H^p SPACES AND EXTREMAL FUNCTIONS IN H^1

BY
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Let A be a uniform algebra on a compact Hausdorff space X, i.e., A is a uniformly closed separating subalgebra of $C(X)$ which contains the constant functions. Let dm be a representing measure on X for a complex-valued homomorphism ϕ of A. A_0 will denote the kernel of ϕ, and H^p and H_0^p will denote respectively the closures of A and A_0 in $L^p(dm)$, $0 < p < \infty$.

A function $f \in H^1$ is *extremal* if $f \not\equiv 0$ and $f/\|f\|_1$ is an extreme point of the unit ball of H^1. DeLeeuw and Rudin [1] proved that if $\Delta = \{|z| \leqq 1\}$ and A is the algebra of continuous functions on Δ which are analytic on the interior of Δ, and if $f \in H^1(d\theta/2\pi)$, then Af is dense in $H^1(d\theta/2\pi)$ if and only if f is extremal. It is the purpose of this note to prove the following generalization. ***Here, and*** *throughout the paper, we assume that dm is a Szegö measure for* ϕ (defined later).

THEOREM. *Suppose* $f \in H^1$. *Then* $A_0 f$ *is dense in* H_0^1 *if and only if* f *is extremal.*

The proof of the theorem is based on the idea from [2] of projecting L^1 into H^p, where $0 < p < 1$, together with a technique of Hoffman and Wermer [9] which allows one in certain situations to modify H^p-convergence to obtain pointwise bounded convergence. Professor Forelli tells us he has used the projection of L^1 into H^p, together with some special function theory, to obtain the theorem for the H^1 spaces associated with algebras of almost periodic functions.

The Hoffman-Wermer technique is used in subsequent sections to study H^p spaces for $0 < p < \infty$. Here proofs are given of some standard results, all known for $p \geqq 1$, which also cover the case $0 < p < 1$. An invariant subspace theorem is proved in the final section which shows that once the invariant subspaces of L^2 are understood, the invariant subspaces of the other L^p spaces, $0 < p < \infty$, offer no difficulty.

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Remarks. According to a theorem of Wermer [12], [6], the Gleason part of ϕ on H^∞ is either an analytic disc or just the one point ϕ, depending on whether or not there is an inner function F such that $FH^1 = H_0^1$. If such an inner function

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exists, it is easy to see that the only extremal functions in H^1 are the outer functions. In the other case, it is not known whether there exist functions $f \in H_0^1$ such that fA is dense in H_0^1. In [5, p. 183], this problem is raised in an equivalent form, in connection with evanescent stochastic processes.

A function f such that $A_0 f$ is dense in H_0^1 has the following property: if $g \in H_0^1$ and $|g| \leqq |f|$, then $g/f \in H^\infty$. In particular, if $|g| = |f|$, then $g = Ff$ expresses g as the product of an inner function and an extremal function, and the factorization is unique, up to a constant multiple of modulus one. Consequently, the existence of extremal functions in H_0^1 would lead to a wider factorization theory than now available.

Proof of the theorem. A *Szegö measure* is a representing measure dm for ϕ such that Szegö's theorem is valid: for any function $h \in L^1$ such that $h \geqq 0$,

$$\inf_{f \in A_0} \int |1 - f|^2 h \, dm = \exp\left\{ \int \log h \, dm \right\}.$$

Equivalently, dm is a Szegö measure if whenever dv is another representing measure for ϕ which is absolutely continuous with respect to dm, then $dv = dm$. Also, dm is a Szegö measure if and only if the algebra $H^\infty(dm)$ of bounded functions in $H^1(dm)$ is a logmodular algebra on its Šilov boundary Y [6].

We will be more interested in the logmodular algebra H^∞ on Y rather than in A. The results of [6] carry over to H^∞. In particular, dm is the unique representing measure on Y for the extension of ϕ to H^∞ (also denoted by ϕ). Since ϕ then has a unique norm-preserving extension from H^∞ to $C(Y)$, the following approximation lemma is valid [10].

Lemma 1. *If u is any real-valued continuous function on Y, then*

$$\int u \, dm = \inf \left\{ \int v \, dm \colon v \in \mathrm{Re}(H^\infty),\ v \geqq u \right\}.$$

The proof of the theorem begins with the main lemma used by deLeeuw and Rudin.

