INTEGRAL REPRESENTATION ALGEBRAS(1)

BY

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1. **Introduction.** Let RG denote the group ring of a finite group G over a commutative ring R. By an RG-module we shall mean a left finitely generated module which is R-torsion free. The *representation ring* $a(RG)$ is an abelian additive group defined by generators and relations: the generators are the symbols $[M]$, where M ranges over a full set of representatives of the isomorphism classes of RG-modules, with relations

$$[M] = [M'] + [M'']$$

whenever $M \cong M' \oplus M''$. Multiplication in $a(RG)$ is defined by forming tensor products of modules:

$$[M][N] = [M \otimes_R N],$$

where as usual G acts on the tensor product by the formula

$$g(m \otimes n) = gm \otimes gn, \qquad g \in G, m \in M, n \in N.$$

Let C be the complex field, and define the *integral representation algebra* $A(RG)$ by the formula

$$A(RG) = C \otimes_Z a(RG).$$

Such representation algebras have recently been studied by Conlon [1], Green [3], [4], and O'Reilly [10], for the special case in which R is a field. They have shown that under suitable hypotheses, the algebra $A(RG)$ is semisimple.

The present author investigated $a(RG)$ when R is a ring of integers (see [11], [12]). Of particular interest are the following choices for R:

Z (the ring of rational integers),

Z_p (the p-adic valuation ring in the rational field Q),

Z_p^* (the ring of p-adic integers in the p-adic completion of Q),

$Z' = \bigcap_{p \mid [G:1]} Z_p$, a semilocal ring of integers in Q.

To give the reader the proper perspective, we quote two earlier results.

Received by the editors September 27, 1965.

(1) This research was supported in part by the National Science Foundation.

THEOREM 1. *If K is a field of characteristic p, and if G has a cyclic p-Sylow subgroup, then $A(KG)$ is a finite dimensional semisimple C-algebra (O'Reilly* [10]).

THEOREM 2. *Suppose that for some prime p, the group G contains an element of order p^2. Then both $a(Z_pG)$ and $a(Z_p^*G)$ contain at least one nonzero nilpotent element. The same is therefore true for $A(Z_pG)$ and $A(Z_p^*G)$ (Reiner* [12]).

The aim of the present paper is to present a partial converse to the latter theorem. We shall prove here:

THEOREM 3. *Let p be a fixed prime, and suppose that the p-Sylow subgroups of G are cyclic of order p. Then $A(Z_pG)$ and $A(Z_p^*G)$ contain no nonzero nilpotent elements.*

THEOREM 4. *If $[G:1]$ is squarefree, then $A(Z'G)$ contains no nonzero nilpotent element.*

In the course of the proof we shall establish the following fact, which is of independent interest, and is an immediate consequence of Theorem 5 below.

PROPOSITION. *Let p be an odd prime, and let G have a normal p-Sylow subgroup which is cyclic of order p. Let M and N be Z_pG-modules. Then $M \cong N$ if and only if $M/pM \cong N/pN$ as $(Z/pZ)G$-modules.*

2. **Preliminary remarks.** We collect here some definitions, remarks, and previously established results which will be needed in the paper.

(a) If R is a field, or if $R = Z_p^*$, the Krull-Schmidt theorem holds for RG-modules. (See [2, Theorem 14.5 and Theorem 76.26].)

(b) For R an arbitrary ring, every element of $a(RG)$ is expressible in the form $[M]-[N]$, where M and N are RG-modules, but is not uniquely so expressible. Furthermore, $[M]=[N]$ in $a(RG)$ if and only if there exists an RG-module X such that $M \oplus X \cong N \oplus X$. If the Krull-Schmidt Theorem holds for RG-modules, this last isomorphism implies that $M \cong N$. Furthermore, in this case $a(RG)$ has a Z-basis consisting of the symbols $[L]$, where L ranges over a full set of representatives of the isomorphism classes of indecomposable RG-modules.

(c) Let $A \to B$ be a monomorphism of abelian additive groups. Then also

$$C \otimes_Z A \to C \otimes_Z B$$

is a monomorphism. (See MacLane [7, p. 152, Theorem 6.2].)

