_2(\xi, x_1, y_1, t)$ be as above; then the following estimates hold uniformly for $t \geq t_0 > 0$

- (1) $g(\xi, x_1, y_1, t) = O((1 + |\xi|^{4m})^{-1})$,
- (2) $h_1(\xi, x_1, y_1, t) = O((1 + |\xi|^{4m})^{-1})$,
- (3) $h_2(\xi, x_1, y_1, t) = O((1 + |\xi|^{4m})^{-1})$.

Proof. Making the change of variable $\xi = t^{1/2m} \xi'$ and taking into account the results of Lemmas 3.2, 3.3, we get the above estimates immediately. We return to the proof of the theorem. If $4m > n$, we may take the inverse Fourier transform of $F_{\hat{x}}G_{(2)}$ with respect to $\hat{\xi}$. We obtain:

$$G_{(2)}(x, y, t) = (2\pi)^{-n} \int_{E^n} \exp(-i\langle x - y, \xi \rangle) [a(\xi) + t]^{-2} d\xi \\ + (2\pi)^{(1-n)/2} \int_{E^{n-1}} \exp(i\langle \hat{x}, \hat{\xi} \rangle) \{h_1(\xi, x_1, y_1, t) + h_2(\xi, x_1, y_1, t)\} d\hat{\xi}.$$

If $4m < n$, we construct $F_{\hat{x}}G_{(3)}(x, y, t)$ etc. step by step. We take only the first term and we have to make an estimate of the error involved in terms of the parameter t , for large t .

From the proof of Theorem 3.3, we have:

$$g(\xi, x_1, y_1, t) = O(1) \tau^{-4m+1-\varepsilon} (1 + |\xi|^{4m-1})^{-1}, \\ h_1(\xi, x_1, y_1, t) = O(1) \tau^{-4m+1-\varepsilon} (1 + |\xi|^{4m-1})^{-1}, \\ h_2(\xi, x_1, y_1, t) = O(1) \tau^{-4m+1-\varepsilon} (1 + |\xi|^{4m-1})^{-1}.$$

Therefore if $4m > n$, we have:

$$G_{(2)}(x, y, t) = (2\pi)^{-n} \int_{E^n} \exp(i\langle x - y, \xi \rangle) [a(\xi) + t]^{-2} d\xi + O(1) t^{-2+(n-\varepsilon)/2m}.$$

More generally if $2mp > n$,

$$G_{(p)}(x, y, t) = (2\pi)^{-n} \int_{E^n} \exp(i\langle x - y, \xi \rangle) [a(\xi) + t]^{-p} d\xi + O(1) t^{-p+(n-\varepsilon)/2m}.$$

The conclusion of the theorem follows immediately.

4. In this section the Green's function $\mathcal{G}(x, y, t)$ corresponding to the elliptic boundary value problem $\{A + tI, B_j; j = 1, \dots, m\}$ where A and B_j are defined respectively on S and on Γ and have infinitely differentiable coefficients is constructed. We will:

(1) Construct the Green's function G associated with the elliptic boundary value problem $\{A + tI; B_j; j = 1, \dots, m\}$ where A and B_j are defined on E_+^n, E^{n-1} respectively, with infinitely differentiable coefficients.

(2) Seek an integral representation of a function $u(x)$, infinitely differentiable function with compact support in $E_+^n \cup E^{n-1}$ in terms of $(A + tI)u, B_j u$.

(3) Get the function $\mathcal{G}$ using (1) and (2).

LEMMA 4.1. Let $H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)$ be given by the expression:

$$H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) = \sum_{r=1}^m \int_{E^{n-1}} \exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) Q_{rj}(\hat{y}, \hat{\xi}, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\hat{y}, \zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 d\hat{\xi}$$

where $a(\hat{y}, \zeta_1, \hat{\xi})$ is the characteristic form of the homogeneous regularly elliptic operator $A_{\hat{y}}$ with coefficients evaluated at $\hat{y}$; $C_{\hat{\xi}, t}$ is a closed curve in the ζ_1 upper half plane surrounding the roots of $a(\hat{y}, \zeta_1, \hat{\xi}) + t = 0$ for fixed $\hat{\xi}, t$.

$Q_{rj}(\hat{y}, \hat{\xi}, t)$ are the elements of the transpose of the inverse of the matrix:

$$\left(c_{rj}(\hat{y}, \hat{\xi}, t) = \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1} b_j(\hat{y}, \zeta_1, \hat{\xi}) [a(\hat{y}, \zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 \right).$$

$b_j(\hat{y}, \hat{\xi})$ is the characteristic form of the differential operator $B_{j\hat{y}}$ of order r_j and with coefficients evaluated at $\hat{y}$.

$H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)$ is infinitely differentiable for $\hat{x} \neq \hat{y}$ and:

$$(B_k - B_{k\hat{y}})H_{j\hat{y}}(0, \hat{x} - \hat{y}, t) = O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon}, \quad k, j = 1, \dots, m,$$

$$AH_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) = O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon}, \quad x_1 > 0.$$

Proof. $H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)$ is well defined and for $x_1 > 0$ is infinitely differentiable (Lemmas 3.2, 3.3). We study the case when $x_1 \rightarrow 0$. Let p be a positive integer and consider:

$$\begin{aligned} & (\hat{x} - \hat{y})^p \hat{D}_1^\alpha D_1^\beta H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) \\ &= \sum_{r=1}^m \int_{E^{n-1}} \exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) \\ & \quad \cdot D_{\hat{\xi}}^\beta \left\{ \hat{\xi}^\alpha Q_{rj}(\hat{\xi}, \hat{y}, t) \int_{C_{\hat{\xi}, t}} \zeta_1^{r-1+\beta} \exp(i\zeta_1 x_1) [a(\hat{y}, \zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 \right\} d\hat{\xi}. \end{aligned}$$

The expression $D_{\xi}^p\{ \}$ is integrable at the origin when $x_1 \rightarrow 0$. We have only to consider it at infinity.

$$\begin{aligned} D_{\xi}^p \left\{ \hat{\xi}^{\alpha} Q_{rj}(\hat{y}, \hat{\xi}, t) \int_{C_{\xi, t}} \zeta_1^{\beta+r-1} [a(\hat{y}, \zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 \right\} \\ = t^{(|\alpha|+|\beta|-r_j-p)/2m} D_{\xi}^p \left\{ \hat{\xi}^{\alpha} Q_{rj}(\hat{y}, \hat{\xi}, 1) \int_{C_{\xi, 1}} \zeta_1^{r-1+\beta} [a(\hat{y}, \zeta_1, \hat{\xi}) + 1]^{-1} d\zeta_1 \right\}. \end{aligned}$$