Lemma 2. *A nonzero function $f \in H^1$ is not extremal if and only if there exists a bounded real-valued function k such that $\int k \, dm = 0$, $kf \in H^1$, and $kf \not\equiv 0$.*

Corollary. *If the Gleason part of ϕ on H^∞ is an analytic disc, then the extremal functions in H^1 are the outer functions.*

In this case f is not outer if and only if we can write $f = Fg$, where $g \in H^1$ and F is a nonconstant inner function. If $\int f \, dm \neq 0$, this follows from [6], and if $\int f \, dm = 0$, from the remarks preceding the proof. If $k = F + \bar{F}$, then the function $k - \int k \, dm$ satisfies the requirements of Lemma 2.

If f is outer, then 1 is in the L^1-closure of Af, so any k satisfying the requirements of Lemma 2 would belong to H^1. But H^1 contains no nonconstant real-valued functions [6]. This proves the corollary.

LEMMA 3 (HOFFMAN). *If E is a measurable subset of Y such that $0 < m(E) < 1$, there is a function $g \in H^\infty$ such that g is real on E and g is not a constant a.e. on E.*

To prove this, let h be an outer function in H^∞ such that $|h| = 1$ on E and $|h| = e$ on $Y - E$. h is invertible in H^∞, and $g = h + 1/h$ is real on E. Also, $|\phi(h)| > 1$, since

$$\log|\phi(h)| = \int \log|h| \, dm > 0.$$

Suppose that g is a constant on E. Then h assumes at most two values on E, so h is a constant on a subset D of E of positive measure. If $h = \lambda$ on D, $h - \lambda$ vanishes on a set of positive measure, so $\phi(h - \lambda) = 0$ by Jensen's inequality. Consequently, $|\phi(h)| = |\lambda| = 1$, a contradiction.

COROLLARY. *If $f \in H^1$ vanishes on a set of positive measure, then f is not extremal.*

If $f \not\equiv 0$ vanishes on a set of positive measure, let E be the set where f does not vanish, and let g be the function of Lemma 3. If k is defined to be equal to 0 where f vanishes and equal to g where f does not, then $k - \int k \, dm$ satisfies the requirements of Lemma 2.

LEMMA 4. *$fA_0 + \overline{fA_0} + C$ is dense in L^1 if and only if f is extremal.*

Here C is the field of complex constants, and the bar denotes complex conjugation. Lemma 4 is a simple consequence of Lemma 2 and the corollary to Lemma 3. The function k appears as a linear functional on L^1 which is orthogonal to $fA_0 + \overline{fA_0} + C$. Since $H + \overline{H^1}$ is dense in L^1, we have proved the following half of the theorem.

COROLLARY. *If fA_0 is dense in H_0^1, then f is extremal.*

The next lemma first appears in a function algebra setting in [4]. The proo there is valid for dirichlet algebras, and a minor adjustment using Lemma 1 covers the case at hand.

LEMMA 5. *If $0 < p < 1$, there is a constant $K(p)$ such that*

$$\|f\|_p \leqq K(p) \|f + \bar{g}\|_1 \tag{1}$$

for all $f \in H^\infty$ and $g \in H_0^\infty$. There is a constant $J(p)$, $0 < p < 1$, such that

$$\| v \|_p \leqq J(p) \| u \|_1 \tag{2}$$

for all real-valued functions u and v such that $u + iv \in H^\infty$ and $\int v \, dm = 0$.

It is easy to show that inequalities of the forms (1) and (2) are equivalent (cf. [7], [13]). To prove (2), one first assumes that $u \in \mathrm{Re}(H^\infty)$ is positive, and proceeds as in [4], or as in [13, p. 254]. If $w \in \mathrm{Re}(H^\infty)$ is arbitrary, Lemma 1 produces a $u \in \mathrm{Re}(H^\infty)$ such that $u \geqq \max(0, w)$ and $\| u \|_1 \leqq \| w \|_1$. Then w is expressed as the difference of the positive functions u and $u - w$, and the inequality is extended to w.

Now suppose $f \in H^1$, and let $h = \max(|f|, 1)$. h and $\log h$ are integrable, so there is an outer function $G \in H^1$ such that $|G| = h$ [6]. In particular, $|f/G| \leqq 1$. If g_n is a sequence in A such that $g_n G \to 1$ in H^1, then

$$\int |f g_n - f/G| \, dm \leqq \int |g_n G - 1| \, dm \to 0,$$

so $f/G \in H^\infty$.

