ON MILNOR'S INVARIANT FOR LINKS

BY
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1. **Introduction.** Let C_n be the space consisting of n disjoint (oriented) circles. By an (oriented) *link* of multiplicity n in 3-space R^3 is meant a homeomorphic image of C_n in R^3.

Two links l and l' of multiplicity n and n' in R^3 are said to be of the *same type* if there exists an orientation preserving homeomorphism of R^3 onto itself that maps l onto l' (and hence necessarily $n = n'$). A link is *polygonal* if it has a polygonal representative. Further, two links l and l' are said to be *isotopic* if there exists a continuous family h_t, $0 \leqq t \leqq 1$, of homeomorphisms of C_n into R^3 with $h_0(C_n) = l$ and $h_1(C_n) = l'$. These two concepts are not necessarily equivalent. In fact, the group of a link l, $G = \pi_1(R^3 - l)$, is an invariant of the link type but is not an isotopy invariant of the link.

In 1952, K.T. Chen investigated the lower central series $\{G_q\}$ of the group G of a polygonal link and proved that G/G_q is an isotopy invariant for all q [1]. (G_q is defined inductively by $G_1 = G$, $G_{i+1} = [G, G_i]$, $i = 1, 2, \cdots$, where $[G, G_i]$ is the subgroup generated by all $aba^{-1}b^{-1}$ with $a \in G$, $b \in G_i$.) Later J. Milnor generalized this result for the group $\mathscr{G} = \pi_1(\mathscr{M} - l)$, where $\mathscr{M}$ is an arbitrary open orientable 3-manifold and l is a link in $\mathscr{M}$ which is not necessarily polygonal [6]. In particular, for a link in 3-space R^3, Milnor defined a numerical invariant $\bar{\mu}(i_1 \cdots i_k)$, where $i_1 \cdots i_k$ is a sequence of positive integers between 1 and n. $\bar{\mu}$ will be called *Milnor's invariant* in this paper. $\bar{\mu}(ij)$, $i \neq j$, is the linking number of the ith component l_i and jth component l_j of l.

On the other hand, R.H. Fox defined the polynomial $\Delta(x_1, \cdots, x_n)$ with integral coefficients for a given link l of multiplicity n based on a presentation of the group G of l by means of his own free differential calculus [2], [3]. It is now called the *Alexander polynomial* of l. This is the natural generalization of the so called Alexander polynomial of a knot, (a knot being a link of multiplicity one). As is well known, the Alexander polynomial $\Delta(x_i, x_j)$ of a link $l_i \cup l_j$ of multiplicity two evaluated at $x_i = x_j = 1$ coincides up to sign with the linking number of l_i and l_j [7]. Therefore, we can write

$$\left| \Delta(x_i, x_j)_{x_i = x_j = 1} \right| = \left| \bar{\mu}(ij) \right| .$$

This immediate relation between $\Delta(x_i, x_j)$ and $\bar{\mu}$ is the only one obtained up to the present time.

Received by the editors August 12, 1965.

In this paper we shall establish some relations between the *partial derivatives* of $\Delta(x_1,\cdots,x_n)$ and $\bar{\mu}(i_1\cdots i_k)$ (Theorems 4.1–4.3). Since $\Delta(x_1,\cdots,x_n)$ is an invariant of $G/[G_2,G_2]$, it follows that for sequences treated in our theorems $\bar{\mu}(i_1\cdots i_k)$ depends on $G/[G_2,G_2]$ rather than on G/G_q.

Noting that $\bar{\mu}(i_1\cdots i_k)$ is defined by means of free differentiation, our relations seem to be quite natural. However, the delicate differences arise from the fact that while the usual partial differentiation is commutative, free differentiation is not. Fortunately, these differences do not cause any serious difficulties when links of multiplicity two are treated. Nevertheless the proof of Theorem 4.1 (§§6–8) is quite complicated. The form of Theorems 4.2 and 4.3 is chosen to avoid unnecessary complications. However the direction in which these relations can be generalized will be indicated.

2. **The Alexander polynomial.** Let $l=l_1\cup l_2\cup\cdots\cup l_n$ be an oriented polygonal link of multiplicity n ($\geqq 2$) in 3-space R^3. Let $G=\pi_1(R^3-l)$ be the group of l and let $\mathscr{P}$ be the *Wirtinger presentation* determined by a link projection:

$$\mathscr{P}=(x_{i,j}\colon r_{i,j},\ 1\leqq i\leqq n, 1\leqq j\leqq \lambda_i),$$

where $x_{i,j}$ is represented by a loop going once around an arc of the ith component l_i of l in the positive direction and $r_{i,j}=u_{i,j}x_{i,j}u_{i,j}^{-1}x_{i,j+1}^{-1}$(1), $u_{i,j}=x_{p,q}$ or $x_{p,q}^{-1}$.

Now consider a set of elements $s_{i,j}$ defined by $s_{i,j}=v_{i,j}x_{i,1}v_{i,j}^{-1}x_{i,j+1}^{-1}$, where $v_{i,j}=u_{i,j}u_{i,j-1}\cdots u_{i,1}$. Then it is an elementary matter to show that the set $\{r_{i,j}\}$ can be replaced by the set $\{s_{i,j}\}$ so that a new presentation $\mathscr{S}$ of G is obtained [2]:

$$\mathscr{S}=(x_{i,j}\colon s_{i,j},\ 1\leqq i\leqq n, 1\leqq j\leqq \lambda_i).$$

$\mathscr{S}$ will be called the *standard presentation* of G with respect to $\mathscr{P}$.

Let F be the free group generated by $x_{i,j}$, $1\leqq i\leqq n$, $1\leqq j\leqq\lambda_i$. Let ϕ be the canonical homomorphism of F onto G. ϕ can be uniquely extended to the ring homomorphism of the integral group rings JF onto JG(2).

Let A_n be a free abelian group of rank n with a basis $\{x_1,\cdots,x_n\}$. Then a homomorphism $\psi\colon G\to A_n$ defined by $x_{i,j}^{\psi}=x_i$(3), $1\leqq i\leqq n$ and $1\leqq j\leqq\lambda_i$, can be uniquely extended to the ring homomorphism $JG\to JA_n$.

Let M be the Jacobian matrix of $\mathscr{S}$ at $\psi\phi$ [3, II],

$$M=\left\|\left(\frac{\partial s_{i,j}}{\partial x_{k,l}}\right)^{\phi\psi}\right\|_{i,j,k,l},$$

$$\left(\frac{\partial s_i}{\partial x_{k,l}}\right)^{\phi\psi}=(1-x_l)\left(\frac{\partial v_{i,j}}{\partial x_{k,l}}\right)^{\phi\psi}+\delta_{i,k}(\delta_{1,l}v_{i,j}^{\phi\psi}-\delta_{j+1,l}),\tag{2.1}$$

where $\delta_{p\,q}$ is Kronecker's delta.

(1) The second index is mod λ_i.

(2) Any group homomorphism: $G\to H$ determines uniquely the ring homomorphism of integral group rings: $JG\to JH$. These two homomorphisms always denoted by the same letter.

(3) $x_{i,j}^{\psi}$ denotes the image of $x_{i,j}$ under ψ and $x_{i,j}^{\phi\psi}=(x_{i,j}^{\phi})^{\psi}$.