(d) For M a Z_pG-module, let $M^* = Z_p^* \otimes_{Z_p} M$. The map $[M] \to [M^*]$ gives a ring homomorphism $a(Z_pG) \to a(Z_p^*G)$, which we claim is a monomorphism. For let M and N be Z_pG-modules such that $[M^*]=[N^*]$. Then $M^* \cong N^*$, which implies that $M \cong N$ (see Maranda [8], or [2, Theorem 76.9]). This proves that the above homomorphism is monic, so by the preceding remark, $A(Z_pG) \to A(Z_p^*G)$ is also monic.

Next, there is a ring homomorphism

$$a(Z'G) \to \prod_{p\mid[G:1]} a(Z_pG),$$

gotten by mapping $[M]$ onto the element whose pth component is $[Z_p \otimes_{Z'} M]$. This map is monic, since if M and N are $Z'G$-modules such that

$$Z_p \otimes_{Z'} M \cong Z_p \otimes_{Z'} N, \qquad p\,|\,[G:1],$$

then by Maranda [9] (see [2, Theorem 81.2]) it follows that $M \cong N$. The map

$$A(Z'G) \to \prod_{p\mid[G:1]} A(Z_pG)$$

is also monic, by (c) above.

(e) Suppose that R is either a field of characteristic p, or $R = Z_p^*$, and let H be a subgroup of G. An RG-module M is called *(G,H)-projective* if M is a direct summand of an induced module L^G for some RH-module L. If H is a p-Sylow subgroup of G, then every RG-module is (G,H)-projective (see [2, §63]). The $(G,\{1\})$-projective modules are just the ordinary projective RG-modules.

For D a subgroup of G, let $a_D(RG)$ be the ideal of $a(RG)$ generated by the set of all (G,D)-projective RG-modules. Denote by $a_D'(RG)$ the ideal generated by the (G,D')-projectives, where D' ranges over the proper subgroups of D. Define

$$w_D(RG) = a_D(RG)/a_D'(RG),$$

and set

$$W_D(RG) = C \otimes_Z w_D(RG), \qquad A_D = C \otimes a_D, \qquad A_D' = C \otimes a_D'.$$

Then we have:

TRANSFER THEOREM. *The algebra $A(RG)$ is semisimple if $W_D(R \cdot N_GD)$ is semisimple for each p-subgroup D of G. Here, N_GD is the normalizer of D in G (Green [4]).*

Using this, O'Reilly [10] was able to prove Theorem 1 by showing:

THEOREM. *If k is a field of characteristic p, and if G has a cyclic p-Sylow subgroup, then $W_D(k \cdot N_GD)$ is semisimple for each p-subgroup D of G.*

(f) Now let H be a normal subgroup of G, and suppose that the Krull-Schmidt Theorem holds for RG-modules. Let L be an RH-module, and let $x \in G$. We may form a new RH-module L^x, called a *conjugate* of L, by letting L^x have the same elements as L, but where each $h \in H$ acts on L^x as does xhx^- on L.

For an RG-module M, denote by $\mathrm{res}_H M$ the RH-module gotten from M by restriction of operators from G to H. From the Mackey Subgroup Theorem (see [2, Theorem 44.2]), it follows at once that $\operatorname{res}_H(L^G)$ is a direct sum of conjugates of L.

We note further that if L_1 and L_2 are RH-modules, then

$$L_1^G \otimes L_2^G \cong \sum^{\oplus} (L_1 \otimes L_2^y)^G,$$

the sum extending over certain elements $y \in G$ (see [2, Theorem 44.3]).

(g) Starting with a (left) RG-module M, we may form another (left) RG-module M^*, called the *contragredient* of M (see [2, §43]). As R-module, M^* is just $\operatorname{Hom}_R(M,R)$. Each $x \in G$ acts on M^* in the same way that x^{-1} acts on the right RG-module $\operatorname{Hom}_R(M,R)$. Then $(M_1 \oplus M_2)^* \cong M_1^* \oplus M_2^*$. Also, if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of RG-modules, then so is

$$0 \to M_3^* \to M_2^* \to M_1^* \to 0.$$

Finally, we have $(RG)^* \cong RG$, and therefore contragredients of projective modules are again projective.

