A CHARACTERIZATION OF THE CUTPOINT-ORDER ON A TREE(1)

BY
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1. **Introduction.** A *tree* is a continuum (compact, connected Hausdorff space) such that every two distinct points are separated by the omission of a third point. Let X be a tree, and let z be an arbitrary, fixed element of X. Let $Q(z)$ be the set of all pairs (a, b) in $X \times X$ such that at least one of the following three conditions is satisfied:

(i) $a = z$,

(ii) $a = b$,

(iii) a separates z and b in X.

It turns out that $Q(z)$ is a continuous partial order on X, and with respect to this partial order z is the unique minimal element. We shall refer to $Q(z)$, for any z in X, as a *cutpoint-order* [1] on the tree X. The purpose of this paper is to give a characterization of the cutpoint-order on a tree (Theorem 2). We also establish a new characterization of a tree from relation-theoretic and cohomological viewpoints (Theorem 1).

2. **Preliminaries.** A *relation* R on a space X is a subset of the cartesian product $X \times X$. If $x \in X$, we write $xR = \{y \mid (x, y) \in R\}$, $Rx = \{y \mid (y, x) \in R\}$, $RA = \bigcup\{Rx \mid x \in A\}$ and $AR = \bigcup\{xR \mid x \in A\}$. Following Wallace [11], we say that R is *left* (*right*) *monotone* if each $Rx(xR)$ is connected. A relation R is a *quasi-order* if it is reflexive and transitive; it is a *partial-order* if it is an anti-symmetric quasi-order. R is *total* if for every (x, y) either $(x, y) \in R$ or $(y, x) \in R$; a *total-order* is a partial-order that is also total. A set $A \subset X$ is an *R-chain* if $R \cap (A \times A)$ is a total-order on A. An *R-minimal* element a is an element such that $(x, a) \in R$ implies $(a, x) \in R$.

Definition 1. A space X is *unicoherent* if and only if X is connected and $X = A \cup B$ (with A and B closed and connected) implies that $A \cap B$ is connected. X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent.

Several characterizations of a tree have been given [2], [3], [14], and [15]. Perhaps the most useful of these characterizations is the following.

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LEMMA 1 [2], [3]. *A continuum X is a tree if and only if it is locally connected and hereditarily unicoherent.*

An excellent proof of this lemma may be found in Ward [15].

DEFINITION 2. A space X is said to be *semi-locally-connected* (abbreviated s.l.c.) *at a point* $x \in X$ if for each open set U in X containing x there exists an open set V such that $x \in V \subset U$ and such that $X - V$ has only a finite number of components. If X is s.l.c. at each of its points, it is said to be *s.l.c.*

The Alexander-Kolmogoroff-Wallace cohomology groups will be used as explicated in [5], [6], [8] and [12]. In what follows, the coefficient group will be arbitrary but fixed, and will therefore not be mentioned. The following Mayer-Vietoris exact sequence will be used.

LEMMA 2. *If X is a compact Hausdorff space and $X = A \cup B$, with A and B losed, then there is an exact sequence*

$$\cdots \xrightarrow{I^*} H^p(A \cap B) \xrightarrow{\Delta} H^{p+1}(X) \xrightarrow{J^*} H^{p+1}(A) \times H^{p+1}(B) \xrightarrow{I^*} \cdots$$

such that $\Delta = 0$ if $p = 0$ and $A \cap B$ is connected.

Following Wallace [10], if $C \subset D$ and $h \in H^p(D)$, we denote by $h \mid C$ the image of h under the natural homomorphism induced by the inclusion map of C into D. Thus, if $h \in H^p(X)$, then $J^*(h) = (h \mid A, h \mid B)$.

DEFINITION 3. If X is a space, $A \subset X$, and h is a nonzero member of $H^p(A)$, then a closed set F ($F \subset A$) is a *floor* for h if and only if $h \mid F \neq 0$ while $h \mid F_0 = 0$ for each closed proper subset F_0 of F.

LEMMA 3 (FLOOR THEOREM [12]). *If A a closed subset of a compact Hausdorff space X and h is a nonzero member of $H^p(A)$, then h has a floor. Moreover, every floor is connected.*

The closure of a set A will be denoted by A^*, and the empty-set by $\square$.

3. **A characterization of trees.** In 1953, A. D. Wallace [9] proved that a one-codimensional compact connected and locally connected topological semigroup with unit and zero is a tree (for the definition and properties of codimension, see Cohen [4]). L. W. Anderson and L. E. Ward, Jr. in 1961 [1] modified Wallace's result by eliminating the necessity of hypothesizing a unit. More precisely, they proved that if X is a compact, connected, locally connected, one-codimensional topological semilattice, then X is a tree. Wallace [10] improved this result by weakening the local connectedness of X to semilocal connectedness of X. These elegant results on topological algebra motivated the following theorem, which bears a relation-theoretic analogy.