Making the change of variables $\hat{\xi} = \tau \hat{\xi}'$, $\zeta_1 = \tau \zeta'_1$ with $\tau = |\hat{\xi}|$; we get:

$$\begin{aligned} D_{\xi}^p \left\{ \hat{\xi}^{\alpha} Q_{rj}(\hat{y}, \hat{\xi}, 1) \int_{C_{\xi, 1}} \zeta_1^{r-1+\beta} [a(\hat{y}, \zeta_1, \hat{\xi}) + 1]^{-1} d\zeta_1 \right\} \\ = \tau^{\alpha+\beta-r_j-p} D_{\xi'}^p \left\{ \hat{\xi}'^{\alpha} Q_{rj}(\hat{y}, \hat{\xi}', \tau^{-2m}) \int_{C_{\xi'}} \zeta_1^{r-1+\beta} [a(\hat{y}, \zeta_1, \hat{\xi}') + \tau^{-2m}]^{-1} d\zeta_1 \right\}. \end{aligned}$$

So:

$$\begin{aligned} D_{\xi}^p \left\{ \hat{\xi}^{\alpha} Q_{rj}(\hat{y}, \hat{\xi}, t) \int_{C_{\xi, t}} \zeta_1^{r-1+\beta} [a(\hat{y}, \zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 \right\} \\ = t^{(\alpha+\beta-r_j-p)/2m} O(1) |\hat{\xi}|^{-r_j-p+|\alpha|+|\beta|} \quad \text{for large } |\hat{\xi}|. \end{aligned}$$

If we take $p = -r_j + n + |\alpha| + |\beta| \geq 0$, the expression $(\hat{x} - \hat{y})^p \hat{D}^{\alpha} D_1^{\beta} H_{j\hat{y}}$ is well defined. When $\hat{x} = \hat{y}$, it is equal to zero. We may replace $\exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle)$ by $\exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) - 1$ which is less than $|\hat{x} - \hat{y}|^{1-\varepsilon} |\hat{\xi}|^{-\varepsilon+1}$. We get:

$$D^{\alpha} H_{j\hat{y}}(0, \hat{x} - \hat{y}, t) = O(1) t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{r_j-n-|\alpha|-|\beta|+1-\varepsilon}.$$

It follows that if $r_k \leq r_j$:

$$(B_k - B_{k\hat{y}}) H_{j\hat{y}}(0, \hat{x} - \hat{y}, t) = O(1) t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon}.$$

When $r_k > r_j$; consider

$$\begin{aligned} B_k H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) &= \sum_{r=1}^m \int_{E^{n-1}} \exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) Q_{rj}(\hat{y}, \hat{\xi}, t) \\ &\quad \cdot \int_{C_{\xi, t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) b_k(\hat{x}, \zeta_1, \hat{\xi}) [a(\hat{y}, \zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 d\hat{\xi}. \end{aligned}$$

It is well defined for $x_1 > 0$, has a discontinuity at $(x_1, \hat{x} - \hat{y}) = 0$ and:

$$B_k H_{j\hat{y}}(x_1, 0, t) \Big|_{x_1=0} = 0.$$

The integrand is nonnull for $\hat{x} = \hat{y}$; $x_1 > 0$.

Consider: $B_k H_{j\hat{y}}(|\hat{x} - \hat{y}|, \hat{x} - \hat{y}, t)$. We have:

$$B_k H_{j\hat{y}}(|\hat{x} - \hat{y}|, \hat{x} - \hat{y}, t) \Big|_{\hat{x}=\hat{y}} = 0.$$

$b_k(\hat{y}, \zeta_1, \hat{\xi})$ is infinitely differentiable; taking the Taylor's development of b_k in powers of $\hat{x} - \hat{y}$ for $\hat{x}$ near $\hat{y}$, up to the order $n + r_k + 2$ and putting in the above integral, we obtain:

$$B_k H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) = B_{k\hat{y}} H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) + \int_{E^{n-1}} \exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) P_{jk}^1(\hat{\xi}, \hat{y}, x_1, t) d\hat{\xi} \\ + \int_{E^{n-1}} \exp(i\langle \hat{x} - \hat{y}, \hat{\xi} \rangle) P_{jk}^2(\hat{\xi}, x, t, \hat{y}) d\hat{\xi}.$$

The integrals are well defined for $x_1 > 0$ and the last expression is majorized by $Mt^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-\varepsilon + r_j + 1}$ for $x_1 \geq 0$. The expression $P_{jk}^1(\hat{\xi}, x_1, \hat{y}, t)$ is not integrable when $x_1 = 0$. Since $B_k H_{j\hat{y}}(|\hat{x} - \hat{y}|, \hat{x} - \hat{y}, t)|_{\hat{x}=\hat{y}} = 0$ we must have:

$$P_{jk}^1(\hat{\xi}, 0, \hat{y}, t) = 0.$$

It follows then that:

$$(B_k - B_{k\hat{y}}) H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) = O(1) t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon}$$

for all j, k .

We note that $(\hat{x} - \hat{y})^{n-2} A H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)$ is uniformly continuous in $\hat{x} - \hat{y}$ for $x_1 > 0$ and is equal to zero for $\hat{x} = \hat{y}$. So for large t when $|\hat{x} - \hat{y}| \leq t^{-\varepsilon/2(2m+2)}$, we have:

$$|(\hat{x} - \hat{y})^{n-2} A H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)| \leq M t^{-s}$$

for some positive number s .

On the other hand, we have:

$$|(\hat{x} - \hat{y})^{n+2m-1+\varepsilon} A H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)| \leq M t^{-\varepsilon/2m}.$$

So for $|\hat{x} - \hat{y}| > t^{-\varepsilon/2(2m+2)}$, we get:

$$|(\hat{x} - \hat{y})^{n-2} A H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t)| \leq M t^{-\varepsilon/4m}.$$

Hence for large t and $x_1 > 0$, we obtain:

$$(\hat{x} - \hat{y})^{n-2} A H_{j\hat{y}}(x_1, \hat{x} - \hat{y}, t) = O(1) t^{-\nu/2m}, \quad 0 < \nu < 1.$$

LEMMA 4.2. Let $H_{j\hat{y}}$ be as above, then:

$$B_{k\hat{y}} H_{j\hat{y}}(0, \hat{x} - \hat{y}, t) = 0 \quad \text{if } k \neq j,$$

$$B_{j\hat{y}} H_{j\hat{y}}(0, \hat{x} - \hat{y}, t) = \delta_{\hat{y}}.$$

Proof. It follows immediately from the definition of H_j .

LEMMA 4.3. Let $\{A; B_j; j = 1, \dots, m\}$ be a uniformly regularly elliptic boundary value problem where A and B_j are defined on E_+^n , E^{n-1} , have infinitely differentiable coefficients and are homogeneous of orders $2m, r_j$ re-

spectively. Let A_z, B_{jz} be the operators obtained from $A; B_j$ by taking the values of the coefficients at the point z . Let $G_{(z)}(x, z, t)$ be the Green's function associated with $\{A_z + tI; B_{jz}; j = 1, \dots, m\}$ constructed in §3, (Theorem 3.2).