(h) Schanuel's Lemma (see Swan [13]). *Suppose we are given two exact sequences of RG-modules*:

$$0 \to U_i \to P_i \to V_i \to 0, \quad i = 1,2,$$

in which P_1 and P_2 are projective. If $V_1 \cong V_2$, then

$$U_1 \oplus P_2 \cong U_2 \oplus P_1.$$

3. **Main theorem.** Throughout this section we fix a prime p, and set $R = Z_p^*$, $\bar{R} = R/pR$. Let G contain a p-Sylow subgroup H which is cyclic of order p. In view of 2(d) above, Theorem 3 will be established as soon as we show that $A(RG)$ contains no nonzero nilpotent element.

Up to isomorphism, there are exactly three indecomposable RH-modules, namely R_H, I_H, and RH (see Heller and Reiner [6]). Here, R_H is the module R on which H acts trivially; I_H is the augmentation ideal of the group ring RH, and there is an exact sequence of RH-modules:

$$0 \to I_H \to RH \to R_H \to 0. \tag{1}$$

By 2(e) each indecomposable RG-module is a direct summand of one of the induced modules $(I_H)^G$, RG, $(R_H)^G$. (We have used the obvious isomorphism: $(RH)^G \cong RG$.) Thus the number of isomorphism classes of indecomposable RG-modules is finite, so $A(RG)$ is a finite dimensional commutative C-algebra. We must show that $A(RG)$ is semisimple, or equivalently, that $A(RG)$ contains no nonzero nilpotent element. To prove this, by 2(e) it is enough to show that $W_D(R \cdot N_G D))$. is semisimple for each p-subgroup D of G. But W_D is unchanged when D is replaced by one of its conjugates, and therefore we need only show that

$$W_{\{1\}}(RG) \text{ and } W_H(R \cdot N_G H)$$

are both semisimple.

The algebra $W_{\{1\}}(RG)$ is generated by the projective RG-modules. As is well known (see [2, §77]), there is a one-to-one isomorphism-preserving correspondence between the indecomposable direct summands of RG and those of $\bar{R}G$. In other words, we have

$$W_{\{1\}}(RG) \cong W_{\{1\}}(\bar{R}G),$$

the isomorphism being given by $[M] \to [M/pM]$. But $\bar{R}$ is a field of characteristic p, so by O'Reilly's Theorem of 2(e) it follows that $W_{\{1\}}(\bar{R}G)$ is semisimple. (This can also be proved easily by use of Brauer characters; see, for example, Conlon [1].)

It remains for us to show that $W_H(R \cdot N_G H)$ is semisimple. Changing notation, we may hereafter assume that G has a normal p-Sylow subgroup H which is cyclic of order p, and we must prove that $W_H(RG)$ is semisimple. For p odd, this is an immediate consequence of O'Reilly's Theorem together with the following result:

THEOREM. 5. *Let G have a normal p-Sylow subgroup H which is cyclic of order p, where p is an odd prime. Then the algebra homomorphism*

$$W_H(RG) \to W_H(\bar{R}G)$$

is monic.

(Before starting the proof, we may remark that the theorem fails to be true when $p = 2$. Nevertheless, most of the details of the proof are valid for $p = 2$, and will be used for that case in the following section.)

Proof. As was pointed out earlier in this section, the nonisomorphic indecomposable direct summands of $(R_H)^G$, $(I_H)^G$, and RG, give a full set of indecomposable RG-modules. The direct summands of RG are RG-projective, and generate the ideal $a_H(RG)$ of $a_H(RG)$. Thus $w_H(RG)$ is generated as Z-module by the indecomposable direct summands of $(R_H)^G$ and $(I_H)^G$. Let us write

$$(R_H)^G = N_1 \oplus \cdots \oplus N_k, \; N_i \text{ indecomposable,} \tag{2}$$

where the summands are numbered so that the first m of them are a full set of nonisomorphic modules from the set of summands. It will turn out that

$$(I_H)^G = L_1 \oplus \cdots \oplus L_k, \; L_i \text{ indecomposable,} \tag{3}$$

with the first m summands a full set of nonisomorphic modules from the set $\{L_1, \cdots, L_k\}$. Thus $\{[N_1], \cdots, [N_m], [L_1], \cdots, [L_m]\}$ form a Z-basis for $w_H(RG)$.

