ON BADLY APPROXIMABLE NUMBERS AND CERTAIN GAMES

BY

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1. **Introduction.** A number α is called *badly approximable* if $|\alpha - p/q| > c/q^2$ for some $c > 0$ and all rationals p/q. It is known that an irrational number α is badly approximable if and only if the partial denominators in its continued fraction are bounded [4, Theorem 23]. In a recent paper [7] I proved results of the following type: *If $f_1, f_2, \cdots$ are differentiable functions whose derivatives are continuous and vanish nowhere, then there are continuum-many numbers α such that all the numbers $f_1(\alpha), f_2(\alpha), \cdots$ are badly approximable.*

Let $0 < \alpha < 1/2$, $0 < \beta < 1$, and consider the following game of two players black and white. First black picks a closed interval B_1. Then white picks a closed interval $W_1 \subset B_1$ whose length is α times the length of B_1. Then black chooses an interval $B_2 \subset W_1$ which is closed and has length β times the length of W_1. Then again white picks a closed interval $W_2 \subset B_2$ of length α times the length of B_2, and so on. Call white the *winner* of a play if the intersection of the intervals W_j is badly approximable; otherwise black is called the winner.

Who will win? Since the badly approximable numbers have Lebesgue measure zero, [4, Theorem 29], one might think that black can always win. It turns out, however, that white can always win (Theorem 3).

We shall show that sets S with this property (namely that white can always play such that the intersection $\bigcap W_j$ is in S) necessarily contain continuum-many elements (Lemma 23), that countable intersections of sets with this property again have this property (Theorem 2), and that if S has this property, and $f(x)$ has a continuous derivative with $f'(x) \neq 0$ everywhere, then the set of α with $f(\alpha) \in S$ again has this property (Theorem 1). These facts imply the result stated at the beginning.

We shall discuss games of much greater generality than those mentioned in this introduction.

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2. **$(\mathfrak{F}, \mathfrak{G})$-games.** Let M, Ω be sets, Ω' a subset of Ω, and α a mapping from Ω into subsets of M. Call *$\mathfrak{H}$-function* any function $\mathfrak{H}$ which assigns to every $B \in \Omega$ a *nonempty* set $\mathfrak{H}(B) \subset \Omega$ such that for $C \in \mathfrak{H}(B)$,

$$\alpha(C) \subset \alpha(B).$$

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Now let $\mathfrak{F}, \mathfrak{G}$ be $\mathfrak{H}$-functions, and let S be an arbitrary subset of M. Consider the following game of two players black and white. First black picks an element $B_1 \in \Omega'$. Then white picks $W_1 \in \mathfrak{F}(B_1)$. Then black picks $B_2 \in \mathfrak{G}(W_1)$. Then again white chooses an element $W_2 \in \mathfrak{F}(B_2)$, and so forth. Put $B_i^* = \alpha(B_i)$, $W_i^* = \alpha(W_i)$ $(i = 1, 2, \cdots)$. One has $B_1^* \supset W_1^* \supset B_2^* \supset W_2^* \supset \cdots$. We call white the *winner* of a play if $\bigcap_{i=1}^{\infty} W_i^* = \bigcap_{i=1}^{\infty} B_i^*$ is contained in S; otherwise black is the winner. This game we call $(\mathfrak{F}, \mathfrak{G}; S)$-game. We call S an $(\mathfrak{F}, \mathfrak{G})$-*winning set* if white can always win the $(\mathfrak{F}, \mathfrak{G}; S)$-game.

Another way of defining an $(\mathfrak{F}, \mathfrak{G})$-winning set is the following. Let F_n $(n = 1, 2, \cdots)$ be the set of functions $f(B_1, B_2, \cdots, B_n)$ defined for elements $B_i \in \Omega$ such that $f(B_1, \cdots, B_n) \in \mathfrak{F}(B_n)$. A sequence $f_1, f_2, \cdots$ where $f_n \in F_n (n = 1, 2, \cdots)$ will be called a *strategy*. A strategy is called $(\mathfrak{F}, \mathfrak{G}; S)$-*winning strategy* if the following holds: *Let $B_1, B_2, \cdots, W_1, W_2, \cdots$ be sets such that $B_1 \in \Omega'$ and*

$$B_n \in \mathfrak{G}(W_{n-1}) \qquad (n = 2, 3, \cdots), \tag{1}$$

$$W_n = f_n(B_1, \cdots, B_n) \qquad (n = 1, 2, \cdots). \tag{2}$$

Then $\bigcap_{n=1}^{\infty} B_n^*$ is necessarily contained in S. Now S is an $(\mathfrak{F}, \mathfrak{G})$-winning set if and only if there is an $(\mathfrak{F}, \mathfrak{G}; S)$-winning strategy. White will win by choosing $W_n = f_n(B_1, \cdots, B_n)$ when $B_1, \cdots, B_n$ are given $(n = 1, 2, \cdots)$.

This means that white bases his decision on how to pick W_n not only on B_n, but also on the previous elements $B_1, \cdots, B_{n-1}$. Is this necessary? In other words, if S is a winning set, does there exist a winning strategy of the type $f_n(B_1, \cdots, B_n) = f(B_n)$ $(n = 1, 2, \cdots)$, where $f \in F_1$? Such a strategy we shall call *positional strategy*. Using the well-ordering principle we shall show in the last section that a winning set does indeed have a positional winning strategy. This result has the following interpretation. In our game both players know the outcomes of all the previous moves. One obtains a different game if one specifies that white at the nth move shall know B_n but shall not remember the number n or the previous elements $B_1, \cdots, B_{n-1}$. Now the result on positional strategies means that if S is a winning set of the original game, it is also a winning set of the new game. A special case of this was proved in [2, §4].

Let $f_1, f_2, \cdots$ be a winning strategy. We call $B_1, B_2, \cdots$ with $B_1 \in \Omega'$, $B_i \in \Omega$ an $f_1, f_2, \cdots$-*chain* if there are elements $W_1, W_2, \cdots$ such that (1) and (2) hold. The intersection of the sets $B_n^* = \alpha(B_n)$ of a chain is in S. We call $B_1, \cdots, B_k$ a *finite* $f_1, f_2, \cdots$-*chain* if there are $B_{k+1}, B_{k+2}, \cdots$ such that $B_1, B_2, \cdots, B_k, B_{k+1}, \cdots$ is an $f_1, f_2, \cdots$-chain.

LEMMA 1. *Let $B_1, B_2, \cdots$ be elements of Ω such that $B_1, \cdots, B_k$ is a finite f_1, $f_2, \cdots$-chain for every k. Then $B_1, B_2, \cdots$ is an $f_1, f_2, \cdots$-chain.*

Proof. *Put* $W_n = f_n(B_1, \cdots, B_n)$ $(n = 1, 2, \cdots)$. Since $B_1, \cdots, B_k$ is a chain,

$$B_j \in \mathfrak{G}(W_{j-1}) \qquad (1 \leqq j \leqq k). \tag{3}$$

Since k is arbitrary, (3) holds in fact for every j. Hence (1) and (2) hold, and $B_1, B_2, \cdots$ is an $f_1, f_2, \cdots$-chain.

We shall repeatedly use this lemma without mention.

It is reasonable to call S a *losing set* if black can always win. It was shown by Gale and Stewart [3] that an $(\mathfrak{F}, \mathfrak{G}; S)$-game need not be *determinate*, and hence in general there are sets which are neither $(\mathfrak{F}, \mathfrak{G})$-winning nor $(\mathfrak{F}, \mathfrak{G})$-losing.

3. (α, β)- **and** (a^*b)**-games.** Already we have to specialize.

For completeness we mention Oxtoby's game [6]. Here M is a topological space, and $\Omega = \Omega'$ consists of a class of subsets with nonempty interiors such that given a nonempty open set O there is a set $C \subset O$, $C \in \Omega$. α is the identity map. $\mathfrak{F}(B) = \mathfrak{G}(B)$ consists of all subsets of B which are in Ω. A special case is the Banach-Mazur-game [5], [8]. Here M is the real line and Ω consists of M and all closed intervals.

Let M be a complete metrical space with distance function $d(x, y)$. Let $\Omega = \Omega'$ consist of all pairs (ρ, c), where ρ is a positive real number and $c \in M$. The elements $B = (\rho, c)$ of Ω will be called *balls* or more precisely balls of M, and we call $\rho = \rho(B)$ the *radius*, $c = c(B)$ the *center* of B. With $B \in \Omega$ we associate the set $B^* = \alpha(B)$ consisting of all points $x \in M$ satisfying $d(x, c) \leqq \rho$. In general, $\alpha(B)$ does not determine B, but it does if M is a Banach space of positive dimension.

We say a ball $B_1 = (\rho_1, c_1)$ *is contained* in a ball $B_2 = (\rho_2, c_2)$ and write $B_1 \subset B_2$ if

$$\rho_1 + d(c_1, c_2) \leqq \rho_2.$$

This implies (but is not implied by) $\alpha(B_1) \subset \alpha(B_2)$. One easily checks that $B_1 \subset B_2$ and $B_2 \subset B_3$ implies $B_1 \subset B_3$.

Let $0 < \gamma < 1$. Given $B \in \Omega$, let B^γ be the set of all balls $B' \subset B$ having $\rho(B') = \gamma\rho(B)$. Let $\mathfrak{F}^\gamma$ be the $\mathfrak{H}$-function defined by $\mathfrak{F}^\gamma(B) = B^\gamma$.

Now let $0 < \alpha < 1$, $0 < \beta < 1$, $S \subset M$. The $(\mathfrak{F}^\alpha, \mathfrak{F}^\beta; S)$-game is well defined. For brevity we will call this the $(\alpha, \beta; S)$-game. Thus in an (α, β)-game, black first picks a ball B_1, then white picks a ball $W_1 \in B_1^\alpha$, then black a ball $B_2 \in W_1^\beta$, and so forth.

Next let M be the real line and $\Omega = \Omega'$ the set of all closed intervals of positive length. For α take the identity mapping. If I is a closed interval and $c > 1$ an integer, write ${}^cI$ for the unique set of c closed intervals whose lengths are c^{-1} times that of I, and which cover I. Let ${}^c\mathfrak{F}$ be the $\mathfrak{H}$-function with ${}^c\mathfrak{F}(I) = {}^cI$.