Let

$$\begin{aligned} \alpha_0(x, z, t) &= (A - A_z)G_{(z)}(x, z, t), & x \neq z; \ x, z \text{ in } B^+, \\ (1) \quad \alpha_j(x, z, t) &= (B_j - B_{jz})G_{(z)}(x, z, t), & j = 1, \dots, m, \\ \alpha(x, z, t) &= (\alpha_0, \dots, \alpha_m), \\ \beta_{0k}(x, \hat{y}, t) &= (A - A_{\hat{y}})H_{k\hat{y}}(x, \hat{y}, t), \\ (2) \quad \beta_{jk}(\hat{x}, \hat{y}, t) &= (B_j - B_{j\hat{y}})H_{k\hat{y}}(\hat{x}, \hat{y}, t), & j = 1, \dots, m, \\ \beta_k(x, \hat{y}, t) &= (\beta_{0k}, \dots, \beta_{mk}), \end{aligned}$$

$H_{k\hat{y}}$ is given by Theorem 4.1.

$$(3) \quad w(x, z, t) = (v(x, z), h(\hat{x}, \hat{z}), \dots, h_m(\hat{x}, \hat{z})).$$

Define the linear transformations:

$$\begin{aligned} T_0 w(x, z, t) &= \int_{B^+} \alpha(x, y, t) v(y, z) dy, \\ T_k w(x, z, t) &= \int_{\Gamma_0} \beta_k(x, \hat{y}, t) h_k(\hat{y}, \hat{z}) d\hat{y}, \end{aligned}$$

with:

$$Tw = T_0 w + \sum_{k=1}^m (T_k w),$$

$$B^+ = \{x: |x| < 1, x_1 > 0\}; \quad \Gamma_0 = \{x: |x| < 1, x_1 = 0\}.$$

Then the integral equation:

$$w(x, z, t) + Tw(x, z, t) + \alpha(x, z, t) = 0$$

may be solved by the Neumann series for sufficiently large t and:

$$\begin{aligned} v(x, z, t) &= O(1)t^{-\varepsilon/2m} |x - z|^{1-n-\varepsilon} (1 + |t^{1/2m}(x - z)|^N)^{-1}, \\ h_j(\hat{x}, \hat{z}, t) &= O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{z}|^{-n+2-\varepsilon} (1 + |t^{1/2m}(\hat{x} - \hat{z})|^N)^{-1}, \end{aligned}$$

$t = \tau^{2m} < \varepsilon < 1; N$ any positive integer.

Proof. We have from Theorem 3.2:

$$\begin{aligned} \alpha_0(x, z, t) &= (A - A_z)G_{(z)}(x, z, t) = O(1)\tau^{-\varepsilon} |x - z|^{1-n-\varepsilon} (1 + |(x - z)|^N)^{-1}, \\ \alpha_j(\hat{x}, z, t) &= (B_j - B_{jz})G_{(z)}(\hat{x}, z, t) = O(1)\tau^{-\varepsilon} |\hat{x} - \hat{z}|^{-n+2-\varepsilon} (1 + |(\hat{x} - \hat{z})|^N)^{-1}. \end{aligned}$$

Finally from Lemma 4.1:

$$\beta_{jk}(\hat{x}, \hat{y}, t) = O(1)t^{-\varepsilon}|\hat{x} - \hat{y}|^{-n+2-\varepsilon}.$$

Consider the series:

$$w(z, x, t) = \alpha(x, z, t) + T\alpha + T^2\alpha + \dots.$$

It may be written as:

$$\begin{aligned} v(x, z, t) &= \alpha_0(x, z, t) + \int_{B^+} \alpha_0(x, y, t) \alpha_0(y, z, t) dy \\ &+ \sum_{k=1}^m \int_{\Gamma_0} \beta_{0k}(x, \hat{y}, t) \alpha_k(\hat{y}, \hat{z}) h_j(\hat{x}, z, t) = \alpha_j(\hat{x}, \hat{z}, t) + \int_{B^+} \alpha_0(\hat{x}, y, t) \alpha_j(y, \hat{z}, t) dy \\ &+ \sum_{k=1}^m \int_{\Gamma_0} \beta_{jk}(\hat{x}, \hat{y}, t) \alpha_k(\hat{y}, \hat{z}) d\hat{y} + \dots, \quad j = 1, \dots, m. \end{aligned}$$

The first series is majorized by:

$$\begin{aligned} O(1)t^{-\varepsilon/2m} |x - z|^{1-n-\varepsilon} + O(1)t^{-\varepsilon/2m} \int_{B^+} |x - y|^{1-n-\varepsilon} |y - z|^{1-n-\varepsilon} dy \\ + O(1)t^{-\varepsilon/m} \int_{\Gamma_0} |\hat{x} - \hat{y}|^{-n+2-\varepsilon} |\hat{z} - \hat{y}|^{-n+2-\varepsilon} d\hat{y} + \dots, \end{aligned}$$

which is uniformly convergent for large t ; x, z in B^+ . The second series is majorized by:

$$\begin{aligned} O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{z}|^{2-n-\varepsilon} + O(1)t^{-\varepsilon/m} \int_{B^+} |x - y|^{1-n-\varepsilon} |y - z|^{1-n-\varepsilon} dy \\ + O(1)t^{-\varepsilon/m} \int_{\Gamma_0} |\hat{x} - \hat{y}|^{-n+2-\varepsilon} |\hat{z} - \hat{y}|^{2-n-\varepsilon} d\hat{y} + \dots, \end{aligned}$$

which is also uniformly convergent for large t .

The proof of the theorem is completed.

THEOREM 4.1. Let $\{A; B_j; j = 1, \dots, m\}$ be a uniformly regularly elliptic boundary value problem where A, B_j are defined on E_+^n, E^{n-1} and have infinitely differentiable coefficients. If $G_{(z)}(x, z, t)$ is the Green's function associated with the constant coefficients problem $\{A_z + tI; B_{jz}; j = 1, \dots, m\}$ constructed in Theorem 3.2; then:

$$G(x, z, t) = G_{(z)}(x, z, t) + \int_{B^+} G_{(y)}(x, y, t) v(y, z, t) dy + \sum_{k=1}^m \int_{\Gamma_0} H_{k\hat{y}}(x_1, \hat{y}, t) h_k(\hat{y}, z)$$

is the Green's function associated with the elliptic boundary value problem $\{A + tI; B_j; j = 1, \dots, m\}$. $H_{k\hat{y}}$ are the kernels defined by Lemmas 4.1, v, h_k are the solutions of the system of integral equations of Lemma 4.3.

$$B^+ = \{x: x_1 > 0; |x| < 1\}; \quad \Gamma_0 = \{x: x_1 = 0; |x| < 1\}.$$

Proof. We verify that

$$(A + tI)G(x, z, t) = \delta_z,$$

$$B_j G(x, z, t) = 0, \quad x_1 = 0; j = 1, \dots, m.$$

(1) Consider $(A + tI)G(x, z, t)$. We have:

$$\begin{aligned} (A + tI)G(x, z, t) &= (A_z + tI)G_z(x, z, t) + (A - A_z)G_z(x, z, t) \\ &\quad + \sum_{k=1}^m \int_{\Gamma_0} (A - A_y)H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)d\hat{y} \\ &\quad + (A + tI) \left(\int_{B^+} G_y(x, y, t)v(y, z, t)dy \right). \end{aligned}$$