Let $\bar{N}_i = N_i/pN_i$, $\bar{L}_i = L_i/pL_i$, viewed as $\bar{R}G$-modules. In order to prove that $W_H(RG) \to W_H(\bar{R}G)$ is monic, it suffices by 2(c) to show that $w_H(RG) \to w_H(\bar{R}G)$ is monic; and for this, we need only show that $\{[\bar{N}_1], \cdots, [\bar{N}_m], [\bar{L}_1], \cdots, [\bar{L}_m]\}$ are linearly independent (over Z) in $w_H(\bar{R}G)$.

By Schur's Theorem [2, Theorem 7.5], there exists a subgroup F of G such that $G = HF$, $F \cong G/H$. It is easily seen that

$$(R_H)^G \cong RF \qquad \text{as } RG\text{-modules},$$

where H acts trivially on the module RF. Indeed, $(R_H)^G = RG \otimes_{RH} R_H$, and the isomorphism is given by

$$\sum_{x \in F; y \in H} \alpha_{x,y}(xy \otimes 1) \to \sum_{x,y} \alpha_{x,y} x, \qquad \alpha_{x,y} \in R.$$

Hence the $\{N_i\}$ occurring in (2) are gotten by decomposing RF into a direct sum of indecomposable left ideals. However, $p \nmid [F:1]$, so if K is the quotient field of R, we have

$$KF = KN_1 \oplus \cdots \oplus KN_k,$$

where the $\{KN_i\}$ are minimal left ideals of KF. Furthermore, it follows from [2, Theorem 76.17 and Theorem 76.23], that each $\bar{N}_i$ is indecomposable, and that for $1 \leqq i, j \leqq k$,

$$N_i \cong N_j \Leftrightarrow KN_i \cong KN_j \Leftrightarrow \bar{N}_i \cong \bar{N}_j.$$

Thus $\{\bar{N}_1, \cdots, \bar{N}_m\}$ are distinct indecomposable $\bar{R}G$-modules, on each of which H acts trivially.

Turning next to the consideration of $(I_H)^G$, we observe first that forming induced modules preserves exactness, and so from (1) we obtain an exact sequence of RG-modules

$$0 \to (I_H)^G \to RG \to (R_H)^G \to 0. \tag{4}$$

Each N_i is a quotient module of $(R_H)^G$, hence also of RG, and so there exist exact sequences

$$0 \to M_i \to RG \to N_i \to 0, \qquad 1 \leqq i \leqq k. \tag{5}$$

If $N_i \cong N_j$, then by 2(h) we have $M_i \cong M_j$. Conversely, if $M_i \cong M_j$, then taking contragredients (see 2(g)) and using 2(h) again, we obtain $N_i^* \cong N_j^*$, and $N_i \cong N_j$.

If $RG^{(k)}$ denotes a direct sum of k copies of RG, then from (5) we obtain an exact sequence

$$0 \to M_1 \oplus \cdots \oplus M_k \to RG^{(k)} \to N_1 \oplus \cdots \oplus N_k \to 0.$$

Comparing this with (4) and using 2(h), we find that

(6) $$M_1 \oplus \cdots \oplus M_k \cong (I_H)^G \oplus RG^{(k-1)}.$$

For each i, $1 \leqq i \leqq k$, let us write

$$M_i = L_i \oplus P_i,$$

where P_i is projective, and L_i has no projective direct summand. It follows from the Krull-Schmidt Theorem for $\bar{R}G$-modules that M_i determines L_i and P_i uniquely, up to isomorphism. By (6), each L_i is a direct summand of $(I_H)^G$. On the other hand, $(I_H)^G$ has no projective direct summand, since $\text{res}_H(I_H)^G$ is a direct sum of conjugates of I_H, hence of copies of I_H, whereas for X a projective RG-module, $\text{res}_H X$ is free. Consequently

$$L_1 \oplus \cdots \oplus L_k \cong (I_H)^G,$$

and $\{L_1, \cdots, L_m\}$ are a full set of nonisomorphic modules from the set $\{L_1, \cdots, L_k\}$. To show that each L_i is indecomposable, we shall establish the stronger result that $\{\bar{L}_1, \cdots, \bar{L}_m\}$ are a set of distinct indecomposable $\bar{R}G$-modules.