M is a square matrix of order $\sum_{i=1}^{n}\lambda_i$. Each row of M corresponds to a relator $s_{i,j}$ and each column to a generator $x_{k,l}$ in $\mathcal{S}$. These will be called the $s_{i,j}$-rows and $x_{k,l}$-columns respectively.

Let $\tilde{M}(s_{i,j}: x_{k,l})$ denote the square matrix obtained from M by deleting the $s_{i,j}$-row and $x_{k,l}$-column(4). Then g.c.d.$_{s_{i,j},x_{k,l}}\{\det \tilde{M}(s_{i,j}: x_{k,l})\}$ is uniquely determined up to a factor $\pm x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ [3, II]. It does not depend on the presentation of G and is denoted by $\Delta(x_1,\cdots,x_n)$. $\Delta(x_1,\cdots,x_n)$ is, in fact, an invariant of link types and is called the *Alexander polynomial* of l. For properties of the Alexander polynomial, the reader should refer to [7].

Now $\det\tilde{M}(s_{i,j}: x_{k,l})$ is a polynomial in $x_1,\cdots,x_n$ with a possibility of having negative exponents. Two such polynomials f and g are said to be *equivalent*, written $f\sim g$, if $f=\varepsilon x_{x_1}^{\alpha_1}\cdots x_n^{\alpha_n}g$ for some integers $\alpha_1,\cdots,\alpha_n$, where $\varepsilon=\pm 1$ or 0.

In the following, we will show that $\det \tilde{M}(s_{1,\lambda_1}: x_{n,1})/(1-x_n)$ may be specified as the Alexander polynomial of l.

First we will prove the following lemma.

LEMMA 2.1.

$$\det \tilde{M}(s_{i,j}: x_{k,l}) \sim P(i,j: k,l: p)\det \tilde{M}(s_{p,\lambda_p}: x_{k,l}),$$

where P is a polynomial in $x_1,\cdots,x_n$ and $P=1$ if $j=\lambda_i$.

Proof. As is well known ([2] or [6, Lemma 5]), given the Wirtinger presentation $\mathcal{P}$ of G, there are elements $w_{i,j}$ of F such that

$$\prod_{i=1}^{n}\prod_{j=1}^{\lambda_i} w_{i,j}r_{i,j}w_{i,j}^{-1}=1 \quad \text{in } F. \tag{2.2}$$

Since any relator $r_{i,j}$ is represented by means of relators $s_{k,l}$ in $\mathcal{S}$ as $r_{i,j}=u_{i,j}s_{i,j-1}^{-1}u_{i,j}^{-1}s_{i,j}$ $(s_{i,0}=1)$ it follows from (2.2) that s_{p,λ_n} is a consequence of the others. Thus $\partial s_{p,\lambda_p}/\partial x_{k,l}$ is a linear combination of other $\partial s_{j,m}/\partial x_{k,l}$. Thus

$$\det \tilde{M}(s_{i,j}: x_{k,l}) \sim P(i,j: k,l: p)\det \tilde{M}(s_{p,\lambda_n}: x_{k,l}).$$

It is obvious that $P=1$ if $j=\lambda_i$.

From (2.1) and Lemma 2.1, we see that $\det\tilde{M}(s_{i,j}: x_{k,l})\equiv 0 \mod 1-x_k$. We define $N(s_{i,j}: x_{k,l})=\det M(s_{i,j}: x_{k,l})/(1-x_k)$. N is a polynomial in $x_1,\cdots,x_n$.

Now the fundamental formula [3, I.(2.3)] implies that $N(s_{i,j}: x_{k,l})\sim N(s_{i,j}: x_{p,q})$. Further, from Lemma 2.1, it follows immediately that $N(s_{i,\lambda_i}: x_{k,l})\sim N(s_{1,\lambda_1}: x_{k,l})$. Thus, g.c.d. $\{\det M(s_{i,j}: x_{k,l})\}\sim N(s_{1,\lambda_1}: x_{n,1})$. In other words, we have proved the following

LEMMA 2.2. $\Delta(x_1,\cdots,x_n)\sim N(s_{1,\lambda_1}: x_{n,1})$.

$N(s_{1,\lambda_1}: x_{n,1})$ is specified, hereafter, as the Alexander polynomial of l and is denoted by $\Delta(x_1,\cdots,x_n)$. Since each element of the s_{n,λ_n}-row in $\tilde{M}(s_{1,\lambda_1}: x_{n,1})$ has

(4) More generally $\tilde{M}(\alpha_1,\ldots,\alpha_p: \beta_1,\ldots,\beta_q)$ denotes the matrix obtained from M by deleting the α_1-,..., α_p-rows and β_1-,..., β_q-columns.

the factor $1-x_n$, the Alexander polynomial is the determinant obtained from $\tilde{M}$ by dividing the s_{n,λ_n}-row by $1-x_n$. In the case $n=2$, our specification is justified by the following

(2.3) $\Delta(1,1)$ *equals the linking number.*

The proof will be given in §4.

The Alexander polynomial of a subset of l is discussed in detail by Torres in his paper [7]. We only mention without proof the following lemma.

Let $l_{i_1} \cup l_{i_2} \cup \cdots \cup l_{i_p}$ be a subset of l and let $\Delta(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$ be its Alexander polynomial. Let $M_{i,j}$ denote the submatrix of M consisting of the $s_{i,1}$-, $\cdots$, s_{i,λ_i}-rows and the $x_{j,1}$-, $\cdots$, x_{j,λ_j}-columns. Then we have

LEMMA 2.3.

$$\pm\ x_{i_1}^{\alpha_1} \cdots x_{i_p}^{\alpha_p} \Delta\ (x_{i_1}, \cdots, x_{i}\ ) \ =\ \det \left\| \tilde{M}_{i_k, i_p}(s_{i_n, \lambda_{ip}} : x_{i,1}^{\phi})^{\psi\Phi} \right\|, \qquad 1 \leqq k, l \leqq p,$$

where Φ is a homomorphism $JA_n \to JA_p$ defined by

$$x_j^{\Phi} \ = \ 0 \quad \textit{for } j \neq i_1, \cdots, i_p,$$

$$x_j^{\Phi} \ = \ x_j \quad \textit{for } j = i_1, \cdots, i_p.$$

3. **Milnor's invariant.** Let $\mathscr{S} = (x_{i,j} : s_{i,j})$ be the standard presentation of $G = \pi_1(R^3 - l)$ with respect to the Wirtinger presentation given in §2. Let F_n be the free group generated by $x_1, \cdots, x_n$, and let A_n be the commutator quotient group of F_n, hence A_n is the free abelian group of rank n with a basis $\{x_1, \cdots, x_n\}$. Let ϕ be the canonical homomorphism of F_n onto A_n. ϕ can be uniquely extended to the ring homomorphism $JF_n \to JA_n$.