(2) Let $\phi(x)$ be an infinitely differentiable function with compact support in B^+ . We have:

$$\begin{aligned} \int_{B^+} (A + tI)G(x, z, t)\phi(x)dx &= \phi(z) + \int_{B^+} (A - A_z)G_z(x, t, z)\phi(x)dx \\ &\quad + \sum_{k=1}^m \int_{B^+} (A - A_y)H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)\phi(x)d\hat{y}dx \\ &\quad + \int_{B^+} (A + tI) \left(\int_{B^+} G_y(x, y, t)v(y, z, t)dy \right) \phi(x)dx. \end{aligned}$$

(3) Consider the last integral. Since $\phi(x) \in C_c^\infty(B^+)$ we may write it as:

$$\begin{aligned} \sum_{|\alpha|=2m} \int_{B^+} \int_{B^+} G_y(x, t, y)v(y, z, t)[t\phi + D_x^\alpha(\bar{a}_\alpha(x)\phi(x))]dydx \\ = \sum_{|\alpha|=2m} \int_{B^+} \int_{B^+} [t\phi(x) + D_x^\alpha(\bar{a}_\alpha(x)\phi(x))]G_y(x, y, t)v(y, z, t)dx dy \end{aligned}$$

by Fubini's theorem. Integrating by parts, we obtain:

$$\begin{aligned} \int_{B^+} \int_{B^+} (A + tI)G_y(x, y, t)v(y, z, t)\phi(x)dx dy \\ = \int_{B^+} \int_{B^+} (A_y + tI)G_y(x, y, t)\phi(x)v(y, z, t)dx dy \\ \quad + \int_{B^+} \int_{B^+} (A - A_y)G_y(x, y, t)v(y, z, t)\phi(x)dx dy \\ = \int_{B^+} \phi(y)v(y, z, t)dy + \int_{B^+} \int_{B^+} (A - A_y)G_y(x, y, t)v(y, z, t)\phi(x)dy dx \end{aligned}$$

by Fubini's theorem.

(4) But v, h_j satisfy the equation:

$$0 = v(x, z, t) + (A - A_z)G_z(x, z, t) + \int_{B^+} (A - A_y)G_y(x, y, t)v(y, z)dy \\ + \sum_{k=1}^m \int_{\Gamma_0} (A - A_y)H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)d\hat{y}.$$

Hence: $(A + tI)G(x, z, t) = \delta_z$.

(5) We show that $B_j G(x, z, t) = 0$ if $x_1 = 0$; $j = 1, \dots, m$. We have with x, z in B^+ :

$$B_j G(x, z, t) = B_{jz}G_z(x, z, t) + (B_j - B_{jz})G_z(x, z, t) + \int_{B^+} B_j G_y(x, y, t)v(y, z)dy \\ + \sum_{k=1}^m \int_{\Gamma_0} B_j H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)d\hat{y}.$$

The differentiations under the integral sign are valid (Theorems 3.2, 4.1). The last integral may be written as:

$$\int_{\Gamma_0} (B_j - B_{jy})H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)d\hat{y} + \int_{\Gamma_0} B_{jy}H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)d\hat{y}.$$

Let $\phi(\hat{x}) \in C_c^\infty(\Gamma_0)$. Consider:

$$\int_{\Gamma_0} \int_{\Gamma_0} B_{jy}H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)\phi(\hat{x})d\hat{x}d\hat{y}.$$

By Fubini's theorem, we have:

$$\int_{\Gamma_0} \int_{\Gamma_0} B_{jy}H_{ky}(x, \hat{y}, t)h_k(\hat{y}, z, t)\phi(\hat{x})d\hat{x}d\hat{y}.$$

We know that $B_{jy}H_{ky}(\hat{x}, \hat{y}) = \delta_{jk}\delta_{\hat{y}}$ so as $x_1 \rightarrow 0$, we get:

$$\delta_{jk} \int_{\Gamma_0} h_k(\hat{y}, z, t)\phi(\hat{y})d\hat{y}.$$

(6) On the other hand, v and h_k satisfy the equation:

$$(B_j - B_{jz})G_z(\hat{x}, \hat{z}, t) + \int_{B^+} (B_j - B_{jy})G_y(\hat{x}, \hat{y}, t)v(\hat{y}, z, t)d\hat{y} \\ + \sum_{k=1}^m \int_{\Gamma_0} B_{jy}H_{ky}(\hat{x}, \hat{y}, t)h_k(\hat{y}, z, t)d\hat{y} + h_j(z, x, t) = 0; \quad j = 1, \dots, m.$$

So $B_j G(\hat{x}, z, t) = 0$, $j = 1, \dots, m$.

LEMMA 4.4. Let $\{A; B_j; j = 1, \dots, m\}$ be a uniformly regularly elliptic boundary value problem where A and B_j are defined on E_+^n, E^{n-1} and have infinitely differentiable coefficients. A and B_j are homogeneous differential operators. Let $G_z(x, z, t)$ be the Green's function corresponding to the constant coefficients problem $\{A_z + tI; B_{jz}; j = 1, \dots, m\}$. G_z is given by Theorem 3.2.

Let $H_{j\theta}(x, \hat{y})$ be the kernels given in Lemma 4.1. Set:

$$\begin{aligned} \alpha_{0j}(x, \hat{y}) &= (A - A_\theta)H_{j\theta}(x, \hat{y}), \\ (1) \quad \alpha_{rj}(\hat{x}, \hat{y}) &= (B_r - B_{r\theta})H_{j\theta}(\hat{x}, \hat{y}), \\ \alpha_j(x, \hat{y}) &= (\alpha_{0j}, \dots, \alpha_{mj}), \\ \beta_0(x, z) &= (A - A_z)G_z(x, z), \\ (2) \quad \beta_r(\hat{x}, z) &= (B_r - B_{rz})G_z(x, z), \\ \beta(x, z) &= (\beta_0, \dots, \beta_m), \\ (3) \quad w_j(x, \hat{y}) &= (v_j, h_j, \dots, h_{mj}). \end{aligned}$$

Define the transformations:

$$\begin{aligned} Tw_j(x, \hat{y}) &= \int_{B^+} \beta(x, z) v_j(z, \hat{y}) dz, \\ T_k w_j(x, \hat{y}) &= \int_{\Gamma_0} \alpha_k(x, z) h_{jk}(z, \hat{y}) dz, \\ Tw_j + \sum_{k=1}^m T_k w_j &= \mathcal{J} w_j(x, \hat{y}). \end{aligned}$$

Then the integral equation:

$$w_j(x, y) + \mathcal{J} w_j(x, y) + \alpha_j(x, y) = 0, \quad j = 1, \dots, m,$$

may be solved by the Neumann series for large t and for x, y in $B^+ \cup \Gamma = \{x: |x| < 1; x_1 \geq 0\}$.