Now let $a > 1$, $b > 1$ be integers, $S \subset M$. The $({}^a\mathfrak{F}, {}^b\mathfrak{F}; S)$-game is well defined. For convenience we call it $(a^*b; S)$-game.

A variant of the (a^*a)-game is the *a-digit-game*. Here Ω' consists of a single element only, namely the unit-interval $0 \leqq x \leqq 1$. This game amounts to the following. First white chooses a digit c_1 to base a, namely $c_1 = 0, 1, \cdots, a-2$ or $a-1$. Then black chooses a digit c_2, and so on. White is the winner if $x = 0, c_1 c_2 \cdots$ (written in the scale of a) is in S. Every (a^*a)-winning set is an a-digit-winning set.

LEMMA 2. *Let* $0<\alpha<1$, $0<\beta<1$, $a>1$, $b>1$, *where* a,b *are integers and where*

$$ab\alpha\beta = 1, \qquad a\alpha \geqq 2.$$

Then every (a,β)*-winning set on the real line is* $(a{*}b)$*-winning.*

Proof. Here $\Omega = \Omega'$ consists of closed intervals of positive length, and α is the identity map. Thus we need not distinguish between $B\in\Omega$ and the set $B^* = \alpha(B) \subset M$. Let $h_1, h_2, \cdots$ be an $(\alpha, \beta; S)$-winning strategy. Given $B_1, \cdots, B_n$, put $\tilde{W}_n = h_n(B_1,\cdots,B_n)$. The length $l(\tilde{W}_n)$ of $\tilde{W}_n$ satisfies $l(\tilde{W}_n) = \alpha l(B_n) \geqq 2a^{-1}\, l(B_n)$. Hence there is a $W_n \in {}^a B_n$, $W_n \subset \tilde{W}_n$. Put $f_n(B_1,\cdots,B_n) = W_n$. We claim $f_1, f_2, \cdots$ to be an $(a{*}b; S)$-winning strategy. Suppose (1), (2) hold with $\mathfrak{G} = {}^b\mathfrak{F}$. Then

$$B_n \in \tilde{W}^{\beta}_{n-1} \qquad (n = 2,3,\cdots), \tag{4}$$

$$\tilde{W}_n = h_n(B_1,\cdots,B_n) \qquad (n = 1,2,\cdots) \tag{5}$$

hold. (4) is true because $B_n \subset W_{n-1} \subset \tilde{W}_{n-1}$, $B_n \in {}^b W_{n-1}$, $W_{n-1} \in \tilde{W}_{n-1}^{(\alpha a)^{-1}}$, and $(\alpha a b)^{-1} = \beta$. By (4), (5), $B_1, B_2, \cdots$ is a $h_1, h_2, \cdots$-chain of the $(\alpha,\beta;S)$-game, and $\bigcap B_n$ is in S.

LEMMA 3. *Let* $0<\alpha<1$, $0<\beta<1$, $a>1$, $b>1$,

$$ab\alpha\beta = 1, \qquad b\beta \geqq 2$$

where a, b *are integers. Then every* $(a{*}b)$*-winning set is also* (α,β)*-winning.*

Proof. Let $h_1, h_2, \cdots$ be an $(a{*}b; S)$-winning strategy. Define f_n by induction on n as follows. Given a closed interval B_1 pick some $\tilde{B}_1 \in B_1^{(b\beta)^{-1}}$, then $W_1 = h_1(\tilde{B}_1)$. Define $f_1(B_1) = W_1$. Given $B_1,\cdots,B_n$, $n>1$, put $W_{n-1} = f_{n-1}(B_1,\cdots,B_{n-1})$, and pick $\tilde{B}_n$ such that $\tilde{B}_n \subset B_n$, $\tilde{B}_n \in {}^b W_{n-1}$. This is possible since $l(B_n) = \beta l(W_{n-1}) \geqq 2b^{-1} l(W_{n-1})$. (Here we used $B_n \subset W^{\beta}_{n-1}$. If this is not the case, the sequence $B_1, B_2, \cdots, B_n$ will not occur in a play, and $f_n(B_1,\cdots,B_n)$ can be defined arbitrarily.) Now put $f_n(B_1,\cdots,B_n) = h_n(\tilde{B}_1,\cdots,\tilde{B}_n)$. We claim $f_1, f_2, \cdots$ is an $(\alpha,\beta;S)$-winning strategy. Suppose (1), (2) hold with $\mathfrak{G} = \mathfrak{F}^{\beta}$. Then

$$\tilde{B}_n \in {}^b W_{n-1} \qquad (n = 2,3,\cdots),$$

$$W_n = h_n(\tilde{B}_1,\cdots,\tilde{B}_n) \qquad (n = 1,2,\cdots).$$

Hence $\tilde{B}_1, \tilde{B}_2, \cdots$ is a $h_1, h_2, \cdots$-chain for the $(a{*}b)$-game. $\bigcap \tilde{B}_n$ is in S and $\bigcap W_n$ is in S.

4. **More about** (α,β)**-winning sets.** Again let Ω be the set of "balls" $B = (\rho, c)$, where $\rho > 0$ and $c \in M$. Given a ball B of center c and radius ρ and a point $x \in M$ write

$$e(x,B) = d(x,c)\rho^{-1}.$$

One has $e(x,B) = 0$ if and only if $x = c$, $e(x,B) \leqq 1$ if and only if $x \in \alpha(B)$.

LEMMA 4. *Let $e = e(x, B) \leqq 1$ and $0 < \gamma < 1$. Every ball $B' \in B^\gamma$ has $e' = e(x, B')$ in the interval*

$$\max(0, (e + \gamma - 1)\gamma^{-1}) \leqq e' \leqq (e + 1 - \gamma)\gamma^{-1}. \tag{6}$$

If $\gamma \leqq 1 - e$, there is always a ball $B' \in B^\gamma$ having $e' = e(x, B') = 0$. Moreover, if M is a Banachspace of positive dimension with distance $d(y, z) = |y - z|$, then for every e' in the interval (6) *there is a $B' \in B^\gamma$ with $e(x, B') = e'$.*

Proof. Let B have center c and radius ρ, $B' \in B^\gamma$ center c' and radius $\rho' = \rho\gamma$.

$$\begin{aligned} e' &= d(x, c')\rho'^{-1} \leqq (d(x, c) + d(c, c'))\rho'^{-1} \leqq (d(x, c) + \rho - \rho')\rho'^{-1} \\ &= (d(x, c)\rho^{-1} + 1)\rho\rho'^{-1} - 1 = (e + 1)\gamma^{-1} - 1 = (e + 1 - \gamma)\gamma^{-1}, \\ e' &\geqq (d(x, c) - d(c, c'))\rho'^{-1} \geqq (d(x, c) - \rho + \rho')\rho'^{-1} = (e + \gamma - 1)\gamma^{-1}. \end{aligned}$$

Hence (6) always holds.

Now let $\gamma \leqq 1 - e$. The ball B' with center x and radius $\gamma\rho$ is in B, since $\gamma\rho + d(x, c) = \gamma\rho + e\rho \leqq \rho$. Here $e' = e(x, B') = 0$. Next, let M be a Banachspace and $\gamma > 1 - e$. Let B' be the ball with center $c' = c - e^{-1}(\gamma - 1)(x - c)$ and radius $\rho' = \gamma\rho$. Now $\gamma\rho + d(c', c) = \gamma\rho + d(x, c)(1 - \gamma)e^{-1} = \gamma\rho + e\rho(1 - \gamma)e^{-1} = \rho$, hence $B' \subset B^\gamma$. Furthermore, $e' = e(x, B') = d(x, c')\rho'^{-1} = d(x, c)(1 + (\gamma - 1)e^{-1})(\gamma\rho)^{-1} = e\rho(1 + (\gamma - 1)e^{-1})(\gamma\rho)^{-1} = (e + \gamma - 1)\gamma^{-1}$. Thus if M is a Banachspace, there is always a $B' \subset B^\gamma$ whose $e' = e(x, B')$ is the left endpoint of the interval (6).

Let $e > 0$, put $c' = c + e^{-1}(\gamma - 1)(x - c)$ and let B' be the ball with center c' and radius $\rho' = \rho\gamma$. $d(c, c') = d(x, c)(1 - \gamma)e^{-1} = (1 - \gamma)\rho = \rho - \rho'$, and therefore $B' \in B^\gamma$. Also $e' = e(x, B') = d(x, c')\rho'^{-1} = d(x, c)\ (1 - (\gamma - 1)e^{-1})\ (\gamma\rho)^{-1} = e\rho(1 - (\gamma - 1)e^{-1})\gamma^{-1}\rho^{-1} = (e + 1 - \gamma)\gamma^{-1}$.

If $e = 0$, let c' be any point having $d(c, c') = \rho - \rho' = (1 - \gamma)\rho$, and let B' be the ball with center c' and radius $\rho' = \gamma\rho$. Then $B' \subset B^\gamma$ and $e' = e(x, B') = d(x, c')\rho'^{-1} = d(c, c')\rho'^{-1} = (1 - \gamma)\rho\gamma^{-1}\rho^{-1} = (1 - \gamma)\gamma^{-1}$.

Hence if M is a Banachspace there is always a ball $B' \in B^\gamma$ whose $e' = e(x, B')$ equals the right endpoint of the interval (6). Since for $B' \subset B^\gamma$, $e' = e(x, B')$ depends continuously on the center c' of B', there is a ball $B' \subset B^\gamma$ whose $e(x, B')$ equals e', where e' is an arbitrary number in the interval (6).

LEMMA 5. *Suppose $0 < \alpha < 1$, $0 < \beta < 1$, $2\alpha \geqq 1 + \alpha\beta$. Then the only (α, β)-winning set is M itself.*

Proof. Let $x \in M$. Black may choose B_1 with center x. Hence $e_1 = e_1(x, B_1) = 0$. Then $W_1 \in B_1^\alpha$ satisfies $e_1' = e(x, W_1) \leqq (1 - \alpha)\alpha^{-1}$ by Lemma 4. Now $\beta \leqq \beta + (2\alpha - 1 - \alpha\beta)\alpha^{-1} = 2 - \alpha^{-1} = 1 - (1 - \alpha)\alpha^{-1} \leqq 1 - e_1'$, hence by Lemma 4 black can choose $B_2 \in W_1^\beta$ with $e_2 = e(x, B_2) = 0$. Thus B_2 also has center x. In this fashion black can enforce that x is the center of every ball B_n.