Now, we shall define the homomorphisms $\theta_p : F \to F_n$ by induction on p as follows [6].

$$x_{i,j}^{\theta_1} = \ x_i \quad \text{for } 1 \leqq i \leqq n, 1 \leqq j \leqq \lambda_i,$$

$$x_{i\,1}^{\theta_{p+1}} = \ x_i, \tag{3.1}$$

$$x_{i,j+1}^{\theta_{p+1}} = \ (v_{i,j} x_{i,1} v_{i,j}^{-1})^{\theta_p} \quad \text{for } 1 \leqq i \leqq n,\ 1 \leqq j < \lambda_i.$$

θ_p will be extended to the ring homomorphism $JF \to JF_n$. The particular homomorphism $\phi\theta_1 : JF \to JA_n$ will be denoted by ρ, i.e. $x_{i,j}^{\rho} = x_i$.

The trivializer $o : JG \to J$ is a homomorphism defined by $(\sum_i a_i x_{i_1}^{\alpha_1} \cdots x_{i}^{\alpha_p})^o = \sum_i a_i$.

Then we have the following commutative diagram:

$$\begin{tikzcd} & JF_n \arrow[dr, "o"] \arrow[dd, "\phi"] & \\ JF \arrow[ur, "\theta_p"] \arrow[dr, "\rho"'] & & J \\ & JA_n \arrow[ur, "o"'] & \end{tikzcd}$$

Let $\eta_k = v_{k,\lambda_k}$. It represents a *parallel* of the kth component of l. Then, following Milnor, we define an integer $\mu(i_1 \cdots i_p k)$ for a sequence $i_1 \cdots i_p k$ ($p \geqq 1$) of integers between 1 and n as follows [6].

$$\mu(i_1 \cdots i_p k) = \left(\frac{\partial^p \eta_k^{\theta_p}}{\partial x_{i_1} \cdots \partial x_{i_p}} \right)^o. \tag{3.2}$$

Let $\bar{\Delta}(i_1 \cdots i_r) = \text{g.c.d. } \mu(j_1 \cdots j_s)$, where $j_1 \cdots j_s$ ($2 \leqq s < r$) is to range over all sequences obtained by cancelling at least one of the indices $i_1, \cdots, i_r$ and permuting the remaining indices cyclically. Then Milnor proved that

$$\bar{\mu}(i_1 \cdots i_p k) \equiv \mu(i_1 \cdots i_p k) \mod \bar{\Delta}(i_1 \cdots i_p k) \tag{3.3}$$

is an isotopy invariant of a link.

$\bar{\mu}$ will be called *Milnor's invariant* in this paper. Further, let

$$\Delta^*(i_1 \cdots i_r) = \text{g.c.d. } \mu(j_1 \cdots j_s), \quad 2 \leqq s < r,$$

where $j_1 \cdots j_s$ is to range over all permutations of proper subsequences of $i_1 \cdots i_r$. Since $\Delta^*(i_1 \cdots i_r)$ divides $\bar{\Delta}(i_1 \cdots i_r)$, it follows that

$$\mu^*(i_1 \cdots i_p k) \equiv \mu(i_1 \cdots i_p k) \mod \Delta^*(i_1 \cdots i_p k) \tag{3.4}$$

is also an isotopy invariant of a link.

μ^* will be called the *weak Milnor invariant.*

REMARK 1. Milnor proved [6] that μ^* and $\bar{\mu}$ are isotopy invariants of (not necessarily polygonal) links. However, we consider only polygonal links in 3-space R^3, because we are concerned about relations with the Alexander polynomials which may not be defined for "wild" links.

REMARK 2. As Milnor pointed out [6], $\mu^*(i_1 \cdots i_r)$ is an invariant of the *homotopy type* of a link in the sense of [5] provided $i_1, \cdots, i_r$ are mutually distinct.

Now many of the relations among $\bar{\mu}(i_1 \cdots i_r)$ and $\mu^*(i_1 \cdots i_r)$ for various sequences $i_1 \cdots i_r$ are found in [6]. Those relations needed to prove our theorems will be collected in §5. The notation used for Milnor's invariant is justified by the following lemma which shows that $\bar{\mu}(i_1 \cdots i_r)$ depends only on the sublink $l_{i_1} \cup \cdots \cup l_{i_r}$.

LEMMA 3.1. *Let l' be a sublink of l multiplicity m. We may assume without loss of generality that $l = l_1 \cup \cdots \cup l_m$, $2 \leqq m < n$. Let $j_1 \cdots j_p k$ be a sequence of integers between 1 and m. Then two integers $\mu(j_1 \cdots j_p k)$ and $\mu'(j_1 \cdots j_p k)$ are defined by means of $\mathcal{S}$, the standard presentation of $G = \pi_1(R^3 - l)$, and $\mathcal{S}'$, one such of $G' = \pi_1(R^3 - l')$. Then*

$$\mu(j_1 \cdots j_p k) = \mu'(j_1 \cdots j_p k). \tag{3.5}$$

The proof is straightforward.

4. **Main theorems.** Let $f(x_1, \cdots, x_n)$ be an element of the integral group ring of the free group F_n. As usual the *free derivative* of $f(x_1, \cdots, x_n)$ with respect

to x_i is denoted by $\partial f/\partial x_i$. On the other hand, since $f(x_1,\cdots,x_n)^\phi$ is an element of the integral polynomial ring JA_n, we can define the *partial derivative* of f with respect to x_i, which will be denoted by df^ϕ/dx_i throughout this paper. By the *trivializer* o of a function $f(x_1,\cdots,x_n)^\phi$ is meant the homomorphism of JA_n into J, the ring of integers, defined by $[f(x_1,\cdots,x_n)^\phi]^o = f(1,\cdots,1)$.

The main theorem of this paper is Theorem 4.1 which establishes a relation between the Alexander polynomial and the weak Milnor invariant for a link of multiplicity two. (The Alexander polynomial is an element of JA_n.)

For the sake of simplicity, hereafter, the particular sequence $1\cdots 12\cdots 2$, where there are a 1's and b 2's, will be denoted by $[a,b]$.

THEOREM 4.1. *Let l be a link of multiplicity two and let $\Delta(x,y)$ be the Alexander polynomial of l, which is specified in* §2. *Then for all $p,q \geqq 0$,*

$$\bar{\mu}([p+1,q+1]) \equiv (-1)^q \frac{1}{p!}\frac{1}{q!}\left[\frac{d^{p+q}}{dx^p dy^q}\Delta(x,y)\right]^o \quad \bmod \Delta^*([p+1,q+1]). \tag{4.1}$$

First observe that Theorem 4.1 is true in the simplest case $p=q=0$. In fact, by [3, I.(2.6)] or Lemma 5.1,

$$\bar{\mu}(12) = \left(\frac{\partial \eta_2^{\theta_1}}{\partial x_1}\right)^o = \sum_{i,j}\left(\frac{\partial \eta_2}{\partial x_{i\,j}}\right)^{\theta_1 o}\left(\frac{\partial x_{i,j}^{\theta_1}}{\partial x_1}\right)^o = \sum_{j=1}^{\lambda_1}\left(\frac{\partial \eta_2}{\partial x_{1\,j}}\right)^o = [\Delta(x,y)]^o,$$

which also proves (2.3). The proof of Theorem 4.1 will be done by induction on $p+q$ in §§6–8.

If $\Delta(x,y)$ is *normalized* by multiplying by $x^r y^s$, Theorem 4.1 remains true. This is verified straightforwardly. Therefore our reason in specifying the Alexander polynomial in §2 is to ensure that $\mu([p+1,\ q+1])$ coincides with the derivative of $\Delta(x,y)$ in Theorem 4.1 including the sign. If we disregard the sign of $\Delta(x,y)$ and consider only the absolute value of $\bar{\mu}$, we can choose $\Delta(x,y)$ arbitrary.