LEMMA 6. *f is extremal if and only if f/G is extremal. fA_0 is dense in H_0^1 if and only if $(f/G)A_0$ is dense in H_0^1.*

The proof of this lemma is straightforward, and will be omitted.

We now complete the proof of the theorem. Let f be an extremal function in H^1. Replacing f by f/G as in Lemma 6, we can assume that f is bounded. Let $g \in A_0$. By Lemma 4, there are sequences p_n, $q_n \in A_0$ and complex numbers λ_n such that $p_n f + q_n f + \lambda_n \to g$ in L^1. Integrating both sides of this limit relation, we see that $\lambda_n \to 0$. So we can assume that $\lambda_n = 0$.

By Lemma 5, $q_n f \to g$ in H^p, $0 < p < 1$. Passing to a subsequence, if necessary, we can also assume that $q_n f \to g$ a.e. The remainder of the proof involves reproducing a technique due to Hoffman and Wermer [9] for modifying the sequence to obtain pointwise convergence.

We can assume that $\| f \|_\infty < 1$ and $\| g \|_\infty < 1$. Let $w_n = \log_+ |q_n f|$, then $w_n \geqq 0$ and $w_n \to 0$ a.e. Let $E_n = \{x : |q_n(x) f(x)| > 1\}$. Since $p \log_+ s \leqq s^p$ for $s \geqq 0$,

$$\begin{aligned} p \int w_n \, dm &= p \int_{E_n} w_n \, dm \leqq \int_{E_n} |q_n f|^p \, dm \\ &\leqq \int_{E_n} |q_n f - g|^p \, dm + \int_{E_n} |g|^p dm \\ &\leqq \| q_n f - g \|_p^p + m(E_n). \end{aligned}$$

Since $\| g \|_\infty < 1$ and $q_n f \to g$ in L^p, $m(E_n) \to 0$. Consequently, $\int w_n \, dm \to 0$.

By Lemma 1, we can find $u_n \in \mathrm{Re}(H^\infty)$ such that $u_n \geqq w_n$ and $\int u_n \, dm \to 0$. Choose v_n real such that $u_n + i v_n \in H^\infty$ and $\int v_n \, dm = 0$. If $g_n = \exp(-u_n - i v_n)$, then $g_n \in H^\infty$ and $\| g_n q_n f \|_\infty \leqq 1$.

Now $\| g_n \|_\infty \leqq 1$ and $\int g_n \, dm = \exp\{\int u_n \, dm\} \to 1$. Passing to a subsequence, we can assume that $g_n \to 1$ a.e. Hence $g_n q_n f \to g$ a.e. In particular, $g_n q_n f \to g$ in

$H^1(dm)$, so every function $g \in A_0$ is in the L^1-closure of $A_0 f$. This proves that $A_0 f$ is dense in H_0^1.

COROLLARY. *If $f \in H^2$, then $A_0 f$ is dense in H_0^1 if and only if $A_0 f$ is dense in H_0.*

Density in H_0^2 trivially implies density in H_0^1. So suppose $A_0 f$ is dense in H_0^1. The technique used in Lemma 6 allows us to assume that f is bounded. The Hoffman-Wermer argument then shows that every $g \in A_0$ is a bounded pointwise limit of functions in $H_0^\infty f$. In particular, $A_0 f$ is dense in H_0^2.

The same proof could be used to study extremal functions in H_0^1, i.e., functions $f \in H_0^1$ such that $f/\|f\|_1$ is an extreme point of ball H_0^1. The analogous result is the following.

THEOREM. *Let $f \in H_0^1$. fA is dense in H_0^1 if and only if f is extremal in H_0^1.*

The altered form of Lemma 4 needed to prove this theorem is that $fA + \overline{fA} + C$ is dense in L^1 if and only if f is extremal in H_0^1.

COROLLARY. *Suppose that the Gleason part of ϕ on H^∞ is the one point $\{\phi\}$. Then the extreme points of ball H^1 are the outer functions in H^1 of norm 1, together with the extreme points of ball H_0^1.*

The problem here is to show that every extreme point f of ball H_0^1 is extremal in H^1. Now Af is dense in H_0^1. If f were not extremal in H^1, then $A_0 f$ would not be dense in H_0^1. Consequently, H_0^1 would be a simply invariant subspace of H^1 (cf. [11], or Theorem 7), and $H_0^1 = FH^1$ for some inner function F. As remarked earlier, this would imply that ϕ is the center of an analytic disc.