Then x is in the intersection of the "ballsets" $\alpha(B_n) = B_n^*$ and every winning set S must contain x. Since x was arbitrary, $S = M$.

LEMMA 6. *Let* $0 < \alpha < 1$, $0 < \beta < 1$, $2\beta \geqq 1 + \alpha\beta$. *Then every dense set* S *is* (α, β)-*winning.*

Proof. Let S be dense, and suppose black picks a ball with center c and radius ρ. There is an $x \in S$ having $d(x, c) \leqq (1 - \alpha)\rho$. White may pick $W_1 \subset B_1^\alpha$ with center x. Now, using the same method black used in Lemma 5, white can enforce that all the balls W_n have center x.

LEMMA 7. *Let* M *be a Banachspace of positive dimension,* $0<\alpha<1$, $0<\beta<1$, $2\alpha < 1 + \alpha\beta$. *Then any set* R *obtained by removing a finite number of points from a winning set* S *is again a winning set.*

Proof. Let R be obtained from the winning set S by removing x. If black picks B_1 such that $x \notin B_1^*$, then of course white can win. If in fact at some stage of a play there occurs a B_n with $x \notin B_n^*$, then white can win. Hence it suffices to show that white can play in such a way that $x \notin B_n^*$ for some n.

Assume $x \in B_1^*$ and set $e_1 = e(x, B_1) \leqq 1$. White can pick a ball W_1 having $e_1' = e(x, W_1)(e_1 + 1 - \alpha)\alpha^{-1}$ by Lemma 4. If $e_1' > 1$, $x \notin W_1^*$, and we are through. Otherwise $e_1' + \beta - 1 = (e_1 + 1 - \alpha)\alpha^{-1} + \beta - 1 = e_1\alpha^{-1} + (1 + \alpha\beta - 2\alpha)\alpha^{-1} > 0$ and $B_2 \in W_1^\beta$ satisfies $e_2 = e(x, B_2) \geqq e_1\alpha^{-1}\beta^{-1} + (1 + \alpha\beta - 2\alpha)(\alpha\beta)^{-1} > e_1(\alpha\beta)^{-1}$ by Lemma 4.

Generally, if $x \in B_n^*$, white can play such that either $x \notin W_n^*$ or

$$e(x, B_{n+1}) > (\alpha\beta)^{-1} e(x, B_n).$$

Since $(\alpha\beta)^{-1} > 1$, there will sooner or later occur a ball B_m with $x \notin B_m^*$.

LEMMA 8. *Let* $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \alpha' < 1$, $0 < \beta' < 1$, $\alpha\beta = \alpha'\beta'$, $\alpha' \leqq \alpha$. *Then every* (α, β)-*winning set is also* (α', β')-*winning.*

Proof. Assume $\alpha' < \alpha$. Let $h_1, h_2, \cdots$ be an $(\alpha, \beta; S)$-winning strategy. Given $B_1, \cdots, B_n$, write $\tilde{W}_n = h_n(B_1, \cdots, B_n)$, pick some $W_n \in \tilde{W}_n^{\alpha'/\alpha}$ and put $f_n(B_1, \cdots, B_n) = W_n$. Suppose (1), (2) hold with $\mathfrak{G} = \mathfrak{F}^{\beta'}$. Then

$$B_n \in \tilde{W}_{n-1}^{\beta'\alpha'/\alpha} = \tilde{W}_{n-1}^{\beta} \qquad (n = 2, 3, \cdots),$$

$$\tilde{W}_n = h_n(B_1, \cdots, B_n) \qquad (n = 1, 2, \cdots).$$

Hence $B_1, B_2, \cdots$ is a $h_1, h_2, \cdots$-chain of the (α, β)-game and $\bigcap B_n^*$ is in S. Therefore $f_1, f_2, \cdots$ is an $(\alpha', \beta'; S)$-winning strategy.

LEMMA 9. *Every* (α, β)-*winning set is* $(\alpha(\beta\alpha)^k, \beta)$-*winning for* $k = 0, 1, 2, \cdots$.

Proof. Suppose in the (α, β)-game, white not only makes his choices of the balls W_n, but also of the balls B_n, except those where $(k + 1) \mid (n - 1)$ (that is, $k + 1$

divides $n-1$). Thus black can pick only every $(k+1)$st ball B_n, namely $B_1, B_{1+(k+1)}, B_{1+2(k+1)}, \cdots$. The balls

$$B_1, W_{k+1}, B_{1+(k+1)}, W_{2(k+1)}, B_{1+2(k+1)}, \cdots$$

are balls of an $(\alpha(\beta\alpha)^k, \beta)$-play. If white can win the (α, β)-game it certainly can win the $(\alpha(\beta\alpha)^k, \beta)$-game.

COROLLARY. *Let $\alpha'\beta' = (\alpha\beta)^k$ for some integer $k > 0$ and $\beta' \geqq \beta$. Then every (α, β)-winning set is (α', β')-winning.*

Proof. Combine Lemmas 8 and 9.

Problem. Is it true that an (α, β)-winning set is necessarily (α', β')-winning if $\alpha' \leqq \alpha$, $\beta' \geqq \beta$? In particular is this true if M is the real line?

5. **Behavior of winning sets under local isometries.** Let M, M' be metrical spaces with distance-functions $d(x, y)$ and $d'(x', y')$, respectively. Assume that for every ball B of M, $\alpha(B) = B^*$ is compact, and make the same assumption on M'. M, M' are then locally compact and complete. Let σ be a homeomorphism from M onto M'. The function

$$\bar{\mu}(x, y) = d'(\sigma(x), \sigma(y))/d(x, y) \tag{7}$$

is defined and continuous for $x \neq y$. We call σ a *local isometry* if $\bar{\mu}$ can be continued to a function μ which is defined, continuous and $\neq 0$ for all x, y in M.

THEOREM 1. *Let σ be a local isometry from M onto M'. Let $S \subset M$ be an (α, β)-winning set. Then $S' = \sigma(S) \subset M'$ is an (α', β')-winning set if $\alpha'\beta' = \alpha\beta$, $\alpha' < \alpha$.*

We first need a lemma. Write τ for the inverse map of σ and define $\nu(x', y')$ for $x', y' \in M'$ by either of the following two equivalent formulae:

$$\nu(x', y') = 1/\mu(\tau(x'), \tau(y')), \qquad \mu(x, y) = 1/\nu(\sigma(x), \sigma(y)). \tag{8}$$

When $x' \neq y'$, $\nu(x', y') = d(\tau(x'), \tau(y'))/d'(x', y')$. Given $\lambda > 0$ and a ball B of M with center c and radius ρ write $\sigma(\lambda, B)$ for the ball B' of M' with center $\sigma(c)$ and radius $\lambda\rho$. Given $\lambda > 0$ and a ball B' of M' with center c' and radius ρ' write $\tau(\lambda, B')$ for the ball B of M with center $\tau(c')$ and radius $\lambda\rho'$. Denote the set of balls $C' \subset B'$ with $\rho'(C') \leqq \delta\rho'(B')$ by $B'^{\delta-}$.

LEMMA 10. *Let B' be a ball of M', and $\varepsilon > 0$. Put $\nu_0 = \max \nu(x', y')$, taken over all $(x', y') \in B'^* \times B'^*$.*

There exists a $\delta = \delta(B', \varepsilon) > 0$ such that every $C' \in B'^{\delta-}$ has the following property.

Put $\nu = \nu(c', c')$ and $\mu = \nu^{-1} = \mu(\tau(c'), \tau(c'))$, where c' is the center of C'. Now for any ball $D' \subset C'$ of M' and any ball D of M,

$$D \subset \tau(\nu(1-\varepsilon), D') \text{ implies } \sigma(\mu(1-\varepsilon), D) \subset D'. \tag{9}$$

On the other hand, if E is a ball of M with $E \subset \tau(v_0, C')$ and E' a ball of M',

$$E' \subset \sigma(\mu(1-\varepsilon), E) \textit{ implies } \tau(v(1-\varepsilon), E') \subset E. \tag{10}$$

Proof. Let $B = \tau(2v_0, B')$. $\mu(x,y)$ is uniformly continuous and bounded from below in $B^* \times B^*$. Hence there is an $\eta = \eta(B', \varepsilon) > 0$ such that

$$\left| \frac{\mu(x_1, y_1)}{\mu(x_2, y_2)} - 1 \right| < \varepsilon \tag{11}$$

if x_1, x_2, y_1, y_2 are in B^* and $d(x_1, x_2) \leqq \eta$, $d(y_1, y_2) \leqq \eta$.

Set $\delta_1 = \eta(3v_0\rho(B'))^{-1}$, let $C' \in B'^{\delta_1 -}$ and suppose D, D' satisfy the hypothesis of (9). Now if $D = (d, \rho_d)$, $D' = (d', \rho_d')$, $B' = (b', \rho_b')$, $C' = (c', \rho_c')$

$$\rho_d + d(d, \tau(d')) \leqq v(1-\varepsilon)\rho_d'. \tag{12}$$

Now since $D' \subset C' \subset B'$, $d'(d', b') \leqq \rho_b'$, whence $d(\tau(d'), \tau(b')) \leqq v_0\rho_b'$. Similarly, $d(\tau(c'), \tau(b')) \leqq v_0\rho_b'$. Finally, by (12), $d(d, \tau(b')) \leqq d(d, \tau(d')) + d(\tau(d'), \tau(b')) \leqq v\rho_d' + v_0\rho_b' \leqq 2v_0\rho_b'$. Since B has radius $2v_0\rho_b'$, the points $\tau(d'), \tau(c'), d$ are all in B^*.

Furthermore, $d(\tau(d'), \tau(c')) \leqq v_0 d'(d', c') \leqq v_0\rho_c' \leqq v_0\delta_1\rho_b' < \eta/2$, $d(d, \tau(c')) \leqq d(d, \tau(d')) + d(\tau(d'), \tau(c')) \leqq v\rho_d' + \eta/2 \leqq v_0\delta_1\rho_b' + \eta/2 \leqq \eta$. Hence by (11),

$$\mu(\tau(d'), d) < \mu(\tau(c'), \tau(c'))(1+\varepsilon) = \mu(1+\varepsilon).$$

Now

$$\begin{aligned} \mu(1-\varepsilon)\rho_d + d'(\sigma(d), d') &< \mu\rho_d + \mu(d, \tau(d'))d(d, \tau(d')) \\ &< \mu(1+\varepsilon)(\rho_d + d(d, \tau(d'))) \\ &\leqq \mu v(1-\varepsilon)(1+\varepsilon)\rho_d' < \rho_d'. \end{aligned}$$

Thus the conclusion of (9) holds.