From (5) we obtain exact sequences

$$0 \to \bar{N}_i^* \to \bar{R}G \to \bar{L}_i^* \oplus \bar{P}_i^* \to 0, \qquad 1 \leqq i \leqq m.$$

If $\bar{L}_i \cong \bar{L}_j$ for some i, j, where $1 \leqq i, j \leqq m$, the above implies (using 2(h)) that $\bar{N}_i^* \oplus \bar{P}_j^* \cong \bar{N}_j^* \oplus \bar{P}_i^*$. But $\bar{N}_i^*$ is indecomposable, and is not projective because H acts trivially on $\bar{N}_i^*$. Hence $\bar{N}_i^* \cong \bar{N}_j^*$, so $N_i \cong N_j$ and $i = j$.

Next, suppose $\bar{L}_i$ decomposable; then so is $\bar{L}_i^*$, and we may write $\bar{L}_i^* = U_1 \oplus U_2$, say. Each U_j is a homomorphic image of $\bar{R}G$, so there exist $\bar{R}G$-modules X_1, X_2 with

$$0 \to X_j \to \bar{R}G \to U_j \to 0, \qquad j = 1,2,$$

exact. Thus

$$0 \to X_1 \oplus X_2 \to \bar{R}G^{(2)} \oplus \bar{P}_i^* \to U_1 \oplus U_2 \oplus \bar{P}_i^* \to 0$$

is exact. By 2(h) it follows that

$$\bar{N}_i^* \oplus \bar{R}G \oplus \bar{P}_i^* \cong X_1 \oplus X_2.$$

But $\bar{N}_i^*$ is indecomposable, and $\bar{P}_i^*$ is projective, so either X_1 or X_2 must be projective; say X_1 is projective. Then $\text{res}_H X_1$ is free. On the other hand, X_1 is a direct summand of $\bar{L}_i^*$, and $\text{res}_H \bar{L}_i^*$ is a direct sum of copies of $\bar{I}_H$ (since $\bar{I}_H^* \cong \bar{I}_H$). This gives a contradiction, and so indeed each $\bar{L}_i$ is indecomposable.

We may remark that in terms of the loop space functor Ω introduced by Heller [5], we have $L_i = \Omega(N_i)$.

Let us show at once that $\bar{N}_i \cong \bar{L}_j$ is impossible, and it is precisely for this purpose that the hypothesis $p > 2$ is needed. We know that $\text{res}_H \bar{N}_i$ is a direct sum of copies of $\bar{R}_H$, whereas $\text{res}_H \bar{L}_j$ is a direct sum of copies of $\bar{I}_H$. But $\bar{I}_H$ is indecomposable, and for $p>2$, $\bar{I}_H$ is not isomorphic to $\bar{R}_H$. Thus $\bar{N}_i \not\cong \bar{L}_j$ for any i,j, and we have shown that $\{\bar{N}_1,\cdots,\bar{N}_m,\bar{L}_1,\cdots,\bar{L}_m\}$ are a set of nonisomorphic indecomposable $\bar{R}G$-modules. Obviously none of them lies in $a_H'(\bar{R}G)$, and so they are Z-linearly independent when considered as elements of $w_H(\bar{R}G)$. This completes the proof of Theorem 5.

Since we have already shown that $W_{\{1\}}(RG) \cong W_{\{1\}}(\bar{R}G)$, it follows that the map $A(RG) \to A(\bar{R}G)$ is monic when restricted to $A_H'(RG)$. Combining this with Theorem 5, we may conclude that the algebra homomorphism

$$A(RG) \to A(\bar{R}G)$$

is also monic, provided the hypotheses of Theorem 5 are satisfied. But this establishes the validity of the proposition stated at the end of §1.