H^p spaces. For $p = 1$ and 2, the results of this section are found in [6] for logmodular algebras. The reduction of the general case of a Szegö measure to the logmodular case is in [8]. Here it is shown that H^∞ is logmodular, and that A is weak* dense in H^∞, so that the H^p spaces associated with H^∞ are the same as those associated with A.

Not all results about logmodular algebras transfer to A. In fact, the Hoffman-Wermer argument establishes the following theorem, which is not valid for arbitrary Szegö measures.

THEOREM 1. *Suppose that dm is a unique representing measure on X for ϕ, considered as a homomorphism of A. If $0 < p < \infty$, and f is a bounded function in H^p, then there is a sequence $f_n \in A$ such that $\|f_n\|_X \leqq \|f\|_\infty$ and $f_n \to f$ a.e.*

Applied to the H^p space of an arbitrary Szegö measure, this yields the following corollary.

COROLLARY. *If $0 < p < \infty$, and f is a bounded function in H^p, then $f \in H^\infty$.*

We will need first some facts about H^p spaces for $p \geqq 1$. Recall that a function $g \in H^1$ is outer if

$$\log \left| \int g \, dm \right| = \int \log |g| \, dm > -\infty,$$

and $g \in H^1$ is inner if $|g| = 1$ a.e.

LEMMA 7. *Suppose $p \geqq 1$. A non-negative function h is the modulus of an outer function f in H^p if and only if $h \in L^p$ and $\log h \in L^1$.*

For $p = 1$ or 2, this is proved in [6]. The general case $1 < p < \infty$ is a consequence of the direct sum decomposition $L^p = H^p \oplus \overline{H_0^p}$. The boundedness of the projection of L^p onto H^p is due, in the classical case, to M. Riesz (cf. [7], [13]). His proof carries over, with some minor adjustments as in Lemma 4, to the general case. An immediate corollary of this direct sum decomposition is that H^p is the orthogonal complement in L^p of A_0. Consequently, $H^p = H^1 \cap L^p$, and the H^p theorem now follows from the H^1 theorem.

LEMMA 8. *Suppose $1 \leqq p < \infty$. $f \in H^p$ is an outer function if and only if Af is dense in H^p.*

Again the theorem is known for H^1 [6], so we assume $1 < p < \infty$. If Af is dense in H^p, then Af is dense in H^1, so f is outer.

Suppose that f is outer. Then Af is dense in H^1. Let q be the conjugate index to p, and let $g \in L^q$ be orthogonal to Af and also to $\overline{H_0^p}$. Then the L^1 function gf is orthogonal both to A and to $\bar{A}_0$. Since $A + \bar{A}_0$ is weak* dense in L^∞ [6], $gf \equiv 0$. f cannot vanish on a set of positive measure, so $g \equiv 0$, and Af must be dense in H^p.

THEOREM 2. *Suppose that $f \in H^p$ and $\log|f|$ is integrable. For some integer n such that $np \geqq 1$, there is an outer function $g \in H^{np}$ such that $f = Fg^n$, where F is an inner function.*

To prove this, choose the integer k such that $n = 2^k \geqq 1/p$. By Lemma 7, $|f|^{1/n}$ is the modulus of an outer function in H^{np}. Choose a sequence $g_j \in A$ such that $g_j \to g$ in L^{np}. Then

$$\int |g_j^2 - g^2|^{p/2} dm \leqq \left\{ \int |g_n - g|^{np} dm \right\}^{1/2} \left\{ \int |g_n + g|^{np} dm \right\}^{1/2},$$

and the right-hand side tends to zero. Then $g_j^2 \to g^2$ in $L^{np/2}$. By induction, $g_j^n \to g^n$ in L^p, so that $g^n \in H^p$.

By Lemma 8, there is a sequence $h_j \in A$ such that $h_j g \to 1$ in L^{np}. By the same estimate as above, we see that $\int |h_j^2 g^2 - 1|^{np/2} dm \to 0$. Proceeding by induction, we see that $h_j^n g^n \to 1$ in L^p. Consequently,

$$\int |f/g^n - f h_j^n| \, dm = \int |1 - g^n h_j^n| \, dm \to 0,$$

and $f/g^n = F$ belongs to H^p. Also, $|F| = 1$ a.e., so F is an inner function in H^∞.