Moreover:

$$\begin{aligned} v_j(x, \hat{y}) &= O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon} (1 + |(x - \hat{y})|^N)^{-1}, \\ h_{jk}(\hat{x}, \hat{y}) &= O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon} (1 + |(\hat{x} - \hat{y})|^N)^{-1}, \end{aligned}$$

$0 < \varepsilon < 1$; N is a positive integer.

Proof. From Lemma 4.1, we have:

$$\begin{aligned} \alpha_{0j}(x, \hat{y}) &= (A - A_\theta)H_{j\theta}(x, \hat{y}) = O(1)t^{-\varepsilon/2m} |x - \hat{y}|^{-n+2-\varepsilon}, \\ \alpha_{rj}(\hat{x}, \hat{y}) &= (B_r - B_{r\theta})H_{j\theta}(\hat{x}, \hat{y}) = O(1)t^{-\varepsilon/2m} |\hat{x} - \hat{y}|^{-n+2-\varepsilon}. \end{aligned}$$

From Theorem 3.2, we get:

$$\beta_0(x, z) = (A - A_z)G_z(x, z) = O(1)t^{-\varepsilon/2m}|x-z|^{-n+1-\varepsilon},$$

$$\beta_r(\hat{x}, z) = (B_r - B_{rz})G_z(\hat{x}, z) = O(1)t^{-\varepsilon/2m}|\hat{x}-z|^{-n+2-\varepsilon}.$$

Consider the series:

$$\alpha_j(x, \hat{y}) + \mathcal{J}\alpha_j(x, \hat{y}) + \mathcal{J}\alpha_j + \dots.$$

It may be written as:

$$\alpha_{0j}(x, \hat{y}) + \int_{B^+} \beta_0(x, z, t) \alpha_{0j}(z, \hat{y}) dz + \sum_{k=1}^m \int_{\Gamma_0} \alpha_{0k}(x, \hat{z}) \alpha_{jk}(\hat{z}, \hat{y}) d\hat{z} + \dots$$

and:

$$\alpha_{rj}(\hat{x}, \hat{y}) + \int_{B^+} \beta_r(\hat{x}, z, t) \alpha_{0j}(z, \hat{y}) dz + \sum_{k=1}^m \int_{\Gamma_0} \alpha_{rk}(\hat{x}, \hat{z}) \alpha_{jk}(\hat{z}, \hat{y}) d\hat{z} + \dots.$$

They are majorized by the series:

$$O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-n+2-\varepsilon} + O(1)t^{-\varepsilon/m} \int_{B^+} |\hat{x}-\hat{z}|^{-n+2-\varepsilon} |\hat{y}-\hat{z}|^{-n+2-\varepsilon} dz \\ + O(1)t^{-\varepsilon/m} \int_{\Gamma_0} |\hat{x}-\hat{z}|^{-n+2-\varepsilon} |\hat{z}-\hat{y}|^{-n+2-\varepsilon} d\hat{z} + \dots$$

which is uniformly convergent for large t .

The lemma is proved.

LEMMA 4.5. Let $H_{j\hat{z}}(x, \hat{z})$ be the kernels constructed in Lemma 4.1 for the constant coefficients problem $\{A_{j\hat{z}} + tI; B_{j\hat{z}}; j = 1, \dots, m\}$. Let $G_z(x, z, t)$ be the Green's function associated with the elliptic boundary value problem $\{A_z + tI; B_{jz}; j = 1, \dots, m\}$. The differential operators A, B_j are homogeneous and have infinitely differentiable coefficients.

Let:

$$H_j(x, \hat{y}) = H_{j\hat{y}}(x, \hat{y}) + \int_{B^+} G_z(x, z, t) v_j(z, \hat{y}) dz + \sum_{k=1}^m \int_{\Gamma_0} H_{k\hat{z}}(x, \hat{z}) h_{kj}(\hat{z}, \hat{y}) d\hat{z}$$

where v_j and h_{kj} satisfy the system of integral equations of Lemma 4.4. Then:

$$(A + tI)H_j(x, \hat{y}) = 0, \quad x \text{ in } B^+ = \{x: x_1 > 0, |x| < 1\},$$

$$B_r H_r(0, \hat{x}, \hat{y}) = \delta_{\hat{y}},$$

$$B_r H_j(0, \hat{x}, \hat{y}) = 0 \quad \text{if } r \neq j.$$

Proof. The proof is long but easy and is similar to that of Theorem 4.2.

THEOREM 4.2. Let $u(x)$ be an infinitely differentiable function with compact support in $E_+^n \cup E^{n-1}$. Then u has the following integral representation:

$$u(x, t) = \int_{E_+^n} G(x, y, t)(A + tI)u(y)dy + \sum_{j=1}^m \int_{E^{n-1}} H_j(x - \hat{y}, t)B_j u(0, \hat{y})d\hat{y}.$$

$\{A, B_j; j = 1, \dots, m\}$ is a uniformly regularly elliptic boundary value problem; A and B_j are defined on E_+^n, E^{n-1} with infinitely differentiable coefficients. $G(x, y, t)$ is the Green's function associated with $\{A + tI; B_j; j = 1, \dots, m\}$ and is given by Theorem 4.1. The kernels H_j are given by Lemma 4.5.

Proof. We consider the boundary value problem:

$$(A + tI)u(x) = f(x) \quad \text{on } E_+^n, \\ B_j u(\hat{x}) = g_j(\hat{x}) \quad \text{on } E^{n-1}, \quad j = 1, \dots, m.$$

Since u is infinitely differentiable and has compact support in $E_+^n \cup E^{n-1}$; f and g_i also have compact supports.

We may write $u(x)$ as $u(x) = v(x) + w(x)$ where $v(x)$ is the solution of $(A + tI)v(x) = f(x)$ on E_+^n , $B_j v(\hat{x}) = 0$ on E^{n-1} , $j = 1, \dots, m$, and $w(x)$ is the solution of the boundary value problem: $(A + tI)w(x) = 0$ on E_+^n , $B_j w(x) = g_j$ on E^{n-1} , $j = 1, \dots, m$. Let $G(x, y, t)$ be the Green's function associated with the elliptic boundary value problem $\{A + tI; B_j; j = 1, \dots, m\}$ given by Theorem 4.2. We get:

$$v(x) = \int_{E_+^n} G(x, y, t)f(y)dy.$$

Now we construct w . Let $H_j(x, y)$ be the kernels given by Lemma 4.5; then w is given by the expression:

$$w(x) = \sum_{j=1}^m \int_{E^{n-1}} h_j(\hat{y})H_j(x - \hat{y}, t)d\hat{y}.$$

The conclusion of the theorem follows immediately.