In the case where the multiplicity of l is greater than two, it is not so easy to establish relations between them. However, the following theorems indicate in what direction Theorem 4.1 will be generalized.

THEOREM 4.2(5).

$$\pm\frac{1}{p!}\left[\frac{d^p}{dx_1^p}\Delta(x_1,\cdots x_n)\right]^o \equiv \bar{\mu}([r_1]2)\bar{\mu}([r_2]3)\cdots\bar{\mu}([r_{n-1}+1]n) \quad \bmod \text{g.c.d.}\{\Delta^*([r_1]2),\Delta^*([r_2]3),\cdots,\Delta^*([r_{n-1}+1]n)\}, \tag{4.2}$$

where $r_1+\cdots+r_{n-1}=p$, $r_i \geqq 0$ and $\bar{\mu}(i)=0$ for any i.

(5) $[a]i$ stands for the sequence $1\cdots 1i$, where there are a 1's.

COROLLARY.

$$\pm \frac{1}{(n-2)!}\left[\frac{d^{n-2}}{dx_1^{n-2}}\Delta(x_1,\cdots,x_n)\right]^o = \bar\mu(12)\bar\mu(13)\cdots\bar\mu(1n).$$

REMARK(6). It is not hard to show that

$$\bar\mu([r]i) \equiv \binom{p}{r} \mod \Delta^*([r]i),$$

where p denotes the linking number of the first and ith components of a link. Therefore (4.2) may be stated independently of the Milnor invariants.

THEOREM 4.3. *Let $\Delta(x,y,z)$ be the Alexander polynomial of l of multiplicity three. Then*

$$(4.3)\quad \begin{aligned}\pm\left[\frac{d^3}{dx\,dy\,dz}\Delta(x,y,z)\right]^o &\equiv \mu(123)^2\\ &+ \bar\mu(112)\bar\mu(233)-\bar\mu(113)\bar\mu(223)-\bar\mu(122)\bar\mu(133) \mod \Delta^*(123).\end{aligned}$$

Every term in the right-hand side other than the first is a product of two Milnor's invariants of types considered in Theorem 4.1. Thus by Lemma 3.1 and (4.1), these invariants can be obtained from the Alexander polynomial of the corresponding link of two components. (See the above remark.) If $\mu^*(i_1\cdots i_p)=0$ for all sequences $i_1\cdots i_p$ of mutually distinct integers between 1 and n, then the link is homotopically trivial [5]. Thus Theorem 4.3 implies immediately the following

COROLLARY. *Let $l=l_1\cup l_2\cup l_3$ and let $\Delta(x_i,x_j)$ denote the Alexander polynomial of $l_i\cup l_j$. If $\Delta(x_i,x_j)=0$, $1\leqq i,j\leqq 3$, $i\neq j$ and $\Delta(x_1,x_2,x_3)=0$, then l is homotopically trivial.*

This corollary can not be generalized to the case $n\geqq 4$. In fact, there exists an almost trivial(7), but not trivial link for which the Alexander polynomial vanishes. The link illustrated by Figure 7 in [5, p. 190] is one of such links.

EXAMPLE. The Alexander polynomial $\Delta(x,y)$ of the link l considered in [6, p. 301] is $(-1)^{m-1}(1-x)^{2m-1}(1-y)$. Therefore

$$\begin{aligned}\bar\mu([2m,2]) &\equiv (-1)\frac{1}{(2m-1)!}\left[\frac{d^{2m}}{dx^{2m-1}dy}\Delta(x,y)\right]^o\\ &= (-1)^m \mod \Delta^*([2m,2]).\end{aligned}$$

In this case, $\Delta^*([2m,2])=0$.

5. **Preliminary lemmas.** In this section, we collect some lemmas which will be used frequently in the following sections. Many of them are formulas appearing in [3], [4], [6] or easy consequences.

(6) The author acknowledges to the referee for pointing out this remark.

(7) For the definition, see [5, p. 189].

LEMMA 5.1 (CHAIN RULE). *Let Y and Z be free groups generated by $y_1\cdots,y_n$ and $z_1,\cdots,z_m$ respectively, and let τ be a homomorphism JY into JZ. Then for any $f\in JY$,*

$$\frac{\partial f^\tau}{\partial z_k}=\sum_{j=1}^{n}\left(\frac{\partial f}{\partial y_j}\right)^\tau\left(\frac{\partial y_j^\tau}{\partial z_k}\right),\qquad 1\leqq k\leqq m \tag{5.1}$$

[3, I.(2.6)].

In the rest of this section, we assume that X is the free group generated by x and y, and ϕ is the canonical homomorphism $JX\to J(X/[X,X])$.

Let $i_1\cdots i_r$ and $j_1\cdots j_s$ be two sequences. By a *proper shuffle* of these two sequences is meant one of the $(r+s)!/r!s!$ sequences obtained by intermeshing $i_1\cdots i_r$ with $j_1\cdots j_s$ [6, p. 294]. Let $S(a,b)$ denote the set of all proper shuffles of two sequences $1\cdots 1$ (a times) and $2\cdots 2$ (b times). For the sake of brevity,

$$\frac{\partial^{a+b}f}{\partial z_{i_1}\cdots\partial z_{i_{a+b}}}$$

is denoted by $\partial^{a+b}f/\partial\omega$, where $\omega=i_1\cdots i_{a+b}\in S(a,b)$ and $z_{i_j}=x$ or y according as $i_j=1$ or 2, and

$$\frac{d^{a+b}f}{dx^a dy^b}$$

is denoted by $D^{a,b}f$.

LEMMA 5.2.

$$\left[\sum_{\omega\in S(a,b)}\frac{\partial^{a+b}f}{\partial\omega}\right]^o=\frac{1}{a!}\,\frac{1}{b!}\,(D^{a,b}f^\phi)^o,\ \textit{for any } f\in JX. \tag{5.2}$$

This follows from [3, I.(3.9)] and [4, (3.3)].

LEMMA 5.3.

$$(D^{n,0}f^\phi)^o=n\left[D^{n-1,0}\left(\frac{\partial f}{\partial x}\right)^\phi\right]^o\quad \textit{for } f\in JX \textit{ and } n\geqq 1. \tag{5.3}$$

This follows from Lemma 5.2.

LEMMA 5.4.

$$(-1)^q\bar\mu([p+1,q+1])\equiv\sum_{\omega\in S(a,b)}\bar\mu(\omega 12)\quad\mathrm{mod}\ \Delta^*([p+1,q+1]) \tag{5.4}$$

[6, (26)].

LEMMA 5.5. *Let $M=(a_{i,j})$ be an $n\times(n+1)$ integral matrix. Then (see(4)) for $1\leqq i\leqq n$,*

$$\det\tilde M(\ :i+1)=(-1)^{i-1}a_{i,1}\sum_{k=2}^{n+1}\det\tilde M(k-1:1,k)$$

$$+\sum_{k=2}^{n+1}(-1)^{i+j}a_{i,j}\sum_{k=2,k\neq j}^{n+1}\varepsilon\det\tilde M(k-1:j,k), \tag{5.5}$$

where $\varepsilon = 1$ or -1 according as $j < k$ or $j > k$.