THEOREM 3. *The functional* $\phi(f) = \int f\, dm$ *has a continuous extension to* H^p, $0 < p < 1$, *which will also be denoted by* ϕ. *Jensen's inequality is valid for functions* $f \in H^p$:

$$\log|\phi(f)| \leqq \int \log|f|\, dm.$$

Suppose $f \in A$ and $\int f\, dm \neq 0$. Then $\int \log|f|\, dm \neq 0$, and by Theorem 2 we can write $f = Fg^n$, where $np \geqq 1$, F is inner, and g is outer. Then

$$\begin{aligned} |\phi(f)| &= |\phi(F)\phi(g)^n| \leqq |\phi(g)|^n \\ &= \left|\int g\, dm\right|^n \leqq \left\{\int |g|^{np} dm\right\}^{1/p} \\ &= \left\{\int |f|^p\, dm\right\}^{1/p} = \|f\|_p. \end{aligned}$$

This estimate shows that ϕ extends continuously to H^p. Now Jensen's inequality holds for functions in A, and the same proof which extends it to functions in H^1 also extends it to functions in H^p, $0 < p < 1$ (cf. [6]).

THEOREM 4. *Let* $f \in H^p$. Then Af *is dense in* H^p *if and only if* $\log|\phi(f)| = \int \log|f|\, dm > -\infty$.

If Af is dense in H^p, then $\phi(f) \neq 0$, so $\int \log|f|\, dm > -\infty$. By Theorem 2, $f = Fg^n$, where F is inner and $g \in H^{np}$ is outer. Since 1 is in the closure of Af, $\bar{F}$ is in the closure of Ag, and $\bar{F} \in H^\infty$. Consequently, F is a constant of modulus 1. Now the equality in Theorem 4 for f follows from the corresponding equality for g, and the fact that $|\phi(f)| = |\phi(g)|^n$.

Conversely, if $\log|\phi(f)| = \int \log|f|\, dm > -\infty$, and $f = Fg^n$ as above, then the inner factor F must be a constant, so we can assume $f = g^n$. As in the proof of Theorem 2, we can approximate the function 1 in H^p by functions in Ag^n. Thus, Ag^n is dense in H^p.

Another consequence of Lemma 7 and Theorem 2 is the following.

THEOREM 5. *If* $f \in H^p$ *and* $|f| \in L^q$, *where* $0 < p < q \leqq \infty$, *then* $f \in H^q$.

Adjusting f by a constant, if necessary, we can assume that $\phi(f) \neq 0$. Then $\int \log|f|\, dm > -\infty$, and we can write $f = Fg^n$ as in Theorem 2. But now $g \in H^{np}$ and $|g| \in L^{nq}$, so by Lemma 7, $g \in H^{nq}$. It follows that $g^n \in H^q$, and so $f \in H^q$.

A function $f \in H^p$, $0 < p < 1$, is said to be *outer* if

$$\log|\phi(f)| = \int \log|f|\, dm > -\infty.$$

The characterization of the moduli of outer functions can now be carried over from [6]. The proof is straightforward.

THEOREM 6. *If* $h \geqq 0$, *and* $0 < p < \infty$, *the following assertions are equivalent*:

(i) $h \in L^p$ *and* $\log h \in L^1$.
(ii) $h = |g|$ *for some outer function* $g \in H^p$.
(iii) $h = |f|$ *for some function* $f \in H^p$ *such that* $\phi(f) \neq 0$.

Invariant subspaces. A (closed) subspace $\mathscr{M}$ of L^p is *invariant* if $A\mathscr{M} \subseteq \mathscr{M}$. $\mathscr{M}$ is *simply invariant* if $\mathscr{M}$ is invariant and $A_0\mathscr{M}$ is not dense in $\mathscr{M}$. In order to carry over the invariant subspace theorems in the form given them by Srinivasan [11], we first state a strengthened form of Theorem 1, established by the same argument.

THEOREM 7. *Suppose that* $f_n \in L^\infty$, $f \in L^\infty$, *and* $f_n \to f$ *in the* L^p *metric for some* p, $0 < p < \infty$. *Then there is a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *and functions* $g_k \in H^\infty$ *such that* $\| g_k f_{n_k} \|_\infty \leqq \|f\|_\infty$ *and* $g_k f_{n_k} \to f$ *a.e. If dm is a unique representing measure on X for* ϕ, *and if the* f_n *are continuous, we can choose the* g_k *to belong to A.*

LEMMA 9. *If* $0 < p \leqq \infty$, *and* $\mathscr{M}$ *is a (weak*) closed subspace of* L^p *such that* $A\mathscr{M} \subseteq \mathscr{M}$, *then* $H^\infty\mathscr{M} \subseteq \mathscr{M}$.

