SOME INCLUSION RELATIONS BETWEEN MATRICES COMPOUNDED FROM CESARO MATRICES

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1. **Introduction.** Let $\mathscr{P}$ denote the family of lower triangular matrices which define regular sequence to sequence transformations and which have nonnegative elements and nonzero elements on the leading diagonal; i.e. $B = (b_{nk}) \in \mathscr{P}$ if and only if $b_{nk} \geqq 0$ $(n = 0, 1, \cdots; k = 0, 1, \cdots)$, $b_{nk} = 0$ $(n < k)$, $b_{nn} > 0$ $(n = 0, 1, \cdots)$ and $b_{nk} \to 0$ $(n \to \infty, k = 0, 1, \cdots)$,

$$\lim_{n \to \infty} \sum_{k=0}^{n} b_{nk} = 1.$$

Let $\{A(r)\}$ $(r = 1, 2, \cdots)$ be any sequence of infinite matrices. If $A(r) = (a_{nk}(r))$ $(r = 1, 2, \cdots)$ and if $\{r_n\}$ is any sequence of positive integers, the matrix $A\{r_n\} = (a_{nk}(r_n))$, which has as its nth row the nth row of $A(r_n)$, is said to be compounded from the sequence $\{A(r_n)\}$ or to be a compounded matrix.

The results set out in the following theorem concern some properties of compounded matrices.

THEOREM A. (i) *If $A(r) \in \mathscr{P}$ $(r = 1, 2, \cdots)$ then there is an increasing sequence $\{R_n\}$ of positive integers such that if $1 \leqq r_n \leqq R_n$ the compounded matrix $A\{r_n\}$ is regular.*

(ii) *If $A(r) \in \mathscr{P}$ and $A(r+1)\,(A(r))^{-1} \in \mathscr{P}$ $(r = 1, 2, \cdots)$, if $\{r_n\}$ and $\{r'_n\}$ are sequences of positive integers such that $A\{r_n\}$ and $A\{r'_n\}$ are regular and if $r'_n \leqq r_n$ for all sufficiently large values of n then*[1] *$A\{r'_n\} \subseteq A\{r_n\}$.*

(iii) *If $A(r) \in \mathscr{P}$ and $A(r)\,(A(r+1))^{-1} \in \mathscr{P}$ $(r = 1, 2, \cdots)$, if $\{r_n\}$ and $\{r'_n\}$ are sequences of positive integers such that $A\{r_n\}$ and $A\{r'_n\}$ are regular and if $r'_n \leqq r_n$ for all sufficiently large values of n then $A\{r'_n\} \subseteq A\{r_n\}$.*

Of these results (i)[2] and (ii) are due to Agnew ([1], Theorems 3.1 and 3.2 and the remarks in §5; cf. also the references given there) and (iii) may be obtained by making simple changes in the arguments used to prove (ii).

Now suppose that $A(r) \in \mathscr{P}$ $(r = 1, 2, \cdots)$ that $A(1) \subseteq A(2) \subseteq \cdots$ (this is certainly

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(1) Throughout this note we write $A \subseteq B$ if $s_n \to s(A)$ implies $s_n \to s(B)$, and $A \subset B$ if $A \subseteq B$ but there is a sequence $\{s_n\}$ such that $s_n \to s(B)$ but $s_n \not\to s(A)$. If $A \subseteq B$ and $B \supseteq A$ we write $A \equiv B$.

(2) This result is stated in [1] under the additional hypothesis $A(r+1)\,(A(r))^{-1} \in \mathscr{P}$ but inspection of the proof shows that this condition is not in fact used.

the case if $A(r+1)\ (A(r))^{-1}\in\mathscr{P}$) and that $A(r)\subseteq B$ $(r=1,2,\cdots)$ where B is some regular matrix. It is natural to inquire what relation exists, if any, between B and a compounded matrix $A\{r_n\}$. If $A\{r_n\}$ is regular an attractive conjecture is that $A\{r_n\}\subseteq B$. This can fail to happen in a rather spectacular way, as the following example shows. For $r=1,2,\cdots$ let $A(r)$ be the matrix which transforms a sequence $\{s_n\}$ into the sequence $\{t_n\}$ where

$$t_n = r^{-1}s_n + (1-r^{-1})(n+1)^{-1}\sum_{\nu=0}^{n} s_\nu.$$

Then each $A(r)$ is a Mercerian matrix [5, p. 104] and is equivalent to convergence so that trivially $A(1)\subseteq A(2)\subseteq\cdots\subset C(\alpha)$, $0<\alpha\leqq 1$. If $r_n=n$ the compounded matrix $A\{r_n\}$ transforms $\{s_n\}$ into $\{t'_n\}$ where

$$t'_n = n^{-1}s_n + (1-n^{-1})(n+1)^{-1}\sum_{\nu=0}^{n} s_\nu$$

and so $A\{r_n\}$ is regular. On the other hand if $s_n\to sC(1)$ then $(n+1)^{-1}\sum_{\nu=0}^{n}s_\nu\to s$ and [5, p. 101] $s_n-s=o(n)$. It follows that $n^{-1}s_n=o(1)$ and hence that $t'_n\to s$ as $n\to\infty$. Consequently $C(\alpha)\subset A\{r_n\}$ for $0<\alpha<1$.