4. **The case** $p=2$. In this section we shall prove Theorem 3 for the case $p=2$. We use the notation of the preceding section, and we are assuming now that G has a cyclic 2-Sylow subgroup H of order 2. As we have seen, we need only show that $W_H(RG)$ contains no nonzero nilpotent element, and it suffices to prove this for the case where H is normal in G. Furthermore, in order to prove that $W_H(RG)$ has no nilpotent elements except 0, it is enough to show that if $x \in W_H(RG)$ satisfies $x^2=0$, then necessarily $x=0$.

As in §3, we let $\{N_1,\cdots,N_m\}$ be the nonisomorphic indecomposable summands of $(R_H)^G$, and $\{L_1,\cdots,L_m\}$ those of $(I_H)^G$. Now $R_H \not\cong I_H$, even for $p=2$, so by considering restrictions to H it is clear that $N_i \not\cong L_j$ for any i,j. Hence $W_H(RG)$ has C-basis $\{[N_1],\cdots,[N_m],\ [L_1],\cdots,[L_m]\}$. Furthermore, we know from §3 that $\{\bar{N}_1,\cdots,\bar{N}_m\}$ are the nonisomorphic indecomposable summands of $(\bar{R}_H)^G$, while $\{\bar{L}_1,\cdots,\bar{L}_m\}$ are those of $(\bar{I}_H)^G$. However, since $p=2$ we have $\bar{R}_H \cong \bar{I}_H$, and so the $\bar{L}$'s are a rearrangement of the $\bar{N}$'s. Thus the maps $W_H(RG) \to W_H(\bar{R}G)$, $A(RG) \to A(\bar{R}G)$, are no longer monomorphisms.

In this case we have $[G:F]=2$, so F is normal in G, and $G/F \cong H$. If h is the generator of H, we may form the RH-module Y having the same elements as R, but where

$$h\alpha = -\alpha, \qquad \alpha \in Y.$$

Then use the homomorphism of G onto H to turn Y into an RG-module, that is, let F act trivially on Y. The RG-module thus obtained will also be denoted by Y. Obviously

$$\bar{Y} \cong \bar{R}_G, \qquad Y \otimes Y \cong R_G,$$

where R_G is the trivial RG-module.

Consider now the RG-modules $Y \otimes N_1,\cdots,Y \otimes N_m$. Each is indecomposable, since

$$\overline{Y \otimes N_i} \cong \bar{Y} \otimes \bar{N}_i \cong \bar{N}_i.$$

Furthermore, it cannot happen that $Y \otimes N_i \cong N_j$, since h acts on $Y \otimes N_i$ as multiplication by -1, whereas h acts trivially on N_j. Thus, the modules $\{Y \otimes N_i : 1 \leqq i \leqq m\}$ coincide with the modules $\{L_i : 1 \leqq i \leqq m\}$ in some order.

Let us set $Q_i = Y \otimes N_i$, $1 \leqq i \leqq m$. The above discussion shows that $\{[N_1], \cdots, [N_m], [Q_1], \cdots, [Q_m]\}$ is a Z-basis for $w_H(RG)$, hence also a C-basis for $W_H(RG)$. Furthermore, the kernel of the algebra homomorphism $W_H(RG) \to W_H(\bar{R}G)$ has C-basis $\{[N_i] - [Q_i] : 1 \leqq i \leqq m\}$.

We shall now investigate $N_i \otimes N_j$. Since N_i and N_j are direct summands of $(R_H)^G$, their tensor product is a direct summand of $(R_H)^G \otimes (R_H)^G$. By 2(f) we see that this latter module is a direct sum of modules of the form $(R_H \otimes R_H^y)^G$, for some elements $y \in G$. However, $R_H^y \cong R_H$ and $R_H \otimes R_H \cong R_H$. Therefore $N_i \otimes N_j$ is a direct sum of copies of $N_1, \cdots, N_m$; suppose that N_s occurs with multiplicity α_{ijs} as a direct summand of $N_i \otimes N_j$. We have then