Proof. Let

$$\begin{aligned} L_k = a_{i,1} \det \tilde{M}(k-1: 1, k) - a_{i,2} \det \tilde{M}(k-1: 2, k) \\ + \cdots + (-1)^{k-2} a_{i,k-1} \det \tilde{M}(k-1: k-1, k) \\ + (-1)^{k-1} a_{i,k+1} \det \tilde{M}(k-1: k+1, k) \\ + \cdots + (-1)^{n} a_{i,n+1} \det \tilde{M}(k-1: n+1, k). \end{aligned}$$

Then the right-hand side of (5.5) is equal to $\sum_{k=2}^{n+1}(-1)^{i-1}L_k$. Let M' be the $(n+1) \times (n+1)$ matrix obtained from M by adjoining a row which is identical to the ith row. Then it is evident that $(-1)^{n-1}L_k$ is the expansion of $\tilde{M}'(k-1: k)$ by minors of the last row which is adjoined. Thus $L_k = 0$ if $k \neq i+1$ and $(-1)^{n-1}L_{i+1} = (-1)^{n-i} \det \tilde{M}(\ : i+1)$. Therefore

$$\sum_{k=2}^{n+1} (-1)^{i} L_k = (-1)^{i-1} L_{i+1} = \det \tilde{M}(\ :\ i+1), \quad \text{q.e.d.}$$

6. **Proof of Theorem 4.1(1).** To simplify notation involved, throughout §§6–8, we will use x_j, y_k instead of $x_{1,j}$, $x_{2,k}$: R_j, S_k instead of $s_{1,j}$, $s_{2,k}$; A_j, B_k instead of $v_{1,j}$, $v_{2,k}$; ξ, η instead of η_1, η_2; α, β instead of λ_1, λ_2; and x, y instead of $x_{1,\lambda_1}, x_{2,\lambda_2}$. Therefore F is the free group with free generators $x_1, \cdots, x_\alpha$, $y_1, \cdots, y_\beta$; F_2 is a free group with two free generators x and y; A_2 is a free abelian group of rank two with a free basis $\{x, y\}$; and ϕ is a canonical homomorphism $JF_2 \to JA_2$. With these notations, we can write

$$\mu([p+1, q+1]) = \left(\frac{\partial^{p+q+1} \eta^{\theta_{n+\ \ +1}}}{\partial x^{p+1} \partial y^{q}}\right)^{o}.$$

Further, to avoid unnecessary effort we introduce the following abbreviated symbols.

$$\begin{aligned} X_i^{\theta_\lambda}(p,q) &= D^{p,q}\left(\frac{\partial x_i^{\theta_\lambda}}{\partial x}\right)^{\phi}, \\ Y_k^{\theta}(p,q) &= D^{p,q}\left(\frac{\partial y_k^{\theta_\lambda}}{\partial x}\right)^{\phi}, \\ C^{i}(p,q) &= D^{p,q} C_i^{\rho}, \\ C_z^{i}(p,q) &= D^{p,q}\left(\frac{\partial C_i}{\partial z}\right)^{\rho}, \\ H_z(p,q) &= D^{p,q}\left(\frac{\partial \eta}{\partial z}\right)^{\rho}, \end{aligned}$$

where $C = A$ or B and $z = x_j$ or y_k.

Now the proof of Theorem 4.1 will be done as follows.

First we note that $(-1)^q\bar{\mu}([p+1,q+1])$ is the sum of $\bar{\mu}(\omega 12)$, $\omega \in S(p,q)$ (§5(5.4)). Thus from (5.2), it follows that $(-1)^q\bar{\mu}([p+1,q+1])$ can be represented as the sum of $\sum_i H_{x_i}(p,q)$ and a linear combination of $H_{x_i}(r,s)$ and $H_{y_k}(r,s)$ with coefficients X_i and Y_k respectively. (§6, Lemma 6.1.) On the other hand,

$$(1/p!)\,(1/q!)\,D^{p,q}\Delta(x,y)$$

is the derivative of the determinant which is specified in §2. Then by using the inductive hypothesis, we will show in §7 that it is also represented as the sum of $\sum_i H_{x_i}(p,q)$ and a linear combination of $H_{x_i}(r,s)$ and $H_{y_k}(r,s)$ with coefficients Γ_i and Λ_k respectively (§7 (7.3)). Hence, the final step of the proof of Theorem is to show the equalities between X and Γ, and Y and Λ. This will be done in §8.

Now the first lemma to be proved is

LEMMA 6.1[8].

$$(-1)^q\,\bar{\mu}([p+1,q+1]) \equiv \frac{1}{p!}\,\frac{1}{q!}\sum_{i=1}^{\alpha} H_{x_i}(p,q)^o$$

$$+\frac{1}{p!}\,\frac{1}{q!}\sum_{(0,0)\leqq(r,s)<(p,q)}\sum \binom{p}{r}\binom{q}{s}\Bigg[\sum_{=2}^{\alpha} H_{x_i}(r,s)X_i^{\theta}(p-r,q-s)$$

$$(6.1) \qquad + \sum_{k=2}^{\beta} H_{y_k}(r,s)Y_k^{\theta}(p-r,q-s)\Bigg]^o$$

$$\mod \Delta^*([p+1,q+1]),$$

where $\theta = \theta_{p+q+1}$.

Proof. From (5.2), putting $f = \partial\eta^{\theta}/\partial x$, and (5.4), it follows that

$$(-1)^q\,\mu([p+1,q+1]) \equiv \frac{1}{p!}\,\frac{1}{q!}\left[D^{p,q}\left(\frac{\partial\eta^{\theta}}{\partial x}\right)^{\phi}\right]^o \mod \Delta^*([p+1,q+1]).$$

To obtain (6.1), we have only to apply the chain rule on $\partial\eta^{\theta}/\partial x$ and the rule for differentiating product, noting that $f^{\theta\phi} = f^{\rho}$ for any $f \in JF$, q.e.d.

In the following lemma, the recursive formulas for X_i and Y_k will be given.

LEMMA 6.2. *For* $i,k \geqq 1$, $\lambda \geqq 2$, $(p,q) \geqq (0,0)$,

$$(6.2)\ \text{(i)}\quad [X_{i+1}^{\theta_\lambda}(p,q)]^o = A^i(p,q)^o - p\sum_{j=1}^{\alpha} A_{x_j}^i(p-1,q)^o$$

$$-p\sum_{(0,1)\leqq(r,s)\leqq(p-1,q)}\sum \binom{p-1}{r}\binom{q}{s}\Bigg\{\sum_{j=1}^{\alpha-1} A_{x_{j+1}}^i(p-1-r,q-s)^o X_{j+1}^{\theta_{\lambda-1}}(r,s)^o$$

$$+\sum_{l=1}^{\beta-1} A_{y_{l+1}}^i(p-1-r,q-s)^o Y_{l+1}^{\theta_{\lambda-1}}(r,s)^o\Bigg\}.$$

[8] $(r,s) < (p,q)$ (or $(r,s) \leqq (p,q)$) means that $r \leqq p$, $s \leqq q$ and $r+s < p+q$ (or $r+s \leqq p+q$).

$$\text{(6.2) (ii)}\quad Y^{\theta_\lambda}_{k+1}(p,q)^o = -q\sum_{j=1}^{\alpha} B^k_{x_j}(p,q-1)^o$$

$$-q \sum_{(0,1)\leqq(r,s)\leqq(p,q-1)}\sum \binom{p}{r}\binom{q-1}{s}\left\{\sum_{j=1}^{\alpha-1} B^k_{x_{j+1}}(p-r,\, q-1-s)^o X^{\theta_\lambda-1}_{j+1}(r,s)^o\right.$$

$$\left. + \sum_{l=1}^{\beta-1} B^k_{y_{l+1}}(p-r,q-1-s)^o Y^{\theta_\lambda-1}_{l+1}(r,s)^o\right\}.$$

All the proofs are straightforward, hence omitted.