Hence to obtain (9), one may take $\delta = \delta_1$. Now, by symmetry, one may treat B as we did B'. There is a $\delta_2 > 0$, such that (10) holds if $C \in B^{\delta_2 -}$, $E \subset C$, and if E' is a ball of M'. We set $\delta = \min(\delta_1, \delta_2)$ and show (10) holds for $E \subset \tau(v_0, C')$. By what we just said it suffices to verify $\tau(v_0, C') \in B^{\delta_2 -}$.

We have $C' \subset B'^{\delta -}$, hence $\rho_c' \leqq \delta\rho_b'$ and $\rho_c' + d'(c', b') \leqq \rho_b'$. Since b', c' are in B'^*, $v_0\rho_c' + d(\tau(c'), \tau(b')) \leqq v_0\rho_{c'} + v_0 d'(c', b') \leqq v_0\rho_b'$, whence $\tau(v_0, C') \subset B$. Finally, the radius of $\tau(v_0, C')$ is $v_0\rho_c' \leqq \delta v_0\rho_b' < \delta_2\rho_b$.

Proof of Theorem 1. Let the hypotheses of the theorem be satisfied. There is an $\varepsilon > 0$ such that $\alpha' = \alpha(1-\varepsilon)^2$, $\beta' = \beta(1-\varepsilon)^{-2}$. Suppose black starts with a ball B_1'. Choose $\delta = \delta(B_1', \varepsilon)$. Since $\alpha'\beta' < 1$, some ball B' of the play will be in B_1'. Let $v = v(b_j', b_j')$, where b_j' is the center of B_j', and $\mu = v^{-1}$. Then (9) will hold for balls $D' \subset B_j'$ of M' and balls D of M, while (10) will hold for balls $E \subset \tau(v_0, B_j')$

of M and E' of M', where $v_0 = \max(x', y')$, taken over all $(x', y') \in B_1'^* \times B_1'^*$. We may assume this already to be true for $j = 1$.

Let $f_1, f_2, \cdots$ be an $(\alpha, \beta; S)$-winning strategy. We claim the functions $f_1', f_2', \cdots$ defined by

$$f_n'(B_1', \cdots, B_n') = \sigma(\mu(1-\varepsilon), f_n(\tau(v(1-\varepsilon), B_1'), \cdots, \tau(v(1-\varepsilon), B_n'))) \qquad (n = 1, 2, \cdots)$$

are an $(\alpha', \beta'; S')$-winning strategy for plays beginning with our particular B_1'.

First we have to verify $f_n'(B_1', \cdots, B_n') \in B_n'^{\alpha'}$. For this purpose set $B_i = \tau(v(1-\varepsilon), B_i')$ $(i = 1, \cdots, n)$ and $W_n = f_n(B_1, \cdots, B_n)$. Now $W_n \subset B_n = \tau(v(1-\varepsilon), B_n')$, whence $f_n'(B_1', \cdots, B_n') = \sigma(\mu(1-\varepsilon), W_n) \subset B_n'$ by (9). (After all, $B_n' \subset B_1'$). A comparison of radii actually shows $f_n'(B_1, \cdots, B_n) \in B_n'^{\alpha'}$.

Now let balls $B_1', B_2', \cdots; W_1', W_2', \cdots$ of M' satisfy

$$\begin{aligned} B_n' &\in W_{n-1}'^{\beta'} && (n = 2, 3, \cdots), \\ W_n' &= f_n'(B_1', \cdots, B_n') && (n = 1, 2, \cdots). \end{aligned}$$

Put $B_n = \tau(v(1-\varepsilon), B_n')$, $W_n = f_n(B_1, \cdots, B_n)$ $(n = 1, 2, \cdots)$. One has $B_{n-1}' \subset B_1'$, hence $\rho_{b_{n-1}}' + d'(b_1', b_{n-1}') \leqq \rho_{b_1}'$, whence $v_0\rho_{b_{n-1}}' + d(\tau(b_1'), \tau(b_{n-1}')) \leqq v_0\rho_{b_1}'$, which gives $\tau(v_0, B_{n-1}') \subset \tau(v_0, B_1')$. $W_{n-1} \subset B_{n-1} = \tau(v(1-\varepsilon), B_{n-1}') \subset \tau(v_0, B_{n-1}') \subset \tau(v_0, B_1')$. Hence we may apply (10) with $E = W_{n-1}$ and see that $B_n' \subset W_{n-1}' = f_{n-1}'(B_1', \cdots, B_{n-1}') = \sigma(\mu(1-\varepsilon), W_{n-1})$ implies

$$B_n = \tau(v(1-\varepsilon), B_n') \subset W_{n-1} \qquad (n = 2, 3, \cdots).$$

Using this and $W_n = f_n(B_1, \cdots, B_n)(n = 1, 2, \cdots)$, we conclude $x = \bigcap B_n^* \in S$.

Let $x' = \bigcap B_n'^*$. This implies $d'(x', b_n') \leqq \rho_{b_n}'$, whence $d(\tau(x'), b_n) = d(\tau(x'), \tau(b_n')) \leqq v_0\rho_{b_n}'$. Since b_n tends toward x and $\rho_{b_n} \to 0$, $\tau(x') = x$, whence $x' = \sigma(x) \in \sigma(S) = S'$.

6. **α-winning sets.** Let $0 < \alpha < 1$. Call a subset S of a complete metrical space α-winning, if it is (α, β)-winning for every β, $0 < \beta < 1$.

LEMMA 11. *Let $0 < \alpha' < \alpha < 1$. Then every α-winning set is α'-winning.*

Proof. Given any β', $0 < \beta' < 1$, there exists a β, $0 < \beta < 1$, such that $\alpha\beta = \alpha'\beta'$. S is (α, β)-winning, hence (α', β')-winning by Lemma 8.

LEMMA 12. *The only α-winning set $S \subset M$ with $\alpha > 1/2$ is $S = M$ itself.*

Proof. There is a β, $0 < \beta < 1$, having $2\alpha \geqq 1 + \alpha\beta$. The result is now an immediate consequence of Lemma 5.

Let $S \subset M$. Define the *winning dimension* of S,

$$\text{windim } S, \tag{13}$$

as follows. Windim $S = 0$ if S is α-winning for no $\alpha > 0$. Otherwise windim is the least upper bound of all α in $0 < \alpha < 1$ such that S is α-winning. It follows from Lemma 12 that windim $S = 1$ if and only if $S = M$; otherwise $0 \leqq \text{windim } S \leqq 1/2$, LEMMA 13. *Let M, M' be metrical spaces such that for balls B of M and B' of M', $\alpha(B)$ and $\alpha(B')$ are compact. Let σ be a local isometry from M onto M' and let $S \subset M$. Then*

$$\operatorname{windim} \sigma(S) = \operatorname{windim} S.$$

Proof. Apply Theorem 1.

THEOREM 2. *The intersection of countably many α-winning sets is α-winning.*

COROLLARY. Windim $(\bigcap_{j=1}^{\infty} S_j) =$ g.l.b. (windim S_j).

Proof of Theorem 2. We have to show that $S = \bigcap S_j$ is α-winning if each of the sets S_j is. We show that S is (α, β)-winning, say. White will win by playing according to the following rule.

At the first, third, fifth, $\cdots$ move, white moves according to an $(\alpha, \alpha\beta\alpha; S_1)$-winning strategy. Since $B_{2l+1} \in B_{2l-1}^{\alpha\beta\alpha}$, white can enforce in this way that $\bigcap B_n^*$ is in S_1, no matter what strategy he uses in his second, fourth, sixth, $\cdots$ move.

At the second, sixth, tenth, $\cdots$ move, white uses an $(\alpha, \alpha(\beta\alpha)^3; S_2)$-winning strategy. Generally, at the kth move, where $k \equiv 2^{l-1} (\operatorname{mod} 2^l)$, white moves as if he were playing the $(\alpha, \alpha(\beta\alpha)^{2^l - 1} : S)$-game, and thus can enforce that $\bigcap B_n^*$ is in S.

Our rule amounts to this: Let $f_1^l, f_2^l, \cdots (l = 1, 2, \cdots)$ be an $(\alpha, \alpha(\alpha\beta)^{2^l - 1}; S_l)$-winning strategy. We now define a strategy $f_1, f_2, \cdots$ as follows. For $k \equiv 2^{l-1} (\operatorname{mod} 2^l)$ and $k = 2^{l-1} + (t-1)2^l$, set

$$f_k(B_1, \cdots, B_k^l) = f_t(B_{2^{l-1}}, B_{2^{l-1}+2^l}, \cdots, B_{2^{l-1}+(t-1)2^l}).$$

Then

$$B_{2^{l-1}+(t-1)2^l} \in W_{2^{l-1}+(t-2)2}^{\alpha(\beta\alpha)2^{l-1}} \qquad (t = 2, 3, \cdots),$$

$$W_{2^{l-1}+(t-1)2^l} = f_t^l(B_{2^{l-1}}, \cdots, B_{2^{l-1}+(t-1)2^l}) \quad (t = 1, 2, \cdots),$$

and the intersection

$$\bigcap_{t=1}^{\infty} B^*_{2^{l-1}+(t-1)2^l}$$

is in S_l $(l = 1, 2, \cdots)$.

LEMMA 14. *Let M be a Banachspace. Let T be be obtained from an α-winning set S, $S \subset M$, $0 < \alpha \leqq 1/2$, by deleting at most countably many points. Then T is also α-winning.*

Proof. Let T be obtained by removing the points $x_1, x_2, \cdots$ from S, and let T_j be obtained by removing $x_1, \cdots, x_j$ from S. Each of the sets T_j is α-winning by Lemma 7, hence T is α-winning by Theorem 2.

7. **Badly approximable numbers.** In this section M is the space of real numbers with the usual metric, and the badly approximable numbers are considered as subset of M.