THEOREM 1. *If X is an s.l.c. compact Hausdorff space of codimension one equipped with a relation R such that*

(i) $R = R^*$ *and* $RX = X$,

(ii) $H^1(Rx) = 0$ *for each* x *in* X,

(iii) *the collection* $\{Rx \mid x \in X\}$ *has the finite intersection property (abbreviated f.i.p.), and*

(iv) $Ra \cap Rb$ *is connected for each pair* (a, b) *in* $X \times X$,

then X *is a tree. Conversely, a tree satisfies all of the hypotheses described above.*

LEMMA 4. *If* X *is regular and is s.l.c. at* $x \in X$, *and if* x *does not separate two points* a *and* b *in* X, *then there exists a closed and connected subset* N *of* X *such that* $\{a, b\} \subset N \subset X - x$.

This lemma was first proved by G. T. Whyburn [16] for the particular case in which X was assumed to be a metric continuum. The general case was implicit in a paper by Wallace [10].

Proof of Theorem 1. We shall show that X is hereditarily unicoherent and then using this, together with Lemma 4, to show that X is a tree. It follows from (i), (ii), and (iii) that

$$X = \bigcup \{Rx \mid x \in X\}$$

is connected, and thus X is a continuum.

We first show $H^1(X) = 0$. If there were a nonzero $h \in H^1(X)$, then there would be a nonvoid maximal tower τ of closed subsets A of X such that $h \mid RA \neq 0$. Let $A_0 = \bigcap \{A \mid A \in \tau\}$. Then $h \mid RA_0 \neq 0$, for if $h \mid RA_0 = 0$, then by the reduction theorem [8] there would be an open $V \supset RA_0$ such that $h \mid V^* = 0$. Let $U = \{x \mid Rx \subset V\}$; then it follows from (i) that U is an open set containing A_0 such that $RU \subset V$. Thus, there is an A in τ with $A \subset U$ and $RA \subset RU$ $(\subset V^*)$; therefore $h \mid RA = 0$, a contradiction.

Case 1. Card $A_0 = 1$; that is, $A_0 = \{x\}$. By (ii), $H^1(RA_0) = 0$, a contradiction.

Case 2. Card $A_0 > 1$. Write $A_0 = A_1 \cup A_2$, where both A_1 and A_2 are proper closed subsets of A_0. We consider the part

$$H^0(RA_1 \cap RA_2) \xrightarrow{\Delta} H^1(RA_0) \xrightarrow{J^*} H^1(RA_1) \times H^1(RA_2)$$

of the Mayer-Vietoris exact sequence (Lemma 2). Since by (iii) and (iv) the set

$$RA_1 \cap RA_2 = \bigcup \{Ra \cap Rb \mid (a, b) \in A_1 \times A_2\}$$

is connected, $\Delta = 0$ by Lemma 2, and

$$h \mid RA_0 \in \operatorname{Ker} J^* = \operatorname{Im} \Delta = 0,$$

a contradiction.

Since X is a continuum and $H^1(X) = 0$, X is unicoherent ([2] and [3]). X being of codimension one and $H^1(X) = 0$ imply that $H^1(K) = 0$ for every closed subset K of X [4], and thus every subcontinuum of X is unicoherent.

We now prove that every two points of X are separated in X by a third point.

Suppose there were two points a and b such that no point separates a and b in X. Then by Lemma 4, for any p different from both a and b, there would exist a continuum P that is irreducible from a to b and does not contain p. If q were an element of P distinct from a and b, there would also exist a continuum Q, irreducible from a to b, that does not contain q. But then $P \cup Q$ would be a subcontinuum of X that is not unicoherent, since (by our selection of P and Q) $P \cap Q$ would obviously not be connected. This contradiction proves that X is a tree. Our proof depends heavily on Wallace [10].

The proof for the converse of this theorem is included in the proof of the next theorem.

4. **A characterization of the cutpoint-order on a tree.** The main purpose of the next theorem is to characterize the cutpoint-order on a tree from relation-theoretic and cohomological stand-points. A relation R on a space X is said to be *closed* if it is closed in the product $X \times X$.

THEOREM 2. *If X is a compact Hausdorff space and P is a relation on X, then the conditions*

(i) *X is of codimension one and s.l.c.,*

(ii) *P is a closed partial order,*

(iii) *P is both left and right monotone and $H^1(Px) = 0$ for every x in X, and*

(iv) *$\{Px \mid x \in X\}$ has the f.i.p.*

are necessary and sufficient that X be a tree and that P be a cutpoint-order.

Proof. We first prove the sufficiency. Conditions (ii), (iv) and the first half of (iii) imply that

$$Pa \cap Pb = \bigcup \{Px \mid x \in Pa \cap Pb\}$$

is connected, and thus Theorem 1 implies that X is a tree.