7. **Proof of Theorem 4.1 (II).** In §2, the Alexander polynomial $\Delta(x,y)$ is defined as the determinant $N(R_\alpha\colon y_1)$. Now let L be the determinant obtained from $N(R_\alpha\colon Y_1)$ by adding the x_2-, $\cdots, x_\alpha$-columns to x_1-column. Since

$$L = N(R_\alpha\colon y_1),$$

we can write $L = \Delta(x,y)$. Thus $D^{p,q}\Delta(x,y)$ is obtained by differentiating each row of L w.r.t. x and/or y such that the total number of differentiations w.r.t. x and y are exactly p and q respectively.

Let us write (see (4))

$$D_j(r,s) = [D^{r,s}\tilde{L}(S_\beta\colon x_j)]^o,$$
$$E_k(r,s) = [D^{r,s}\tilde{L}(S_\beta\colon y_k)]^o.$$

Now by expanding $D^{p,q}L$ by minors of the S_β-row, we have the following formula. (Note that $D_1(0,0) = (-1)^{\alpha+\beta-2}$.)

$$[D^{p,q}\Delta(x,y)]^o = \sum_{j=1}^{\alpha} H_{x_j}(p,q)^o$$

$$\text{(7.1)}\quad + (-1)^{\alpha+\beta-2}\sum_{(0,0)\leqq(r,s)<(p,q)}\sum \binom{p}{r}\binom{q}{s}\left\{\sum_{j=1}^{\alpha} H_{x_j}(r,s)^o D_1(p-1,q-s)\right.$$

$$+ \sum_{i=2}^{\alpha}(-1)^{i-1}H_{x_i}(r,s)^o D_i(p-r,q-s)$$

$$\left. + \sum_{k=2}^{\beta}(-1)^{\alpha+k}H_{y_k}(r,s)^o\, E_k(p-r,q-s)\right\}.$$

Since (4.1) is true for $(r,s) < (p,q)$ by the inductive hypothesis, it follows that

$$\text{(7.2)}\quad \frac{1}{r!}\,\frac{1}{s!}\,[D^{r,s}\Delta(x,y)]^o \equiv (-1)^s\,\bar{\mu}([r+1,s+1]) \equiv 0 \mod \Delta^*([p+1,q+1]).$$

Thus $\sum_{j=1}^{\alpha} H_{x_j}(r,s)^o$ can be represented as a linear combination on $H_{x_j}(t,u)^o$, $H_{y_k}(t,u)^o$ and $\sum_{i=1}^{\alpha} H_{x_i}(t,u)^o$, where $j,k \geqq 2$ and $(t,u) < (r,s)$. Hence, by using this fact repeatedly, we can conclude that $\sum_{i=1}^{\alpha} H_{x_i}(r,s)^o$ for $(r,s) < (p,q)$ may be represented as a linear combination of $H_{x_j}(t,u)^o$ and $H_{y_k}(t,u)^o$, $j,k \geqq 2$. (It should be noted that $\sum_{j=1}^{\alpha} H_{x_j}(0,0)^o \equiv \mu(12) \equiv 0 \bmod \Delta^*([p+1,q+1])$.) Therefore we obtain the following formula.

$$
\begin{aligned}
\frac{1}{p!}\,\frac{1}{q!}\,[D^{p,q}\Delta(x,y)]^o \equiv\ & \frac{1}{p!}\,\frac{1}{q!}\sum_{i=1}^{\alpha} H_{x_i}(p,q)^o \\
&+ \frac{1}{p!}\,\frac{1}{q!}\sum_{(0,0)\leqq(r,s)<(p,q)}\sum \binom{p}{r}\binom{q}{s}\left\{\sum_{j=2}^{\alpha} H_{x_j}(r,s)^o\Gamma_j^{p,q}(r,s)\right. \\
&\left.+\sum_{k=2}^{\beta} H_{y_k}(r,s)^o\Lambda_k^{p,q}(r,s)\right\} \\
&\bmod \Delta^*([p+1,q+1]),
\end{aligned}
\tag{7.3}
$$

where $\Gamma_j^{p,q}(r,s)$ and $\Lambda_k^{p,q}(r,s)$ are the resulting coefficients of $H_{x_j}(r,s)^o$ and $H_{y_k}(r,s)^o$ respectively. In particular, we define $\Gamma_j^{p,q}(p,q) = 1$ and $\Lambda_k^{p,q}(p,q) = 0$.

The above process of obtaining (7.3) from (7.1) leads us to the following recursion formulas for Γ_j and Λ_k. (The proofs are straightforward, hence omitted.)

LEMMA 7.1. *For* $i,k \geqq 2$, $(p,q) > (r,s) \geqq (0,0)$,

(7.4)

(i) $$\Gamma_i^{p,q}(r,s) = (-1)^{\alpha+\beta+i-1}D_i(p-r,q-s) + (-1)^{\alpha+\beta-1}\sum_{(r,s)<(t,u)<(p,q)}\sum \binom{p-r}{p-t}\binom{q-s}{q-u}\Gamma_i^{t,u}(r,s)D_1(p-t,q-u).$$

(ii) $$\Lambda_k^{p,q}(r,s) = (-1)^{\beta+k}E_k(p-r,q-s) + (-1)^{\alpha+\beta-1}\sum_{(r,s)<(t,u)<(p,q)}\sum \binom{p-r}{p-t}\binom{q-s}{q-u}\Lambda_k^{t,u}(r,s)D_1(p-t,q-u).$$

8. **Proof of Theorem 4.1 (Conclusion).** In this section all values will be considered mod $\Delta^*([p+1,q+1])$.

By comparing (6.1) and (7.3), we see that in order to prove Theorem 4.1 it is sufficient to show the following:

(8.1) *For* $i,k \geqq 2$, $(p,q) \geqq (r,s) > (0,0)$,

(i) $$X_i^{\theta_{p+q+1}}(r,s)^o \equiv \Gamma_i^{p,q}(p-r,q-s),$$

(ii) $$Y_k^{\theta_{p+q+1}}(r,s)^o \equiv \Lambda_k^{p,q}(p-r,q-s).$$

Now it is easy to check that (8.1) is true for $(p,q) < (1,1)$. Thus we may assume as the first inductive hypothesis that (8.1) is true for $(l,m) < (p,q)$. Further, since (8.1) is also true for $(r,s) < (1.1)$, we can assume as the second inductive hypothesis that (8.1) is true for $(t,u) < (r,s)$.