From here until the end of §12, α will be 1-1. Hence we need not and will not distinguish between elements $B \in \Omega$ and sets $B^* = \alpha(B) \subset M$.

THEOREM 3. *The set S of badly approximable numbers is (α, β)-winning for every α, β having $0 < \alpha < 1$, $0 < \beta < 1$, $2\alpha < 1 + \alpha\beta$.*

COROLLARY. Windim $S = 1/2$.

REMARK. Badly approximable numbers can be generalized to n-tuples, and then the analogous theorem holds. See [1] or [7].

PROPOSITION. *Let $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma = 1 + \alpha\beta - 2\alpha > 0$. Suppose black begins his play with a ball of radius ρ (= interval of length 2ρ). Put $\delta = (\gamma/2)\min(\rho, \alpha^2\beta^2\gamma/8)$. Then white can enforce that $x = \bigcap B_n$ satisfies $|x - p/q| > \delta q^{-2}$ for all integers p and $q \neq 0$.*

Obviously this proposition implies the theorem. Note that reals x with $|x - p/q| > \delta q^{-2}$ have partial denominators $\leqq \delta^{-1}$.

LEMMA 15. *Let α, β, γ be like in the proposition. Let the integer t satisfy $(\alpha\beta)^t < \gamma/2$. Assume a ball B_k with center b_k and radius ρ_k occurs in some (α, β)-play. Then white can play in such a way that B_{k+t} is contained in the "halfline" $x > b_k + \rho_k\gamma/2$.*

Proof. Let $g^+ \in F_1^\alpha$ be the function which assigns to an interval B of center c and length 2ρ the interval with center $c + \rho(1-\alpha)$ and length $2\alpha\rho$. Now for given B_{k+i}, $0 \leqq i < t$, white chooses $W_{k+i} = g^+(B_{k+i})$. Denote the center of B_n by b_n, the center of W_n by w_n $(n = 1, 2, \cdots)$. Then $w_k = b_k + \rho_k(1-\alpha)$, $b_{k+1} \geqq w_k - \alpha\rho_k(1-\beta) \geqq b_k + \rho_k\gamma > b_k$, and $b_{k+t} \geqq b_k + \rho_k\gamma$. Since B_{k+t} has radius $(\alpha\beta)^t\rho_k < \rho_k\gamma/2$, B_{k+t} is in the halfline $x > b_k + \rho_k\gamma/2$.

Proof of the Proposition. We may assume black starts with a ball B_1 of radius $\rho \leqq \alpha\beta\gamma/8$. Otherwise, if $\rho > \alpha\beta\gamma/8$, there will be a first B_j in the course of the play having radius $\rho_j \leqq \alpha\beta\gamma/8$, and then $\alpha^2\beta^2\gamma/8 < \rho_j \leqq \alpha\beta\gamma/8$. Since both $\rho > \alpha^2\beta^2\gamma/8$ and $\rho_j > \alpha^2\beta^2\gamma/8$, it does not matter whether δ is defined using ρ or ρ_j, and white can play as if B_j were the first black ball. Hence assume

$$\rho \leqq \alpha\beta\gamma/8.$$

Choose the integer t such that $\alpha\beta\gamma/2 \leqq (\alpha\beta)^t < \gamma/2$ and define $R > 0$ by

$$R^2(\alpha\beta)^t = 1.$$

To prove the proposition it will suffice to show that white can play in such a way that

$$|x - p/q| > \delta q^{-2} \tag{14}$$

whenever $(p,q) = 1$ (that is, p and q are relatively prime),

$$x \in B_{nt+1} \text{ and } 0 < q < R^n \tag{15}$$

for some integer $n \geqq 0$.

Clearly (14) holds if (15) holds for $n = 0$, since $0 < q < R^0 = 1$ has no integral solution q. Suppose $B_1, B_{t+1}, B_{2t+1}, \cdots, B_{(k-1)t+1}$ are already such that (14) holds if (15) holds for $0 \leqq n \leqq k-1$. Now in the next moves white has to worry only over fractions p/q where $R^{k-1} \leqq q < R^k$. In fact white has to worry over at most one such fraction: If $|x - p/q| < \delta/q^2$, $|x' - p'/q'| < \delta/q'^2$, where $R^{k-1} \leqq q < R^k$, $R^{k-1} \leqq q' < R^k$, $p/q \neq p'/q'$, x, x' both in $B_{(k-1)t+1}$, then $|p/q - p'/q'| \leqq \delta/q^2 + \delta/q'^2 + 2\rho(B_{(k-1)t+1}) \leqq 2\delta R^{2-2k} + 2\rho(\alpha\beta)^{(k-1)t} = 2(\rho + \delta)R^{2-2k} < 4\rho R^{2-2k} \leqq \frac{1}{2}\alpha\beta\gamma R^{2-2k} = \frac{1}{2}\alpha\beta\gamma(\alpha\beta)^{-t}R^{-2k} \leqq R^{-2k}$, while on the other hand $|p/q - p'/q'| \geqq 1/(qq') > R^{-2k}$, which gives a contradiction.

Hence white has to worry over at most one subinterval C of $B_{(k-1)t+1}$ of length $2\rho(C) \leqq 2\delta/q^2 \leqq 2\delta R^{2-2k}$. Now if C has its center to the left or on the center b of $B_{(k-1)t+1}$, C is contained in the halfline $x \leqq b + \delta R^{2-2k} = b + \delta(\alpha\beta)^{(k-1)t} = b + \delta\rho_{(k-1)t+1}/\rho \leqq b + \rho_{(k-1)t+1}\gamma/2$, where $\rho_{(k-1)t+1}$ is the radius of $B_{(k-1)t+1}$. By Lemma 15 white can enforce that B_{kt+1} is contained in $x > b + \rho_{(k-1)t+1}\gamma/2$, and B_{kt+1} has empty intersection with C. The reasoning is similar if the center of C is to the right of the center of $B_{(k-1)t+1}$.

There is an analogy of α-winning sets with residual sets (= complements of sets of first category) in so far, as countable intersections of residual sets are again residual sets. By definition, residual sets are sets T with the property that every intersection $T \cap O$ with a nonempty open set O contains a nonempty open set O', as well as countable intersections of sets with this property.

The numbers with unbounded partial denominators in their continued fraction are a residual set. This set is the intersection of the sets T_k of numbers with at least one partial denominator $\geqq k$, and it is easy to see that every intersection of T_k with an open interval contains an open interval.

Thus the set of numbers with unbounded partial denominators is a residual set but not a winning set, and the set of numbers with bounded partial denominators is α-winning for $0 < \alpha \leqq 1/2$, but is a set of first category. This is in contrast to the situation for the Banach-Mazur game [6, Theorem 1].

8. **Anormal numbers.**

THEOREM 4. *Let $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma = 1 + \alpha\beta - 2\alpha > 0$. Let g be an integer so large that*

$$g > 4(\alpha\beta\gamma)^{-1} \tag{16}$$

and let d be a digit in the scale of g, i.e., $d = 0, 1, \cdots, g-2$ or $g-1$. The set S

of reals x in whose "decimal" expansion to scale g the digit d occurs at most finite number of times is (α,β)-winning.

COROLLARY. *The sets S_s^* of numbers x which are not normal to base s is (α,β)-winning, and therefore* windim $S_s^* = 1/2$.

Proof of the Corollary. Some integral power g of s satisfies (16). The set of numbers with only finitely many zeros in their expansion to scale g is contained in the set of numbers not normal to scale g, which is the same as the set of numbers not normal to scale s. Hence this latter set is (α,β)-winning by the theorem.

Proof of Theorem 4. Let black begin with the ball B_1 of radius ρ. Choose integers $k \geqq 1$, $n_0 \geqq 1$ such that

$$g^{-k}/4 > (\alpha\beta)^{n_0-1}\rho \geqq (\alpha\beta)g^{-k}/4. \tag{17}$$

Define integers $n_1, n_2, \cdots$ by

$$g^{-k-j}/4 > (\alpha\beta)^{n_j-1}\rho \geqq (\alpha\beta)g^{-k-j}/4 \qquad (j = 1,2,\cdots). \tag{18}$$

Then, since $\alpha\beta > 4/(g\gamma)$,

$$g^{-k-j}/4 > (\alpha\beta)^{n_j-1}\rho > g^{-k-j-1}/\gamma > g^{-k-j-1}/4 \qquad (j = 1,2,\cdots). \tag{19}$$

Thus $n_0 < n_1 < n_2 < \cdots$.

We are going to show that white can play such that every $x \in B_{n_j}$, $j \geqq 1$, has its $(k+j)$th digit different from d. Let $B_{n_{j-1}}$ be given. The numbers x whose $(k+j)$th digit equals d are in intervals of length g^{-k-j} whose distance is $\geqq g^{1-k-j}(1 - 1/g) \geqq g^{1-k-j}/2 > 2(\alpha\beta)^{n_{j-1}-1}\rho = 2\rho(B_{n_{j-1}})$. Hence white has to worry over at most one interval $C \subset B_{n_{j-1}}$ of length $\leqq g^{-k-j}$. Let us assume without loss of generality that the center of C is less or equal to the center b of $B_{n_{j-1}}$. Then C is contained in the halfline

$$x < b + g^{-k-j}/2 < b + (\alpha\beta)^{n_{j-1}-1}\rho\gamma/2 = b + \rho(B_{n_{j-1}})\gamma/2.$$

Put $t = n_j - n_{j-1}$.

$$(\alpha\beta)^t = ((\alpha\beta)^{n_j-1}\rho)/((\alpha\beta)^{n_{j-1}-1}\rho) < (g^{-k-j}/4)/(g^{-k-j}/\gamma) = \gamma/4$$

and Lemma 15 applies. White can enforce $B_{n_j} = B_{n_{j-1}+t}$ to be in the set of x having $x > b + \rho(B_{n_{j-1}})\gamma/2$, and hence can enforce that $C \cap B_{n_j}$ is empty.

9. Numbers with infinitely many zeros in their decimal.

THEOREM 5. *Let $g > 2$ be integral and let S_g be the set of reals which have infinitely zeros in their "decimal" to base g. Then S_g is $\alpha_g = ((g-1)^2+1)^{-1}$-winning but not α-winning for $\alpha > \alpha_g$. Hence*

$$\text{windim } S_g = \alpha_g = ((g-1)^2+1)^{-1}.$$

Let $k \geqq 1$, n be integers. Write $I_k(n)$ for the interval $[ng^{-k}, ng^{-k} + (g-1)^{-1}g^{-k}]$, K_k for the union of all intervals $I_k(n)$, $n = 0, \pm 1, \pm 2, \cdots$.