$$[N_i][N_j] = \sum_{s=1}^{m} \alpha_{ijs}[N_s] \qquad \text{in } W_H(RG).$$

Furthermore we obtain

$$Q_i \otimes Q_j = (Y \otimes N_i) \otimes (Y \otimes N_j) \cong (Y \otimes Y) \otimes (N_i \otimes N_j) \cong N_i \otimes N_j,$$

so

$$[Q_i][Q_j] = \sum_{s=1}^{m} \alpha_{ijs}[N_s] \qquad \text{in } W_H(RG).$$

Finally we note that

$$[Q_i][N_j] = [Y \otimes (N_i \otimes N_j)] = \sum_{s} \alpha_{ijs}[Y \otimes N_s] = \sum_{s} \alpha_{ijs}[Q_s] \qquad \text{in } W_H(RG).$$

Suppose now that $x \in W_H(RG)$ and $x^2 = 0$; we are trying to prove that x must be 0. Since the image of x in $W_H(\bar{R}G)$ is also nilpotent, and since $W_H(\bar{R}G)$ contains no nonzero nilpotent element, it follows that x lies in the kernel of the map $W_H(RG) \to W_H(\bar{R}G)$. Thus we may write

$$x = \sum_{i=1}^{m} c_i([N_i] - [Q_i]), \qquad c_i \in C.$$

Then

$$x^2 = \sum_{i,j=1}^{m} \{c_i c_j [N_i][N_j] - 2c_i c_j [N_i][Q_j] + c_i c_j [Q_i][Q_j]\}$$

$$= \sum_{i,j,s=1}^{m} 2c_i c_j \alpha_{ijs}([N_s] - [Q_s]).$$

But $x^2 = 0$, and so

$$2 \cdot \sum_{i,j=1}^{m} c_i c_j \alpha_{ijs} = 0, \qquad 1 \leqq s \leqq m.$$

Therefore $\sum_{i,j} c_i c_j \alpha_{ijs} = 0$, which shows that $\sum_{i=1}^{m} c_i[N_i]$ has square 0. Thus $\sum_i c_i[\bar{N}_i] = 0$ in $W_H(\bar{R}G)$, and consequently each $c_i = 0$. This proves that $x = 0$, and completes the demonstration of Theorem 3 for the case $p = 2$.

5. **Concluding remarks.** Let us show that Theorem 4 is an easy consequence of Theorem 3. Suppose that $[G:1]$ is squarefree; then by using Theorem 3 for each prime p dividing $[G:1]$, we see that each $A(Z_p^* G)$ contains no nonzero nilpotent element. Hence also the product

$$\prod_{p \mid [G:1]} A(Z_p^* G)$$

contains no nonzero nilpotent element. But by 2(d) the algebra $A(Z'G)$ may be embedded in the above product, and hence also $A(Z'G)$ contains no nonzero nilpotent element.

It would be of interest to consider the corresponding question for $A(ZG)$. The difficulty seems to arise from the fact that the map $A(ZG) \to A(Z'G)$ need not be monic.

CONJECTURE 1. The kernel of the map $A(ZG) \to A(Z'G)$ is a torsion Z-module.

As remarked in Theorem 2, if G contains an element of order p^2, then $A(Z_p^* G)$ contains nonzero nilpotent elements. On the other hand, we have shown that if the p-Sylow subgroup of G is cyclic of order p, then $A(Z_p^* G)$ is semisimple. We are left with a large class of groups which fall into neither category, for example an elementary abelian (p, p) group.

CONJECTURE 2. If the p-Sylow subgroup of G is not cyclic of order p, then $A(Z_p^* G)$ contains nonzero nilpotent elements.

We may remark that Theorem 5 is best possible, in the following sense. Let $R = Z_p^*$, and let H be a p-Sylow subgroup of G. If H is not normal in G, or if H is not cyclic of order p, then the maps

$$W_H(RG) \to W_H(\bar{R}G), \quad A(RG) \to A(\bar{R}G),$$

are not monic. Indeed, even when G is cyclic of order p^2, the map $A(RG) \to A(\bar{R}G)$ is not monic.

Finally, the proof of Theorem 5 suggests that the proposition at the end of §1 may be a special case of a more general result. This will be investigated more fully in a future work (to appear in Michigan Math. J.)

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