Since X is compact and $\{Px \mid x \in X\}$ has the f.i.p., $\bigcap \{Px \mid x \in X\}$ is is a single point, the unique P-minimal element of X. Let us denote by $\{0\}$ the set

$$\bigcap \{Px \mid x \in X\}.$$

We prove that $P = Q(0)$. If $(a, b) \in Q(0)$ and $a = 0$ or $a = b$, then clearly (a, b) is also in P. If a separates 0 and b in X, then since Pb is a continuum containing 0 and b, it must contain a, and we again conclude that (a, b) is in P. Thus $Q(0) \subset P$. Conversely, if (a, b) is in P, then since a is in $aP \cap Pb$, and since both aP and Pb are continua, $aP \cup Pb$ is a subcontinuum of the tree X, and therefore by Lemma 1 it is unicoherent. Thus $aP \cap Pb$ is also a continuum. Now, by virtue of Hausdorff's Maximality Principle, $aP \cap Pb$ has a maximal P-chain C; such a P-chain willl be shown to be closed, connected, and unique. The closedness of C is proved in [7]. Suppose C were not connected, then there would exist two non-void disjoint closed sets A and B such that $C = A \cup B$ and $b \in B$. The set A contains a maximal element m. Define A' and B' by the equations

$$A' = Pm \cap C \text{ and } B' = C - Pm.$$

Then $B' \subset mP$, and since $A \subset A'$, it follows that $B' \subset B$. Now

$$A' \cap B'^* \subset Pm \cap (mP \cap B) = (Pm \cap mP) \cap B = \square,$$

therefore

$$C = A' \cup B'$$

is a separation. If b_0 designates the minimal element in B', then by the maximality of C

$$mP \cap Pb_0 = \{m, b_0\};$$

this contradicts the connectedness of $mP \cap Pb_0$. Therefore, any maximal P-chain in $aP \cap Pb$ is connected. We now show that C is unique. For, if C and C' were two distinct maximal P-chains in $aP \cap Pb$, then both C and C' would contain a and b; therefore, $C \cup C'$ would be connected, and hence $C \cap C'$ would be connected. But for $x \in C - C'$,

$$\begin{aligned} C \cap C' &= (px \cup xP) \cap C \cap C' \\ &= (px \cap C \cap C') \cup (xP \cap C \cap C') \end{aligned}$$

is obviously a separation; this is a contradiction. *Throughout the rest of proof, the unique P-chain containing a and b will be denoted by $C_P(a, b)$.*

Since $(0, b) \in Q(0) \subset P$ and X is a tree, there exists a unique connected Q-chain [14] $C_Q(0, b) \subset Pb$ that contains both 0 and b. Pb must also have a connected P-chain containing both 0 and b, and this P-chain must be unique. We denote by $C_P(0, b)$ the unique connected P-chain in Pb contining 0 and b. Since a Q-chain is also a P-chain,

$$C_P(0, b) = C_Q(0, b).$$

Similarly, there is a unique connected P-chain $C_P(0, a)$ in Pa containing both 0 and a. Clearly,

$$C_P(0, a) \cup C_P(a, b) = C_P(0, b) = C_Q(0, b).$$

As a consequence, $a \in C_Q(0, b)$, and hence $(a, b) \in Q$, which was to be proved.

We next prove the necessity. Let X be tree, and let P be the cutpoint-order on X with respect to a point z in X. We shall prove that X and P satisfy the conditions (i), (ii), (iii), and (iv) stated in the theorem.

Proof of (i). By Ward [15], a tree is a compact connected commutative idempotent semigroup with zero; therefore, it is acyclic [10]. Hence, in particular, $H^1(X) = 0$. We now show that $H^1(A) = 0$ for every $A = A^* \subset X$ and thus X is of codimension one, unless X is degenerate. Suppose on the contrary that

$H^1(A) \neq 0$ for some closed subset A of X. If h is a nonzero member of $H^1(A)$, then by the Floor Theorem (Lemma 3) there exists a floor $F \subset A$ for h, which is connected. The set F being a subcontinuum of a tree is itself a tree and hence is acyclic. Therefore $H^1(F) = 0$, which contradicts the fact that F is a floor, and thus $H^1(A) = 0$. The semilocal connectedness of X follows from the fact that X is compact and locally connected (Lemma 1).

Proof of (ii). This is proved in Ward [15].

Proof of (iii). The cutpoint-order P is order dense [14], and since $P = P^*$ by (ii), every maximal P-chain in Px is connected [13]; thus Px is connected. Similarly, each xP is connected. Indeed, Px itself is a tree and therefore, as has been proved in (i), $H^1(Px) = 0$.

Proof of (iv). This is obvious, since $z \in Px$ for every $x \in X$.

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