LEMMA 16. *Let I be a closed interval of length $l(I) \leqq (\alpha_g(g^2 - g))^{-1}$. There is a $k \geqq 1$ and an interval $J \in I^{\alpha_g}$, $J \subset K_k$.*

Proof. Choose $k \geqq 1$ satisfying

$$(\alpha_g(g-1)g^{k+1})^{-1} < l(I) \leqq (\alpha_g(g-1)g^k)^{-1}. \tag{20}$$

We are going to construct an interval $J^* \subset I \cap K_k$ of length $l(J^*) \geqq \alpha_g l(I)$.

K_k consists of intervals of length $(g-1)^{-1}g^{-k}$, the complement of K_k of intervals of length $(g-2)(g-1)^{-1}g^{-k}$. The worst situation is when the midpoint c of I coincides with the midpoint of one of the intervals of the complement of K_k. In this case I contains all numbers x in

$$c \leqq x \leqq c + \frac{1}{2}l(I),$$

and K_k contains all x in

$$\begin{aligned} c + \frac{1}{2}(g-2)(g-1)^{-1}g^{-k} \leqq x &\leqq c + \frac{1}{2}(g-2)(g-1)^{-1}g^{-k} + (g-1)^{-1}g^{-k} \\ &= c + \frac{1}{2}(g-1)^{-1}g^{1-k}. \end{aligned}$$

Let J^* consist of all x in

$$c + \frac{1}{2}(g-2)(g-1)^{-1}g^{-k} \leqq x \leqq c + \min\left(\frac{1}{2}l(I), \frac{1}{2}(g-1)^{-1}g^{1-k}\right).$$

Obviously $J^* \subset I \cap K_k$. Furthermore,

$$\begin{aligned} &l(J^*) - \alpha_g l(I) \\ &= \min\left(\frac{1}{2}l(I), \frac{1}{2}(g-1)^{-1}g^{1-k}\right) - \frac{1}{2}(g-2)(g-1)^{-1}g^{-k} - \alpha_g l(I) \\ &= \min\left(\left(\frac{1}{2} - \alpha_g\right)l(I) - \frac{1}{2}(g-2)(g-1)^{-1}g^{-k}, (g-1)^{-1}g^{-k} - \alpha_g l(I)\right) \\ &\geqq \min\left(\frac{1}{2}g(g-2)\alpha_g l(I) - \frac{1}{2}(g-2)(g-1)^{-1}g^{-k}, 0\right) \geqq 0 \end{aligned}$$

by (20).

Proof of Theorem 5. Let $ng^{-k} = c_0 + c_1 g^{-1} + \cdots + c_k g^{-k}$ where $c_0, c_1, \cdots, c_k$ are integers, $0 \leqq c_j \leqq g-1$ $(j = 1, \cdots, k)$. The interval $I_k(n)$ now consists of all x satisfying

$$c_0 + c_0 g^{-1} + \cdots + c_k g^{-k} \leqq x \leqq c_0 + c_1 g^{-1} + \cdots + c_k g^{-k} + g^{-k-1} + g^{-k-2} + \cdots.$$

Hence if x is in the interior of $I_k(n)$, at least one of the digits $c_{k+1}, c_{k+2}, \cdots$ of x is zero. In fact if x is in a closed subset C of the interior of $I_k(n)$, then at least one of the digits $c_{k+1}, \cdots, c_{k+m}$ of x is zero, where $m = m(C)$.

We now are going to show that S_g is α_g-winning. Let $0 < \beta < 1$. In the (α_g, β)-game white plays arbitrarily until a ball B_{ji}, with $2\rho(B_{j_1}) \leqq (\alpha_g(g^2 - g))^{-1}$ occurs. Now by the lemma, white can pick $W_{j_1} \subset K_{k_1}$, say $W_{j_1} \subset I_{k_1}(n_1)$. At his next move white can enforce that W_{j_1+1} is in the interior of $I_{k_1}(n_1)$. There is an m_1 such that every $x \in W_{j_1+1}$ has at least one of the digits $c_{k_1+1}, \cdots, c_{k_1+m_1}$ equal to zero. White can play arbitrarily again until $2\rho(B_{j_2}) \leqq \alpha_g^{-1}(g-1)^{-1} g^{-k_1-m_1}$. By the lemma, white can pick $W_{j_2} \subset K_{k_2}$ for some k_2, and obviously $k_2 \geqq k_1 + m_1$. At his next move white chooses W_{j_2+1} in the interior of some $I_{k_2}(n_2)$, and so on.

This proves the first part of the theorem.

Let $\alpha > \alpha_g$. Choose m integral and so large that

$$\alpha > (1 + 2(g-1)g^{3-m})\alpha_g, \tag{21}$$

and let $\beta = \alpha^{-1} g^{-m}$. We are going to show that S_g is (α, β)-losing.

Black can adopt the following strategy. First he picks the ball B_1 to consist of all x in

$$g^{-1} + g^{-2} \leqq x \leqq 2g^{-1} + g^{-3} + g^{-4} + \cdots = 2g^{-1} + g^{-2}(g-1)^{-1}.$$

B_1 has length $g^{-1} - g^{-2} + g^{-2}(g-1)^{-1} = \alpha_g^{-1} g^{-2}(g-1)^{-1}$.

For $u = 0, 1, \cdots, (g-1)^2 + 1$ put $y_u = g^{-1} + g^{-2} + u(g-1)^{-1} g^{-2}$. The numbers y_u are at distances $(g-1)^{-1} g^{-2}$, they are contained in B_1, and y_0, $y_{(g-1)2+1}$ are the endpoints of B_1.

W_1 will have length

$$\begin{aligned} l(W_1) = \alpha l(B_1) = \alpha\alpha_g^{-1} g^{-2}(g-1)^{-1} &> g^{-2}(g-1)^{-1}(1 + 2(g-1)g^{3-m}) \\ &= g^{-2}(g-1)^{-1} + 2g^{1-m} \end{aligned}$$

by (21). Let $\tilde{W}_1$ be the closed interval of length $g^{-2}(g-1)^{-1}$ and with the same midpoint as W_1. $\tilde{W}_1$ will contain one of the points y_u, $1 \leqq u \leqq (g-1)^2$; say $y_{u_0} \in \tilde{W}_1$. Hence W_1 will contain the interval

$$y_{u_0} - g^{1-m} \leqq x \leqq y_{u_0} + g^{1-m}. \tag{22}$$

First consider the case (a) where $g-1$ divides u_0, say $u_0 = l(g-1)$. Now $y_{u_0} = g^{-1} + g^{-2} + lg^{-2}$, where $1 \leqq l \leqq g-1$. If $l < g-1$, $y_{u_0} = 0,1(l+1)000\cdots$ when written as a decimal in scale g, and $y_{u_0} - g^{-m} = 0,1l(g-1)\cdots(g-1)000\cdots$, that is, $y_{u_0} - g^{-m}$ will have the digits $1, l$, then $m-2$ times $g-1$, then zeros. If $l = g-1$, $y_{u_0} = 0.2000\cdots$ and $y_{u_0} - g^{-m} = 0,1(g-1)\cdots(g-1)000\cdots$, that is, it will have digits $1, m-1$ times $g-1$, then zeros. Hence any x in the interval

$y_{u_0} - g^{-m} \leqq x < y_{u_0}$ has its first m digits different from zero. Now black picks B_2 to be the interval

$$y_{u_0} - g^{-m} + g^{-m}(g^{-1} + g^{-2}) \leqq x \leqq y_{u_0} - g^{-m} + g^{-m}(2g^{-1} + g^{-3} + g^{-4} + \cdots).$$

B_2 is contained in (22), hence in W_1, it has length

$$l(B_2) = g^{-m} l(B_1) = \alpha^{-1} g^{-m} l(W_1) = \beta l(W_1),$$

and every $x \in B_2$ has its first m digits different from zero.

Next take the case (b) where u_0 is not a multiple of $g - 1$, say $u_0 = l(g - 1) + r$, $1 \leqq r \leqq g - 2$, $l \leqq g - 2$. Now

$$y_{u_0} = g^{-1} + g^{-2} + lg^{-2} + rg^{-2}(g-1)^{-1} = g^{-1} + (l+1)g^{-2} + r(g^{-3} + g^{-4} + \cdots),$$

hence $y_{u_0} = 0,1(l+1)rrr\cdots$. Put $\bar{y} = y_{u_0} + g^{-m} - g^{-m-1} - g^{-m-2} - \cdots = y_{u_0} + g^{-m} - g^{-m}(g-1)^{-1}$. $\bar{y} = 0,1(l+1)rr\cdots r(r+1)000\cdots$, that is, $\bar{y}$ has digits $1, l+1, m-3$ times $r, r+1$, then zeros. Any x in the interval $\bar{y} \leqq x < \bar{y} + g^{-m}$ has its first m digits different from zero. Now black picks B_2 to consist of all x satisfying

$$\bar{y} + g^{-m}(g^{-1} + g^{-2}) \leqq x \leqq \bar{y} + g^{-m}(2g^{-1} + g^{-3} + g^{-4} + \cdots).$$

B_2 is in (22) hence in W_1, its length is $\beta l(W_1)$, and every $x \in B_2$ has its first m digits different from zero.

Black does not have to worry over the first m digits any more. Since B_2 is congruent to $g^{-m}B_1$ modulo g^{-m}, black can apply the same strategy to ensure that the next m digits of any $x \in B_3$ again are all different from zero. Continuing in this way black can enforce that $x = \bigcap B_n$ has no zeros among its digits.

10. **a^*-winning sets.** Let $a > 1$ be integral. A set of reals is called a^*-winning if it is (a^*b)-winning for every integer $b > 1$.

LEMMA 17. *Let $a'b' = ab$, and a divides a'. Then every (a^*b)-winning set is (a'^*b')-winning.*

Proof. Just as for Lemma 8.

LEMMA 18. *Every (a^*b)-winning set is $(a(ba)^{k*}b)$-winning for every integer $k \geqq 0$.*

Proof. Just as for Lemma 9.