THEOREM 4.3. Let $\mathcal{G}(x, y, t)$ be the Green's function associated with the uniformly regularly elliptic boundary value problem (A, γ) where A is defined on a bounded open subset S of E^n with infinitely differentiable coefficients; $\gamma = (B_1, \dots, B_m)$ is a family of differential operators defined on the boundary Γ of S with infinitely differentiable coefficients. A and B_j are homogeneous differential operators.

(A, γ) is assumed to be formally positive in the sense of Definition 2.1. Let $G(x, y, t)$ be the Green's function of Theorem 4.1 (i.e. corresponding to the case of a half space). Then:

$$\mathcal{G}(x, y, t) = G(x, y, t) - u(x, y, t), \quad y \text{ in } S, \\ u(x, y, t) = \sum_k \tilde{u}(\phi_k^{-1}(x), \phi_k^{-1}(y), t),$$

ϕ_k are the diffeomorphisms corresponding to the uniform regularity of S and:

$$\tilde{u}(x, y, t) = \sum_{j=1}^m \int_{\Gamma_0} H_j(x, \hat{z}, t) B_j G(y, \hat{z}, t) d\hat{z}$$

$H_j(x, \hat{z}, t)$ is given by Lemma 4.5

$$\Gamma_0 = \{z: z_1 = 0; |z| < 1\}.$$

Proof. There is no loss of generality in assuming that $y = 0$ is in S . Let $G(x, t)$ be as in Theorem 4.1. Then $G(x, t)$ is a fundamental solution of the elliptic operator $A + tI$.

If u is the solution of the boundary value problem:

$$(A + tI)u(x) = 0 \quad \text{on } S,$$

$$B_j(x)u = B_j G(x, t) \quad \text{on } \Gamma; j = 1, \dots, m,$$

then $\mathcal{G}(x, t) = G(x, t) - u(x, t)$.

S is a bounded domain which is uniformly regular. It may be covered by a finite number of open sets N_k and there exists a family of infinitely differentiable functions η_k with compact supports in N_k , and such that:

$$\sum_k \eta_k^2(x) = 1, \quad x \text{ in } S.$$

We have:

$$(A + tI)(u(x)\eta_k^2(x)) = \eta_k^2(x)(A + tI)u(x) + \sum_{|\alpha|+|\beta|=2m; |\alpha|<2m} a_{\alpha\beta}(x) D^\alpha u D^\beta \eta_k^2(x).$$

Similarly for $B_j(u\eta_k^2)$.

We consider the boundary value problem:

$$(A + tI)u\eta_k^2 = f_k \quad \text{on } N_k \cap S,$$

$$B_j(u\eta_k^2) = g_{jk} + h_{jk} \quad \text{on } N_k \cap \Gamma; j = 1, \dots, m,$$

where:

$$f_k(x) = \sum_{|\alpha|<2m; |\alpha|+|\beta|=2m} a_{\alpha\beta}(x) D^\alpha u D^\beta \eta_k^2(x),$$

$$h_{jk}(x) = \sum_{|\alpha|<r_j; |\alpha|+|\beta|=r_j} b_{j\beta k}(x) D^\alpha u D^\beta \eta_k^2(x).$$

Using the diffeomorphisms $\phi_k(x)$ we map N_k into the positive half ball. Set: $\tilde{u}_k(x) = (\eta_k^2(u))(\phi_k(x)); \tilde{f}_k(x); \tilde{g}_{jk}(x); \tilde{h}_{jk}$ are similarly defined.

Using the same notations for the transplanted operators, we get:

$$(A + tI)\tilde{u}_k(x) = \tilde{f}_k(x) \quad \text{on } B^+ = \{x: x_1 > 0; |x| < 1\},$$

$$B_j \tilde{u}(0, \hat{x}) = \tilde{g}_{jk}(0, \hat{x}) + \tilde{h}_{jk}(0, \hat{x}), \quad \text{on } \Gamma; j = 1, \dots, m.$$

$\tilde{f}_k$ is an infinitely differentiable function with compact support in $B^+ \cup \Gamma_0$. Applying Theorem 4.2; we obtain:

$$\tilde{u}_k(x) = \int_{B^+} \tilde{f}_k(y) G(x, y, t) dy + \sum_{j=1}^m \int_{\Gamma_0} H_j(x - \hat{y}, t) \{ \tilde{g}_{jk}(\hat{y}) + \tilde{h}_{jk}(\hat{y}) \} d\hat{y}.$$

Since $\sum_k \eta_k^2(x) = 1$, we have:

$$\tilde{u}(x) = \sum_k \tilde{u}_k(x) = \sum_{j=1}^m \int_{\Gamma_0} H_j(x - \hat{y}, t) B_j G(0, \hat{y}, t) d\hat{y}.$$

The theorem is proved.

5. THEOREM 5.1. Let $\mathcal{G}_{2p}$ be the $2p$ th iterate of the Green's function $\mathcal{G}$ defined in Theorem 4.3. Let A_γ be the realization of A under null boundary conditions γ as an operator on $L^2(S)$. If (A, γ) is formally self-adjoint and λ_j, ϕ_j are respectively the eigenvalues and eigenfunctions of A_γ ; then: for $2mp > n$:

$$D_x^\alpha D_y^\beta \mathcal{G}_{2p}(x, y, t) = \sum_j D^\alpha \phi_j(x) D^\beta \phi_j(y) (\lambda_j + t)^{-2p}; \quad |\alpha|, |\beta| \leq 2m.$$

Proof. Let $2mp > n$, then $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type. Since (A, γ) is formally self-adjoint, it follows from Theorem 2.1 that A_γ is self-adjoint; $\lambda_j + t > 0$, we have a complete orthonormal system of eigenfunctions ϕ_j .

Consider:

$$((A_\gamma + tI)^{-p} f, \phi_j) = \int_S \int_S \mathcal{G}_{(p)}(x, y, t) f(y) \phi_j(z) dy dz, \quad f \text{ in } L^2(S).$$

We get:

$$(\lambda_j + t)^{-p} \phi_j(y) = \int_S \mathcal{G}_{(p)}(z, y, t) \phi_j(z) dz.$$

Using Parseval's formula, we obtain:

$$\begin{aligned} \sum_j (\lambda_j + t)^{-2p} \phi_j(x) \phi_j(y) &= \int_S \mathcal{G}_{(p)}(z, x, t) \mathcal{G}_{(p)}(z, y, t) dz \\ &= \mathcal{G}_{(2p)}(x, y, t). \end{aligned}$$

Let $\mathcal{G}_{2p,k}(x, y, t) = \sum_{j=1}^k (\lambda_j + t)^{-2p} \phi_j(x) \phi_j(y)$ then:

$$\begin{aligned} \|\mathcal{G}_{2p,k} - \mathcal{G}_{2p}\|_{W^{2m,2} \times W^{2m,2}} &\leq \sum_k^l (\lambda_j + t)^{-2p} \|\phi_j\|_{W^{2m,2}}, \\ &\leq \sum_k^l (\lambda_j + t)^{-2p+2} \rightarrow 0 \quad \text{as } l, k \rightarrow \infty, \end{aligned}$$

$$\mathcal{G}_{2p,k}(x, y, t) \rightarrow \mathcal{G}_{(2p)}(x, y, t) \quad \text{in } W^{2m,2} \times W^{2m,2}.$$