If dm were a unique representing measure, Lemma 8 would be a consequence of Theorem 7. In case dm is only a Szegö measure for ϕ we use the fact [8] that A is weak* dense in H^∞. Hence Lemma 8 is true if $p = \infty$.

Suppose $1 \leqq p < \alpha$. Given $f \in H^\infty$ and $g \in \mathscr{M}$, choose $f_n \in A$ such that $f_n \to f$ weak*. If $h \in (L^p)^*$ and $h \perp \mathscr{M}$, then $0 = \int f_n gh \, dm \to \int fgh \, dm$, so $h \perp fg$. Hence $fg \in < \mathscr{M}^\perp)^\perp = \mathscr{M}$, and $H^\infty\mathscr{M} \subseteq \mathscr{M}$.

Now suppose $0 < p < 1$. By induction we can assume the theorem is true for L^{2p}. Let $f \in H^\infty$ and $g \in \mathscr{M}$. Write $g = g_0 g_1$, where g_0 and g_1 belong to L^{2p}. Then fg_0 is in the L^{2p}-closure of Ag_0, so there is a sequence $f_n \in A$ such that $f_n g \to fg$ in L^{2p}. From

$$\int |f_n g - f_g(|^p dm \leqq \left\{ \int |f_n g_0 - f g_0|^{2p} dm \right\}^{1/2} \left\{ \int |g_1|^{2p} dm \right\}^{1/2},$$

we see that $f_n g \to fg$ in L^p. Again $fg \in \mathscr{M}$, and $H^\infty \mathscr{M} \subseteq \mathscr{M}$.

LEMMA 10. *If* $0 < p < \infty$, *and* $\mathscr{M}$ *is an invariant subspace of* L^p, *then* $\mathscr{M} \cap L^\infty$ *is dense in* $\mathscr{M}$.

Let $f \in \mathscr{M}$, and let G_n be the outer function in H^p whose modulus is $|G_n| = \max(1, |f|/n)$. Then $|f/G| = \min(n, |f|)$, and f/G is bounded. If $h_k \in A$ is a sequence such that $\lim_{n \to \infty} h_k G_n = 1$ in L^p, then

$$\int |f/G_n - f h_k|^p \, dm \leqq n^p \int |1 - h_k G_n|^p \, dm \to 0,$$

as $k \to \infty$. So $f/G_n \in \mathscr{M} \cap L^\infty$.

Now $\{|G_n|\}_{n=1}^{\infty}$ is a monotone decreasing sequence of real functions in L^p, and $|G_n| \to 1$ a.e. Consequently $\int \log|G_n|\, dm \to 0$, and $|\phi(G_n)| \to 1$. Adjusting by a complex constant of modulus 1, we can assume $\phi(G_n) \geqq 1$, so that $\phi(G_n) \to 1$.

Let $G_n = g_n^k$, when k is a fixed integer as in Theorem 1 with $kp > 1$, g_n is outer in H^{kp}, and $\phi(g_n) \to 1$. Passing to a subsequence, we can assume that $g_n \to g$ weakly in L^{kp}. Since $|g_n| \to 1$ by decreasing, $\|g_n\|_{kp} \to 1$, and $\|g\| \leqq 1$. However, $\|g\| \geqq |\int g\, dm| = \lim_{n\to\infty} \int g_n\, dm = 1$, and we see that $\|g\|_{kp} = 1 = \lim \|g_n\|_{kp}$. It follows that $g_n \to g$ strongly in L^{kp}. Assuming that $g_n \to g$ a.e., we see that $|g| = 1$ a.e. Since $\int g\, dm = 1$, g is identically 1. Then $G_n \to 1$ a.e. also.

Now $|f - f/G_n|^p$ is integrable, $|f - f/G_n|^p \to 0$ a.e., and

$$|f - f/G_n|^p \leqq |f|^p + |f/G_n|^p \leqq 2|f|^p, \qquad 0 < p \leqq 1,$$

with a similar inequality holding if $p > 1$.

It follows from the dominated convergence theorem that $f/G_n \to f$ in L^p .