LEMMA 19. *Let a be a divisor of a'. Then every a^*-winning set is a'^*-winning.*

Proof. This follows from Lemma 17.

Combining Lemma 2 and Theorem 3 one finds that the set of badly approximable numbers is a^*-winning for $a \geqq 4$. A direct examination of the proof of

Theorem 3 shows this set to be a^*-winning for every $a \geqq 2$. A similar remark applies to anormal numbers.

11. **The Hausdorff dimension of winning sets.** The Hausdorff dimension of a set S in a metrical space M is defined as follows. S has Hausdorff dimension ∞ if for some $\eta > 0$ S cannot be covered by countably many balls of radius $< \eta$. Otherwise put $\{S,\eta\}^\alpha$ for the greatest lower bound (possibly ∞) of all the sums

$$\sum_{l=1}^{\infty} \rho(B_l)^\alpha, \tag{23}$$

where $B_1, B_2, \cdots$ is a covering of S by balls B_l of radius $< \eta$. $\{S,\eta\}^\alpha$ is a decreasing function of η.

$$\{S\}^\alpha = \lim_{\eta \to 0} \{S,\eta\}^\alpha$$

(possibly ∞) is called α-dimensional measure of S. Either $\{S\}^\alpha = \infty$ for every α, in which case S again has Hausdorff dimension ∞. Or there is a unique $\delta \geqq 0$ such that $\{S\}^\alpha = \infty$ for $\alpha < \delta$ and $\{S\}^\alpha = 0$ for $\alpha > \delta$, and in this case one defines the Hausdorff dimension of S to be δ.

THEOREM 6. *Let M be a Hilbertspace, and let $0 < \alpha < 1$, $0 < \beta < 1$. Assume there are integers t, m with the following property. Given $h_1, h_2, \cdots, h_t$ with $h_i \in F_i^\alpha$ $(i = 1, \cdots, t)$ and given a ball C_1, there are m functions $g^{(0)}, g^{(1)}, \cdots, g^{(m-1)}$ of F_1^β such that, if $C_2^{(j)}, \cdots, C_{t+1}^{(j)}, D_1^{(j)}, \cdots, D_t^{(j)}$ $(0 \leqq j \leqq m-1)$ are balls defined by*

$$C_i^{(j)} = g^{(j)}(D_{i-1}^{(j)}) \qquad (1 < i \leqq t+1), \tag{24}$$

$$D_i^{(j)} = h_i(C_1, C_2^{(j)}, \cdots, C_i^{(j)}) \quad (1 \leqq i \leqq t), \tag{25}$$

then $C_{t+1}^{(0)}, \cdots, C_{t+1}^{(m-1)}$ have pairwise disjoint interiors.

Under these assumptions, every (α,β)-winning set $S \subset M$ has Hausdorff dimension at least

$$\log m / \left| t \log \alpha\beta \right| .$$

COROLLARY 1. *Let $N(\beta)$ be such that every ball B contains a set of $N(\beta)$ balls of B^β with pairwise disjoint interiors. Then every (α,β)-winning set has Hausdorff dimension at least*

$$\log N(\beta) / \left| \log \alpha\beta \right| .$$

For example on the real line one may put $N(\beta) = [\beta^{-1}]$. (That is, the integral part of β^{-1}.)

Proof. One may use the theorem with $t = 1$, $m = N(\beta)$.

COROLLARY 2. *An α-winning set in n-dimensional Euclidean space E_n has Hausdorff dimension n.*

Proof. One has $N(\beta) \geqq c\beta^{-n}$ for some $c > 0$. This gives the lower bound $(\log c + n|\log\beta|)(|\log\alpha| + |\log\beta|)^{-1}$, which tends to n when β tends to zero.

COROLLARY 3. *Let $1 + \alpha\beta > 2\beta$. Then every (α, β)-winning set in E_n has positive Hausdorff dimension, and an (α, β)-winning set in infinite-dimensional Hilbert-space has infinite Hausdorff dimension.*

Proof. We first take the n-dimensional case. Write $\gamma = 1 + \alpha\beta - 2\beta > 0$, and let the integer $t > 1$ be so large that $(\alpha\beta)^t < \gamma/3$.

Let g^{i+}, g^{i-} $(i = 1, \cdots, n)$ be the functions of F_1^β which assign to a ball B of radius ρ and center $(c_1, \cdots, c_n)$ the ball of radius $\beta\rho$ and center

$$(c_1, \cdots, c_{i-1}, c_i + \rho(1-\beta), c_{i+1}, \cdots, c_n),\ (c_1, \cdots, c_{i-1}, c_i - \rho(1-\beta), c_{i+1}, \cdots, c_n),$$

respectively.

Let $h_1 \in F_1^\alpha, \cdots, h_t \in F_t^\alpha$, and let C_1 have center $c = (c_1, \cdots, c_n)$ and radius ρ. Let $1 \leqq j \leqq n$ and let k denote $+$ or $-$. Define $C_2^{jk}, \cdots, C_{t+1}^{jk}$; $D_1^{jk}, \cdots, D_t^{jk}$ by

$$\begin{aligned} C_i^{jk} &= g^{jk}(D_{i-1}^{jk}) \qquad && (2 \leqq i \leqq t+1), \\ D_i^{jk} &= h_i(C_1, C_2^{jk}, \cdots, C_i^{jk}) \qquad && (1 \leqq i \leqq t). \end{aligned}$$

Denote the center of C_i^{jk} by $(c_{i1}^{jk}, \cdots, c_{in}^{jk})$. Then

$$\begin{aligned} c_{2j}^{j+} &\geqq c_j + \rho(1 - \beta - \beta(1-\alpha)) = c_j + \rho\gamma > c_j, \\ c_{t+1\,j}^{j+} &\geqq c_j + \rho\gamma. \end{aligned}$$

Hence C_{t+1}^{j+}, which has radius $\rho(\alpha\beta)^t < \rho\gamma/3$, is contained in the halfplane $x_j \geqq c_j + 2\rho\gamma/3$. Similarly, C_{t+1}^{j-} is contained in $x_j \leqq c_j - 2\rho\gamma/3$.

Let $l = l(\alpha, \beta)$ be an integer with $(l+1)\gamma^2/9 > 1$. We claim that any ball C_{t+1}^{jk} has nonempty intersection with at most l of the balls C_{t+1}^{jk} $(j = 1, \cdots, n;\ k = +, -)$. To show this it suffices to see that C_{t+1}^{1+} can intersect at most l of these balls (including itself). By what has already been shown it cannot at the same time intersect C_{t+1}^{j+} and C_{t+1}^{j-}. Thus it remains to show that C_{t+1}^{1+} cannot intersect all the balls $C_{t+1}^{2+}, \cdots, C_{t+1}^{l+1+}$, say. If these intersections were nonempty, $c_{t+1\,j}^{1+} \geqq c_j + \rho\gamma/3$ $(j=2, \cdots, l+1)$, and the center of C_{t+1}^{1+} would have distance from the center c of C_1at least $\rho\sqrt{(\gamma^2 + l\gamma^2/9)} > \rho\sqrt{((l+1)\gamma^2/9)} > \rho$.

Thus there exist

$$m = \max(2, 2n/l) \tag{26}$$

of the balls C_{t+1}^{jk} $(j = 1, \cdots, n;\ k = +, -)$ which are pairwise disjoint. We can pick m of the functions g^{ik}, say $g^{(0)}, \cdots, g^{(m-1)}$, which satisfy the conditions of the theorem. Therefore S has Hausdorff dimension at least $\log m / |t \log\alpha\beta| > 0$.

In the case of a Hilbertspace of infinite dimension the argument leading to (26) in the previous case shows that now one may take m arbitrarily large.

REMARK. The results of this section are in contrast to Folgerung 1 of Satz 3 of [8] where a different game is studied.

12. **Proof of Theorem 6.**

LEMMA 20. *Put $\omega = 2/\sqrt{3} - 1$. Let $D, D_1, \cdots, D_e$ be balls in a Hilbertspace such that $\rho(D) < \omega\rho(D_1) = \cdots = \omega\rho(D_e)$. Let D_i, D_j have disjoint interiors for $i \neq j$, and let D, D_i have nonempty intersections for $i = 1, 2, \cdots, e$.*

Then $e \leqq 2$.

Proof. We may assume $\rho(D_1) = \cdots = \rho(D_e) = 1$. We have to show that the assumptions of the Lemma with $e = 3$ lead to a contradiction.

Let D have center 0, D_i center $x_i (i = 1, 2, 3)$. Then $|x_i| < 1 + \omega$, $|x_i - x_j| \geqq 2$ for $i \neq j$. One obtains $|x_i|^2 < (1+\omega)^2 = 4/3$, $4 \leqq |x_i - x_j|^2 = |x_i|^2 + |x_j|^2 - 2x_i x_j$ ($=$ inner product) $< 8/3 - 2x_i x_j$, hence $x_i x_j < -2/3$ $(i \neq j)$. This gives $x_1(x_2 + x_3) < -4/3$. On the other hand, $|x_1(x_2 + x_3)|^2 \leqq |x_1|^2 |x_2 + x_3|^2 < (4/3)(8/3 + 2x_2 x_3) < 16/9$, which gives a contradiction.

Let S be (α, β)-winning, and let a winning strategy $f_1, f_2, \cdots$ be given. Call a sequence of balls $E_1, E_2, \cdots$ a $t - f_1, f_2, \cdots$-chain, if there is an f_1, f_2-chain $B_1, B_2, \cdots$ such that $E_1 = B_1$, $E_2 = B_{1+t}$, $E_3 = B_{1+2t}, \cdots$. Finite t-chains are defined similarly. In other words a t-chain consists of every tth element of a chain.