Now $X^{\theta}(r,s)^{o}$, $\theta=\theta_{p+q+1}$, and $\Gamma_i^{p,q}(r,s)$ are finite sums of products of terms $A_{x_{j+1}}^{i-1}(t,u)^{o}$, $A_{y_{l+1}}^{i-1}(t,u)^{o}$, $B_{x_{j+1}}^{k-1}(t,u)^{o}$ and $B_{y_{l+1}}^{k-1}(t,u)^{o}$. Let $X_i^{\theta}(r,s:\lambda)$ and $\Gamma_i^{p,q}(r,s:\lambda)$ denote the sums of "homogeneous" terms of degree λ. Similarly define $D_i(r,s:\lambda)$ and $E_k(r,s:\lambda)$. Then the proof of (8.1) is reduced to that of the following:

(8.2) *For any* $\lambda \geqq 1$,

(i) $$X_i^{\theta}(r,s:\lambda)^{o} \equiv \Gamma_i^{p,q}(p-r,q-s:\lambda),$$

(ii) $$Y_k^{\theta}(r,s:\lambda)^{o} \equiv \Lambda_k^{p,q}(p-r,q-s:\lambda).$$

(8.2) will be proved by induction on λ.

For the case $\lambda=1$, (8.2)(i) is true, because

$$\begin{aligned}\Gamma_i^{p,q}(p-r,q-s:1) &= (-1)^{\alpha+\beta+i-1}D_i(r,s:1)\\ &= A^{i-1}(r,s)-r\sum_{j=1}^{\alpha}A_{x_j}^{i-1}(r-1,s)\\ &= X_i^{\theta}(r,s:1)^{o}.\end{aligned}$$

Similarly (8.2)(ii) is true for $\lambda=1$.

We may therefore assume as the third inductive hypothesis that (8.2) is true for any $\nu<\lambda$. In the following, we will prove only (8.2)(i), since the other can be proved in the same way.

Now from (7.4)(i), we observe that

$$\begin{aligned}\Gamma_i^{p,q}(p-r,q-s:\lambda) &\equiv (-1)^{\alpha+\beta+i-1}D_i(r,s:\lambda)\\ &\quad+(-1)^{\alpha+\beta-1}\sum_{(p-r,q-s)\leqq(t,u)<(p,q)}\sum\binom{r}{p-t}\binom{s}{q-u}\\ &\quad\cdot\sum_{\nu=1}^{\lambda-1}\Gamma_i^{t,u}(p-r,q-s:\nu)D_1(p-t,q-u:\lambda-\nu).\end{aligned}\tag{8.3}$$

Since $D_j(t,u)^{o}=0$ unless the R_j-row of the determinant L is differentiated w.r.t. x or y, we can prove by means of Lemma 5.5 that[9]

$$\begin{aligned}D_i(r,s:\lambda) = \sum_{(0,0)\leqq(c,d)\leqq(r,s)}\sum(-1)^{i-1}\binom{r}{c}\binom{s}{d}\Big\{&\mathscr{A}^{i-1}(c,d)D_1(r-c,s-d:\lambda-1)\\ +(-c)\Big\{&\sum_{j=2}^{\alpha}(-1)^{j+1}A_{x_j}^{i-1}(c-1,d)D_j(r-c,s-d:\lambda-1)\\ &+\sum_{k=2}^{\beta}(-1)^{\alpha+k}A_{y_k}^{i-1}(c-1,d)\,E_k(r-c,s-d:\lambda-1)\Big\}\Big].\end{aligned}\tag{8.4}$$

By substituting from (8.4) in (8.3), $\Gamma_i^{p,q}(p-r,q-s:\lambda)$ can be represented as a linear combination of $D_1(a,b:\lambda-\nu)$ $(1\leqq\nu\leqq\lambda-1)$, $D_j(a,b:\lambda-1)$ $(j\geqq 2)$ and

---

[9] $\mathscr{A}^{i-1}(c,d)=A^{i-1}(c,d)-c\sum_{j=1}^{\alpha}A_{x_j}^{i-1}(c-1,d)$.

$E_k(a,b:\lambda-1)$ $(k \geqq 2)$ with coefficients $\delta_1(a,b:\lambda-v)$, $\delta_j(a,b:\lambda-1)$ and $\varepsilon_k(a,b:\lambda-1)$, respectively.

On the other hand, it follows from (6.1)(i) that

$$X_i^{\theta_p+q+1}(r,s:\lambda)^o \equiv (-r) \sum_{(0,0)\leqq(c,d)\leqq(r-1,s)} \sum \binom{r-1}{c}\binom{s}{d}$$
$$\cdot\left\{\sum_{j=2}^{\alpha} A_{x_j}^{i-1}(r-1-c,s-d)X_j^{\theta_p+q}(c,d:\lambda-1)^o\right.$$
$$\left.+\sum_{k=2}^{\beta} A_{y_k}^{i-1}(r-1-c,s-d)Y_k^{\theta_p+q}(c,d:\lambda-1)^o\right\}.$$

By the first inductive hypothesis, $X_j^{\theta_p+q}$ and $Y_k^{\theta_p+q}$ in the above formula may be replaced by $\Gamma_j^{p-1,q}$ and $\Lambda_k^{p-1,q}$ respectively. Further, since these Γ and Λ are linear combinations of D_i and E_k by Lemma 7.1, $X_i^{\theta_p+q+1}(r,s:\lambda)^o$ can be represented as a linear combination of $D_1(a,b:\lambda-v)$, $D_j(a,b:\lambda-1)$ and $E_k(a,b:\lambda-1)$ with coefficients $d_1(a,b:\lambda-v)$, $d_j(a,b:\lambda-1)$ and $e_k(a,b:\lambda-1)$ respectively. Thus the proof of (8.2)(i) is reduced to that of the following

(8.5)
$$\text{(i)}\quad \delta_1(a,b:\lambda-v) \equiv d_1(a,b:\lambda-v),$$
$$\text{(ii)}\quad \delta_j(a,b:\lambda-1) \equiv d_j(a,b:\lambda-1),$$
$$\text{(iii)}\quad \varepsilon_k(a,b:\lambda-1) \equiv e_k(a,b:\lambda-1).$$

Proof. (i) *The case* $v=1$.

$$\delta_1(a,b:\lambda-1) = (-1)^{\alpha+\beta+i-1}(-1)^{i-1}\binom{r}{r-a}\binom{s}{s-b}\mathscr{A}^{i-1}(r-a,s-b)$$
$$+(-1)^{\alpha+\beta-1}\binom{r}{a}\binom{s}{b}\Gamma_i^{p-a,q-b}(p-r,q-s:1)=0.$$

On the other hand, $d_1(a,b:\lambda-1)=0$.