THEOREM 8. *If $0 < p < q < \infty$, there is a one-to-one correspondence between invariant subspaces $\mathcal{M}_p$ of L^p and $\mathcal{M}_q$ of L^q, such that $\mathcal{M}_q = L^q \cap \mathcal{M}_p$, and $\mathcal{M}_p$ is the closure in L^p of $\mathcal{M}_q$. There is a one-to-one correspondence between invariant subspaces $\mathcal{M}_p$ of L^p and weak* closed invariant subspaces $\mathcal{M}_\infty$ of L^∞, such that $\mathcal{M}_\infty = L^\infty \cap \mathcal{M}_p$ and $\mathcal{M}_p$ is the closure of $\mathcal{M}_\infty$ in L^p.*

To prove this, it clearly suffices to prove the part dealing with the correspondence between $\mathcal{M}_p$ and $\mathcal{M}_\infty$.

Let $\mathcal{M}$ be an invariant subspace of L^p, and let $\mathcal{M}_\infty = \mathcal{M} \cap L^\infty$. By Lemma 10, the closure of $\mathcal{M}_\infty$ in L^p is $\mathcal{M}$. Also, $\mathcal{M}_\infty$ is weak* closed. In fact, a consequence of the Krein-Schmullian theorem [14] is that the space of bounded functions in any closed subspace of $L^p(dm)$, dm a finite measure, is weak* closed in $L^\infty(dm)$.

To complete the argument, we must show that if $\mathcal{M}_\infty$ is a weak* closed invariant subspace of L^∞, and $\mathcal{M}_p$ is the closure of $\mathcal{M}_\infty$ in L^p, then $\mathcal{M}_p \cap L^\infty = \mathcal{M}_\infty$. By Theorem 7, one can modify any sequence $f_n \in \mathcal{M}_\infty$ converging in L^p to a function $f \in L^\infty$, to obtain a sequence $g_n \in \mathcal{M}_\infty$ converging pointwise boundedly to f. Then g_n converges weak* to f, so $f \in \mathcal{M}_\infty$, and $\mathcal{M}_p \cap L^\infty \subset \mathcal{M}_\infty$. The reverse inclusion is immediate.

THEOREM 9. *Let $0 < p < \infty$, and let $\mathcal{M}$ be a simply invariant subspace of L^p. There exists a function $F \in \mathcal{M}$ such that $|F| = 1$ a.e. and $\mathcal{M} = FH^p$.*

The generalized form of Beurling's theorem is due in this form to Srinivasan. The general case $0 < p < \infty$ is now a consequence of Theorem 7 and the case $p = 2$, since the invariant subspaces $\mathcal{M}_p$ of Theorem 7 are simultaneously simply invariant or not simply invariant.

The proof for $p = 2$ is so beautiful that we cannot resist setting it down again. In this case, let F be a function of norm 1 in $\mathcal{M}$ which is orthogonal to $A_0\mathcal{M}$. In particular, $F \perp A_0F$, so $\int g|F|^2\, dm = 0$, all $g \in A_0$. Since $\int |F|^2\, dm = 1$,

$|F|^2\, dm$ is a representing measure for ϕ. Consequently, $|F|^2\, dm = dm$ and $|F| = 1$ a.e. If $g \in \mathcal{M}$ and $g \perp FH^2$, then $\int fF\bar{g}\,dm = 0$, all $f \in A$. Since $F \perp A_0 g$, $\int \bar{f}F\bar{g}\, dm = 0$, all $f \in A_0$. So $F\bar{g} \perp A + \bar{A}$. Since $A + \bar{A}$ is weak* dense in L^∞ $F\bar{g} \equiv 0$, and $g \equiv 0$. Thus $\mathcal{M} = FH^2$.

Remark added in proof. Let dm be a representing measure for a homomorphism ϕ of a uniform algebra A such that

$$\log|\phi(f)| \leqq \int \log|f|\, dm, \qquad f \in A.$$

Then ϕ extends continuously to all H^p (dm), $p > 0$. Otherwise there would be a sequence $f_n \in A$ such that $\|f_n\|_p \to 0$ while $|\phi(f_n)| \to \infty$. But this is impossible in view of the inequality $\log|\phi(f)| \leqq (\int |f|^p\, dm)/p$, $f \in A$, derived from $\log s \leqq s^p/p$, $p > 0$, $s > 0$.

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