In particular; $D_x^\alpha D_y^\beta \mathcal{G}_{2p,k}(x, y, t) \rightarrow D_x^\alpha D_y^\beta \mathcal{G}_{2p}(x, y, t)$ in $L^2 \times L^2$ and we get:

$$D_x^\alpha D_y^\beta \mathcal{G}_{2p}(x, y, t) = \sum_j (\lambda_j + t)^{-2p} D^\alpha \phi_j(x) D^\beta \phi_j(y).$$

LEMMA 5.1. Let $\mathcal{G}_p(x, y, t)$ be the p th iterate of the Green's function defined in Theorem 4.3. Then if $2mp > n + |\alpha| + |\beta|$,

$$x \neq y, \quad t^{p-(n+|\alpha|+|\beta|)/2m} D_x^\alpha D_y^\beta \mathcal{G}_p(x, y, t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

$$t^{p-(n+|\alpha|+|\beta|)/2m} D^{\alpha+\beta} \mathcal{G}_p(x, x, t) = (2\pi)^{-n} \int_{E^n} \xi^{\alpha+\beta} [a(x, \xi) + 1]^{-p} d\xi$$

as $t \rightarrow +\infty$; for x, y in S .

Proof. We prove the lemma for $|\alpha| = |\beta| = 0$; the general case may be treated in the same fashion. Let G be the Green's function associated with the elliptic boundary value problem $\{A + tI; \gamma\}$ on a half space with infinitely differentiable coefficients. From Theorem 4.2, we have

$$G(x, y, t) = G_y(x, y, t) + \int_{B^+} G_z(x, z, t) v(z, y, t) dz + \sum_{j=1}^m \int_{\Gamma_0} H_{jz}(x - \hat{z}, t) h_j(\hat{z}, \hat{y}) d\hat{z}$$

x, y in B^+ .

G_y is the Green's function associated with the constant coefficients problem $\{A_y + tI; B_{jy}; j = 1, \dots, m\}$ on a half space.

We show that:

$$\lim_{t \rightarrow +\infty} t^{p-n/2m} G_p(x, y, t) = \lim_{t \rightarrow +\infty} t^{p-n/2m} G_{p,y}(x, y, t).$$

First consider the case $2m > n$. With $t = \tau^{2m}$, we have:

$$\tau^{2m-n} \int_{B^+} G_z(x, z, t) v(y, z, t) dz = O(1) \tau^{-\varepsilon} \int_{B^+} |y - z|^{1-n-\varepsilon} (1 + |\tau(x - y)|^N)^{-1} dz,$$

$$\tau^{2m-n} \int_{\Gamma_0} H_{jz}(x - \hat{z}, t) h_j(\hat{z}, y, t) d\hat{z} = O(1) \tau^{-\varepsilon} |x_1|^{-n} \int_{\Gamma_0} |\hat{z} - \hat{y}|^{-n+2-\varepsilon} (1 + |\tau(\hat{z} - \hat{y})|^N)^{-1} d\hat{z}.$$

So:

$$\lim_{t \rightarrow +\infty} t^{1-n/2m} G(x, y, t) = \lim_{t \rightarrow +\infty} t^{1-n/2m} G_{(y)}(x, y, t).$$

Now if $2m < n$, from Theorem 4.1; we have:

$$G(x, y, t) = G_{(y)}(x, y, t) (1 + O(1) \tau^{-\varepsilon})$$

so

$$\lim_{t \rightarrow +\infty} t^{p-n/2m} G_{(p)}(x, y, t) = \lim_{t \rightarrow +\infty} t^{p-n/2m} G_{(p),y}(x, y, t).$$

On the other hand:

$$\mathcal{G}(x, y, t) = G(x, y, t) + \sum_k \tilde{v}(\phi_k^{-1}(x), \phi_k^{-1}(y), t)$$

with

$$\tilde{v}(x, y, t) = \sum_{j=1}^m \int_{\Gamma_0} H_j(x - \hat{z}, t) B_j G(\hat{z}, y, t) d\hat{z}.$$

An argument as above gives:

$$\lim_{t \rightarrow +\infty} t^{p-n/2m} \mathcal{G}_{(p)}(x, y, t) = \lim_{t \rightarrow +\infty} t^{p-n/2m} G(x, y, t).$$

The conclusion of the lemma follows from Theorem 3.4.

THEOREM 5.2. *Let A_γ be the realization of the positively strongly elliptic operator A under null boundary conditions $\gamma = (B_1, \dots, B_m)$ as an operator on $L^2(S)$. The operators A and B_j are defined on a bounded open set S and on the boundary Γ respectively and have infinitely differentiable coefficients.*

(A, γ) is assumed to be uniformly regularly elliptic, formally self-adjoint and formally positive in the sense of Definition 2.1. Let λ_j, ϕ_j be the eigenvalues and eigenfunctions of A_γ . Then:

$$(1) \quad N(t) = \sum_{\lambda_j \leq t} 1 \sim (2\pi)^{-n} t^{n/2m} \int_S \int_{a(x, \xi) < 1} d\xi dx \quad \text{as } t \rightarrow +\infty.$$

$$(2) \quad t^{-(n+|\alpha|+|\beta|)/2m} D_x^\alpha D_y^\beta e(x, y, t) = t^{-(n+|\alpha|+|\beta|)} \sum_{\lambda_j \leq t} D^\alpha \phi_j(x) D^\beta \phi_j(y) \rightarrow 0$$

as $t \rightarrow \infty$ for x, y in S and $x \neq y$.

$$(3) \quad D^{\alpha+\beta} e(x, x, t) \sim (2\pi)^{-n} t^{(n+|\alpha|+|\beta|)/2m} K(n, m, p, \alpha, \beta) \int_{E^n} \xi^{\alpha+\beta} [a(x, \xi) + 1]^{-2p} d\xi$$

as $t \rightarrow \infty$ for x in S and $4mp > n + |\alpha| + |\beta|$.

$$K(n, m, p, \alpha, \beta) = \frac{\Gamma(2p)}{\Gamma\left(1 + \frac{n}{2m}\right) \Gamma\left(2p - \frac{n + |\alpha| + |\beta|}{2m}\right)}.$$

Proof. One can show easily that:

$$\sum_j (\lambda_j + t)^{-2p} = \int_S \mathcal{G}_{(2p)}(x, x, t) dx.$$

Consider the sequence of integrable functions $t^{2p-n/2m} \mathcal{G}_{(2p)}(x, x, t)$. For large t , we have from the previous lemmas: $t^{2p-n/2m} |\mathcal{G}_{(2p)}(x, x, t)| \leq M$ for all x in S and M is a constant independent of x and t . We apply the Lebesgue dominated convergence theorem and we get:

$$t^{2p-n/2m} \int_S \mathcal{G}_{(2p)}(x, x, t) dx \sim (2\pi)^{-n} \int_S \int_{E^n} [a(x, \xi) + 1]^{-2p} d\xi dx$$

as $t \rightarrow +\infty$.