LEMMA 21. *Let all the hypotheses of the theorem be satisfied. Let $E_1, E_2, \cdots, E_k$ be a $t - f_1, f_2, \cdots$-chain. Then there are m balls $E^{(0)}_{k+1}, \cdots, E^{(m-1)}_{k+1}$ with pairwise disjoint interiors such that each of the sequences $E_1, \cdots, E_k, E^{(j)}_{k+1}$ $(j = 0, 1, \cdots, m-1)$ is a finite $t - f_1, f_2, \cdots$-chain.*

Proof. Let $B_1, \cdots, B_{1+(k-1)t}$ be a $f_1, f_2, \cdots$-chain with $E_1 = B_1, \cdots, E_k = B_{1+(k-1)t}$. Define $h_i(C_1, \cdots, C_i)$ $(i = 1, \cdots, t)$ by $h_i(C_1, \cdots, C_i) = f_{i+(k-1)t}(B_1, \cdots, B_{(k-1)t}, C_1, C_2, \cdots, C_i)$. Put $C_1 = B_{1+(k-1)t} = E_k$. Now let $g^{(0)}, \cdots, g^{(m-1)}$ be the functions of the theorem, and define $C^{(j)}_2, \cdots, C^{(j)}_{t+1}, D^{(j)}_1, \cdots, D^{(j)}_t$ $(0 \leqq j \leqq m-1)$ by (24) and (25). Put $E^{(j)}_{k+1} = C^{(j)}_{t+1}$. Then obviously $E_1, \cdots, E_k$, $E^{(j)}_{k+1}$ is a $t - f_1, f_2, \cdots$-chain for $0 \leqq j \leqq m-1$, and the m balls $E^{(j)}_{k+1}$ have pairwise disjoint interiors.

LEMMA 22. *Let all the hypotheses of the theorem be satisfied. There are balls $C_1(i_1)$, $C_2(i_1, i_2), \cdots$, defined for digits $i_j = 0, 1, \cdots, m-1$, such that*

$$C_1(i_1), C_2(i_1, i_2), C_3(i_1, i_2, i_3), \cdots$$

is a $t - f_1, f_2, \cdots$-chain for every sequence of digits $i_1, i_2, \cdots$, and where for given k the m^k balls $C_k(i_1, \cdots, i_k)$ have pairwise disjoint interiors and radius $(\alpha\beta)^{kt}$.

Proof. Let $C_1(i_1)$ be any m disjoint balls of radius $(\alpha\beta)^t$. The construction of $C_2(i_1, i_2)$, $C_3(i_1, i_2, i_3), \cdots$ is by induction, using the previous lemma.

Proof of Theorem 6. Given a sequence of digits $i_1, i_2, \cdots$ there is a unique point $x = x(i_1, i_2, \cdots)$ contained in all the balls $C_k(i_1, \cdots, i_k)$, $k = 1, 2, \cdots$ of Lemma 22. Obviously $x \in S$. The set of all points x so obtained will be denoted by S^*.

Define a possibly many-valued function f from S^* onto the unit-interval $U: 0 \leqq y \leqq 1$, as follows. Given $x \in S^*$, let $f(x)$ consist of all numbers $y = 0, i_1 i_2 \cdots$ (written in scale m) such that $x = x(i_1, i_2, \cdots)$. For a set $T \subset S^*$ let $f(T)$ be the union of all sets $f(x)$ where $x \in T$. For a general set R define $f(R) = f(R \cap S^*)$. Now if balls B_l $(l = 1, 2, \cdots)$ cover S, the sets $B_l \cap S^*$ cover S^*, and the sets $f(B_l) = f(B_l \cap S^*)$ cover U. Hence the exterior Lebesgue measures $\bar{\mu}$ of $f(B_l)$ satisfy

$$\sum_{l=1}^{\infty} \bar{\mu}(f(B_l)) \geqq 1. \tag{27}$$

Let B have radius ρ, and put

$$j = [\log(2\rho\omega^{-1})/(t \log \alpha\beta)]. \tag{28}$$

For small ρ, j is positive, and

$$\rho < \omega(\alpha\beta)^{tj}.$$

Hence by Lemma 20, B has nonempty intersection with at most two of the balls $C_j(i_1, \cdots, i_j)$, say with $C_j(i_1(1), \cdots, i_j(1))$ and $C_j(i_1(2), \cdots, i_j(2))$. $f(B)$ contains only numbers whose first j digits are either $i_1(1), \cdots, i_j(1)$ or $i_1(2), \cdots, i_j(2)$. Thus $f(B)$ is contained in two intervals of length m^{-j}, and $\bar{\mu}(f(B)) \leqq 2m^{-j}$.

Now suppose the balls $B_1, B_2, \cdots$ of radius $\rho_1, \rho_2, \cdots$ cover S. By (27),

$$1 \leqq 2 \sum_{l=1}^{\infty} m^{-j_l},$$

where j_l is defined by a formula like (28). This implies

$$1 \leqq 2m \sum_{l=1}^{\infty} (2\omega^{-1}\rho_l)^{\log m / |t \log \alpha\beta|}.$$

We obtain $\{S\}^{\alpha} > 0$ with $\alpha = \log m / |t \log \alpha\beta|$, and the theorem is proved.

LEMMA 23. *Let $2\beta < 1 + \alpha\beta$ and let S be (a, β)-winning in a Hilbertspace M of positive dimension. The intersection of S with any ball contains continuum-many points.*

Proof. The proof of Corollary 3 of Theorem 6 shows that this theorem is applicable with some $m > 1$ and with $C_{t+1}^{(0)}, \cdots, C_{t+1}^{(m-1)}$ disjoint. Under this assumption the m^j balls $C_j(i_1, \cdots, i_j)$ of Lemma 22 will be pairwise disjoint. One may also require that all the balls $C_j(i_1, \cdots, i_j)$, $j = 1, 2, \cdots$, be contained in an arbitrary fixed ball B if one drops the inessential requirement on the radii of these balls. The points $x(i_1, i_2, \cdots)$ will now be continuum—many distinct points of $S \cap B$.

13. **Positional winning strategies.**

THEOREM 7. *Let $S \subset M$ be an $(\mathfrak{F}, \mathfrak{G})$-winning set. Then there exists a positional $(\mathfrak{F}, \mathfrak{G}; S)$-winning strategy.*

Proof. Introduce a well-ordering $\prec$ into the set Ω. Let $f_1, f_2, \cdots$ be an $(\mathfrak{F}, \mathfrak{G}; S)$-winning strategy. We are going to define $f \in F_1$ as follows.

Let $B \in \Omega$. If B does not occur in any $f_1, f_2, \cdots$-chain, define $f(B)$ arbitrarily. Now assume B does occur in $f_1, f_2, \cdots$-chains

$$B_1, B_2, \cdots, B_k = B.$$

Of all the B_1 which occur in such chains, there is one which is smallest with regard to $\prec$. Denote it by $B_1(B)$. There are chains

$$B_1(B), B_2, \cdots, B_h = B.$$

Of all the B_2 which occur in such chains, there is a smallest one, $B_2(B)$, and so on. Now either

(a) there is a k with $B_k(B) = B$. Then

$$B_1(B), \cdots, B_k(B) = B$$

is an $f_1, f_2, \cdots$-chain.

(aa) There is no f_1, f_2-chain $C_1, C_2, \cdots$ where each C_i is one of the elements $B_1(B), \cdots, B_k(B)$. If for every m there were a finite $f_1, f_2, \cdots$-chain $C_1, \cdots, C_m$ with this property, then because of Lemma 1 there would also be an infinite such chain $C_1, C_2, \cdots$. Let $B_1(B), \cdots, B_k(B), C_1, \cdots, C_{i_0}$ be an f_1, f_2-chain with each C_i among the $B_1(B), \cdots, B_k(B)$, such that there is no longer such chain. Set

$$f(B) = f_{k+i_0}(B_1(B), \cdots, B_k(B), C_1, \cdots, C_{i_0}).$$

If $B' \in \mathfrak{G}(f(B))$, B' differs from $B_1(B), \cdots, B_k(B)$.

(ab) There is an $f_1, f_2, \cdots$-chain $C_1, C_2, \cdots$ with each C_i among $B_1(B), \cdots, B_k(B)$. In this case $\bigcap_{i=1}^k \alpha(B_i(B)) \subset \bigcap_n \alpha(C_n) \subset S$. Hence $\alpha(B) = \alpha(B_k(B)) \subset S$, and $f(B)$ can be any element of $\mathfrak{F}(B)$.

(b) There is no k having $B_k(B) = B$. Then $B_1(B), B_2(B), \cdots$ is an f_1, f_2-chain, and $\alpha(B) \subset \bigcap \alpha(B_n) \subset S$. Again $f(B)$ can be arbitrary in $\mathfrak{F}(B)$.

We are going to show that the functions $\bar{f}_n(B_1, \cdots, B_n)$ defined by $\bar{f}_n(B_1, \cdots, B_n) = f(B_n)$ $(n = 1, 2, \cdots)$ are a winning strategy. Let $B_1 \in \Omega'$,

$$B_n \in \mathfrak{G}(W_{n-1}) \qquad (n = 2, 3, \cdots),$$

$$W_n = f(B_n) \qquad (n = 1, 2, \cdots).$$

If for some B_n, case (ab) or (b) happens, $\alpha(B_n) \subset S$, and we are through. Thus for every B_n, assume (aa) holds.

$$B_1(B_n), \cdots, B_k(B_n),\ C_1, \cdots, C_{i_0},\ B_{n+1}$$

is an $f_1, f_2, \cdots$-chain, and B_{n+1} differs from $B_1(B_n), \cdots, B_k(B_n)$.

$$B_1(B_{n+1}) \prec B_1(B_n) \qquad (n = 1, 2, \cdots). \tag{29}$$

There is an i_1 where $B_1(B_{i_1})$ is smallest with regard to $\prec$. By (29), $B_1(B_i) = B_1(B_{i_1}) = \overline{B_1}$, say, if $i \geqq i_1$. Now for $i > i_1$, B_i differs from $B_1(B_{i-1}) = \overline{B_1} = B_1(B_i)$. Thus $B_2(B_i)$ is defined. There is an $i_2 > i_1$ such that $B_2(B_{i_2}) = B_2(B_i)$ for $i \geqq i_2$. In this fashion one finds $i_1 < i_2 < \cdots$ such that $B_t(B_i)$ is defined for $i > i_{t-1}$ and $B_t(B_i) = B_t(B_{i_t}) = \overline{B_t}$ for $i \geqq i_t$. $\overline{B_1}, \overline{B_2}, \cdots$ is an $f_1, f_2, \cdots$-chain by Lemma 1.

$$\bigcap_{i=1}^{\infty} \alpha(B_i) \subset \bigcap_{i=1}^{\infty} \alpha(\overline{B_i}) \subset S$$

gives the desired conclusion.

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