The case $v \geqq 2$.

$$d_1(a,b:\lambda-v) = (-r)\sum_{(0,0)\leqq(c,d)\leqq(r-1,s)}\sum\binom{r-1}{c}\binom{s}{d}\sum_{j=2}^{\alpha} A_{x_j}^{i-1}(r-1-c,s-d)$$
$$\times\ (-1)^{\alpha+\beta-1}\binom{c}{a}\binom{d}{b}\Gamma_j^{p-1-a,q-b}(p-1-c,q-d:v-1)$$
$$+\ (-r)\sum_{(0,0)\leqq(c,d)\leqq(r-,s)}\sum\binom{r-1}{c}\binom{s}{d}\sum_{k=2}^{\beta} A_{y_k}^{i-1}(r-1-c,s-d)$$
$$\times\ (-1)^{\alpha+\beta-1}\binom{c}{a}\binom{d}{b}\Lambda_k^{p-a-1,q-b}(p-1-c,q-d:v-1)$$

(using (8.2) and the second inductive hypothesis.)

$$= (-1)^{\alpha+\beta-1} \binom{r}{a}\binom{s}{b} X_i^{\theta_{p+q-a+b+1}}(r-a, s-b: \nu)$$

$$= (-1)^{\alpha+\beta-1} \binom{r}{a}\binom{s}{b} \Gamma_i^{p-a,q-b}(p-r, q-s: \nu)$$

$$= \delta_1(a, b: \lambda - \nu).$$

(ii) $\delta_j(a, b: \lambda - 1)$

$$= (-1)^{\alpha+\beta+i-1}\binom{r}{r-a}\binom{s}{s-b} \{-(r-a)\}(-1)^{i+j} A_{x_j}^{i-1}(r-a-1, s-b)$$

$$= (-1)^{\alpha+\beta+j-1}(-r)\binom{r-1}{r-a}\binom{s}{s-b} A_{x_j}^{i-1}(r-a-1, s-b)$$

$$= d_j(a, b: \lambda - 1).$$

(iii) is clear.

Thus the proof of (8.5), hence that of (8.2)(i) is completed.

The proof of Theorem 4.1 is thus completed.

9. **Proof of Theorem 4.2.** In this section, we assume that l is a link of multiplicity $n \geqq 2$, and use the notation of §§2–3.

Let $\mathscr{S}$ be the standard presentation of the group of l w.r.t. a Wirtinger presentation. Let M be the Jacobian matrix of $\mathscr{S}$ at $\psi\phi$ and let N be the matrix obtained from $\tilde{M}(s_{1,\lambda_1}: x_{n,1})$ by adding the $x_{i,2}$-, $\cdots$, x_{i,λ_i}-columns to $x_{i,1}$-column for $1 \leqq i \leqq n-1$. Interchange rows and columns of N to obtain the new matrix N' having the s_{2,λ_2}-, $\cdots$, s_{n,λ_n}-rows and the $x_{1,1}$-, $\cdots$, $x_{n-1,1}$-columns in the top left corner. In other words, N' is of the form, $\| N_{ij} \|$, where

$$N_{11} = \left\| \frac{\partial s_{i,\lambda_i}}{\partial x_{k,1}} \right\| \quad (i \neq 1), \qquad N_{12} = \left\| \frac{\partial s_{i,\lambda_i}}{\partial x_{k,l}} \right\| \quad (l \neq 1),$$

$$N_{21} = \left\| \frac{\partial s_{i,j}}{\partial x_{k,1}} \right\| \quad (j \neq \lambda_i), \qquad N_{22} = \left\| \frac{\partial s_{i,j}}{\partial x_{k,l}} \right\| \quad (l \neq 1).$$

Then it is a straightforward matter to show the following

(9.1)
(i) Each of $(N_{11})^o$, $(N_{12})^o$ and $(N_{21})^o$ is a zero matrix.
(ii) $\det (N_{22})^o = (-1)^{\lambda-n}$, where $\lambda = \sum_{i=1}^n \lambda_i$.

Let L, L_{11} and L_{12} be matrices obtained from N', N_{11} and N_{12} by dividing the s_{n,λ_n}-row by $1 - x_n$ respectively.

Consider a derivative $D^p\Delta(x_1, \cdots, x_n)$(10). It is obtained from det L by differentiating each row w.r.t. x_1 in such a way that the total number of differentiations

(10) $D^p f = d^p f / dx_1^p$.

is exactly equal to p. If the s_{i,λ_i}-row of det $L(2 \leqq i \leqq n-1)$ is not differentiated, then $[D^p\Delta(x_1,\cdots,x_n)]^o = 0$. Therefore we obtain

$$[D^p\Delta(x_1,\cdots,x_n)]^o = 0 \quad \text{for } p < n-2. \tag{9.2}$$

Consider the case $p \geqq n-2$. It is easily verified that, for any q, the nonzero elements of $(D^q \det L_{11})^o$ occur only on the line just above the diagonal and in the last row, and that every element of the first column of $(D^q \det N_{21})^o$ is zero. Thus we obtain that

$$[D^p\Delta(x_1,\cdots,x_n)]^o = \sum_{q=0}^{p} \binom{p}{q} (D^q \det L_{11})^o(D^{p-q} \det N_{22})^o.$$

Explicitly, $(D^q \det L_{11})^o$ is of the form.

$$\begin{aligned}(D^q \det L_{11})^o = (-1)^{n-2} \sum \frac{q!}{r_1!\cdots r_{n-1}!} (D^{r_1}\eta_2^o)^o \cdots (D^{r_{n-2}}\eta_{n-1}^o)^o \\ \cdot \left\{D^{r_{n-1}}\left(\sum_{i=1}^{\lambda_1} \frac{\partial \eta_n}{\partial x_{i,1}}\right)^q\right\}^o,\end{aligned} \tag{9.3}$$

where the summation runs over all sequences $r_1,\cdots,r_{n-1}$ such that $r_1 + \cdots + r_{n-1} = q$. Hence the proof of Theorem 4.2 will be completed if the following Lemma is proved.

LEMMA 9.1.

(i) $$\frac{1}{p!}(D^p \det L_{11})^o \equiv \prod_{j=2}^{n-1} \bar{\mu}([r_{j-1}]j) \cdot \bar{\mu}([r_{n-1}+1]n) \mod \bar{\Delta}.$$

(ii) *For* $0 \leqq r < q$,

$$\frac{1}{r!}(D^r \det L_{11})^o \equiv 0 \mod \Delta,$$

where $\bar{\Delta} = \text{g.c.d.} \{\Delta^*([r_1]2),\cdots,\Delta^*([r_{n-1}+1]n)\}$.

Proof. (i) follows from (5.1), (5.2) and (5.3).

Proof of (ii). From (i), it follows that

$$\frac{1}{r!}(D^r \det L_{11})^o \equiv \prod_{j=2}^{n-1} \bar{\mu}([s_{j-1}]j) \cdot \bar{\mu}([s_{n-1}+1]n) \mod \bar{\Delta}.$$

However, it vanishes mod $\bar{\Delta}$, because at least one of $s_1,\cdots,s_{n-1}$, s_i say, is less than r_i, as is seen from the fact that $s_1 + \cdots + s_{n-1} < r_1 + \cdots + r_{n-1}$.

10. **Proof of Theorem 4.3.** Since the proof of Theorem 4.3 can be obtained in the same way as is used in §9, we omit the details. We have only to notice the following.

(10.1) The nonzero determinants in

$$\left[\frac{d^3}{dx_1 dx_2 dx_3}\Delta(x_1, x_2, x_3)\right]^o \bmod \Delta^*(123)$$

occur only when the s_{2,λ_2}-row is differentiated w.r.t. x_2, or x_1 and x_2, or x_2 and x_3.

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