Applying the Tauberian theorem of Hardy-Littlewood [10], we get the results for $N(t)$.

We may write:

$$\mathcal{G}_{(2p)}(x, y, t) = \sum_j (\lambda_j + t)^{-2p} \phi_j(x) \phi_j(y) = \int_0^\infty (\lambda + t)^{-2p} de(x, y, \lambda)$$

where $e(x, y, t)$ is the spectral function. Taking into account the results of Lemma 5.1 and applying the Tauberian theorem of Hardy-Littlewood again, we get the results stated in the theorem.

6. The case of a nonself-adjoint regular elliptic boundary value problem is considered. The study of the asymptotic distribution of eigenvalues for the nonself-adjoint case has been carried out by Carleman [8] and Keldych [11] for second order elliptic equations.

THEOREM 6.1. *Let $\{A; B_j; j = 1, \dots, m\}$ be a uniformly regularly elliptic boundary value problem where A and B_j are defined on a bounded open subset S of E^n and on the boundary Γ with infinitely differentiable coefficients. (A, γ) with $\gamma = (B_1, \dots, B_m)$ is assumed to be formally positive in the sense of Definition 2.1. Let A_γ be the realization of A under null boundary conditions γ as an operator $L^2(S)$. If $2mp > n$ where $2m$ is the order of A , the operator $(A_\gamma + tI)^{-2p}$ is of trace class. Let λ_j be the eigenvalues of A , then:*

$$\text{tr}(A + tI)^{-2p} = \sum_j (\lambda_j + t)^{-2p} = \int_S \mathcal{G}_{(2p)}(x, x, t) dx.$$

$\mathcal{G}_{(2p)}(x, y, t)$ is the $2p$ th iterate of the Green's function associated with $A_\gamma + tI$ on S .

Proof. With the above hypothesis, it has been proved in §2 that $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type, so $(A_\gamma + tI)^{-2p}$ is of trace class. Let ϕ_j be the generalized eigenfunctions of A_γ . They form an orthonormal basis in $L^2(S)$. Denote by P_j the orthogonal projection of $L^2(S)$ onto the subspace of $L^2(S)$ spanned by $\{\phi_1, \dots, \phi_j\}$; consider the operator: $T_j = P_j(A + tI)^{-2p}P_j$.

It takes the subspace spanned by $\{\phi_1, \dots, \phi_j\}$ into itself. The subspace is of finite dimension and we have:

$$\text{tr}(T_j) = \sum_{k=1}^j (\lambda_k + t)^{-2p},$$

$$\text{tr}(T_n - T_m) = \text{tr}(T_n) - \text{tr}(T_m) = \sum_{j=m}^n (\lambda_j + t)^{-2p}.$$

Denote by $|||T|||$ the trace norm of an operator of finite rank. (Ruston [13].) Then:

$$|||T_n - T_m||| \leq \sum_m^n |\lambda_j + t|^{-2p}.$$

Since $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type: $\sum_j |\lambda_j + t|^{-2p} < \infty$. It follows that $T \rightarrow \mathcal{C}$ in the trace norm topology and:

$$\text{tr}(\mathcal{C}) \sum_j (\lambda_j + t)^{-2p}.$$

We now show that $\mathcal{C} = (A_\gamma + tI)^{-2p}$. We know that $(A_\gamma + tI)^{-2p}$ is a compact operator and $T_j \rightarrow (A_\gamma + tI)^{-2p}$ in the operator norm topology. Since $T_j \rightarrow \mathcal{C}$ in the trace norm, it converges to τ in the operator norm; hence $\mathcal{C} = (A_\gamma + tI)^{-2p}$ and:

$$\text{tr}(A_\gamma + tI)^{-2p} = \sum_j (\lambda_j + t)^{-2p}.$$

We may write $\mathcal{C} = \tau_R + i\mathcal{C}_I$ where $\mathcal{C}_R = (\mathcal{C}^* - \mathcal{C})/2$, $\mathcal{C}_I = (\mathcal{C} + \mathcal{C}^*)/2i$. Since $\mathcal{C}_R, \mathcal{C}_I$ are self-adjoint, we may apply results of §5 to get

$$\text{tr}(A_\gamma + tI)^{-2p} = \sum_j (\lambda_j + t)^{-2p} = \int_S \mathcal{G}_{(2p)}(x, x, t) dx.$$

THEOREM 6.2. *Let (A, γ) be a uniformly regularly elliptic boundary value problem with infinitely differentiable coefficients and formally positive in the sense of Definition 2.1. If A_γ is the realization of the positively strongly elliptic operator A as an operator on $L^2(S)$ under null boundary conditions γ and λ_j are the eigenvalues of A_γ then:*

$$N(t) = \sum_{\text{Re } \lambda_j \leq t} 1 \sim (2\pi)^{-n} t^{n/2m} w_a(S) \quad \text{as } t \rightarrow +\infty$$

where $w_a(S) = \int_S w_a(x) dx$ and $w_a(x) = \int_{a(x, \xi) < 1} d\xi$.

Proof. Set $\lambda_j = \alpha_j + i\beta_j$; $f(t) = \sum_j (\alpha_j + t)^{-2p}$ and $g(t) = \sum_j (\lambda_j + t)^{-2p}$.

Let $h(t) = f(t) - g(t) = \sum_j \{(\alpha_j + t)^{-2p} - (\lambda_j + t)^{-2p}\}$.

It has been proved by Browder [4] that the spectrum of A_γ is contained inside an algebraic curve $|\text{Im } \zeta| \leq c(\text{Re } \zeta)^\mu$ with $\mu = (2m - 1)/2m$, we get:

$$|h(t)| \leq \sum_j (\alpha_j + t)^{-2p-1} |\alpha_j|^\mu.$$

The eigenvalues have an accumulation point at infinity, hence there exists a number N such that:

$$|\alpha_N| < t^\delta \leq \alpha_{N+1}, \quad 0 < \delta < 1.$$

We have:

$$\sum_{N+1} |\alpha_j|^\mu (\alpha_j + t)^{-2p-1} \leq \sum_{N+1} t^{(\mu-1)\delta} (\alpha_j + t)^{-2p}, \quad |h(t)| \leq ct^{(\mu-1)\delta} f(t).$$

It follows that: $\lim t^{2p-n/2m} \sum_j (\alpha_j + t)^{-2p} = \lim t^{2p-n/2m} \sum_j (\lambda_j + t)^{-2p}$.

By an argument as in Theorem 5.2 and applying the Tauberian theorem of Hardy-Littlewood we get:

$$N(t) = \sum_{\operatorname{Re} \lambda \leq t} 1 \sim t^{n/2m} w_a(S) \cdot (2\pi)^{-n} \quad \text{as } t \rightarrow +\infty.$$

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(2) Added in proof.