In this example the condition $A(r+1)\ (A(r))^{-1}\in\mathscr{P}$ is not fulfilled, but, as we shall see below (Theorem 4), even if it is there may still be a regular compounded matrix $A\{r_n\}$ such that $B\subset A\{r_n\}$.

Throughout the rest of this note we consider matrices compounded from Cesaro matrices. We write(3)

$$\varepsilon_n(\alpha) = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \tag{1.3}$$

so that, for $\alpha>-1$, $C(\alpha)$, the Cesaro matrix of order α, is the lower triangular matrix (a_{nk}) where $a_{nk}=\varepsilon_{n-k}(\alpha-1)/\varepsilon_n(\alpha)$. If $\alpha_n>-1$ $(n=0,1,\cdots)$ then $C\{\alpha_n\}$ denotes the lower triangular compounded matrix (b_{nk}) where $b_{nk}=\varepsilon_{n-k}(\alpha_n-1)/\varepsilon_n(\alpha_n)$.

It is well known and easily verified that if $\alpha_n\geqq 0$ $(n=0,1,\cdots)$ then $C\{\alpha_n\}$ is regular if and only if $\alpha_n=o(n)$ as $n\to\infty$.

Agnew [1, Theorem 6.1] has studied the relation between $C\{\alpha_n\}$ $(\alpha_n\uparrow\rightarrow\infty)$ and Abel's method. Here we suppose that $\{\alpha_n\}$ is a monotone sequence converging to a real number α and consider inclusion relations of the form $C\{\alpha_n\}\subseteq C(\alpha)$. We show (for example) that if $\{\alpha_n\}$ increases to α sufficiently rapidly then $C\{\alpha_n\}\equiv C(\alpha)$, and that otherwise $C\{\alpha_n\}\subset C(\alpha)$. In this latter case we show that for a certain class of sequences $\{\alpha_n\}$ the "gap" between $C\{\alpha_n\}$ and $C(\alpha)$ may be filled by certain well-known Nörlund matrices.

In addition to the matrices $C(\alpha)$ and $\{C\alpha_n\}$ defined above we require the lower triangular matrices $C(\alpha,\gamma)$ and $C(\{\alpha_n\},\gamma)$ whose (n,k)th elements are given, for

(3) Certain identities involving the binomial coefficients $\varepsilon_n(\alpha)$ which we use freely can be found in [7, p. 77].

$0 \leqq k \leqq n$ by $\varepsilon_{n-k}(\alpha-1)\varepsilon_k(\gamma)/\varepsilon_n(\alpha+\gamma)$ and $\varepsilon_{n-k}(\alpha_n-1)\varepsilon_k(\gamma)/\varepsilon_n(\alpha_n+\gamma)$ respectively.

2. In this section we prove the following two theorems.

THEOREM 1. *If* $\{\alpha_n\}$ *is a nondecreasing sequence converging to a real number* $\alpha\ (>0)$ *then*

(i) $C\{\alpha_n\} \subseteq C(\alpha)$,

(ii) $C\{\alpha_n\} \equiv C(\alpha)$ *if and only if* $\{(\alpha-\alpha_n)\log n\}$ *is bounded.*

THEOREM 2. *If* $\{\alpha_n\}$ *is a nonincreasing sequence converging to a real number* $\alpha\ (\geqq 0)$ *then*

(i) $C(\alpha) \subseteq C\{\alpha_n\}$,

(ii) $C(\alpha) \equiv C\{\alpha_n\}$ *if* $(\alpha_n - \alpha)\log n$ *is bounded,*

(iii) $C(\alpha) \subset C\{\alpha_n\}$ *if* $(\alpha_n-\alpha)\log n \to \infty$ *as* $n \to \infty$.

We require two lemmas.

LEMMA 1. *If* $\alpha_p \geqq 0\ (p=0,1,\cdots)$ *and* $\{\alpha_p\}$ *is bounded*

$$\varepsilon_n(\alpha_p) = \frac{n^{\alpha_p}}{\Gamma(\alpha_p+1)}\left(1+O\left(\frac{1}{n}\right)\right),$$

uniformly in n and p.

See [7, p. 77]. The proof given there is for constant sequences $\{\alpha_n\}$ but is easily seen to cover the present case.

LEMMA 2(4). *If* $0 \leqq m \leqq n$, *if* $0<\alpha_r \leqq 1\ (r=0,1,\cdots)$ *and if either* (i) $\{\alpha_r\}$ *is a nondecreasing sequence, or* (ii) $\{\alpha_r\}$ *and* $\{r^{-1}(1-\alpha_r)\}$ $(r \geqq 1)$ *are nonincreasing sequences, then for any sequence* $\{s_n\}$ *there is an integer p such that* $0 \leqq p \leqq m$ *and*

$$(\varepsilon_n(\alpha_n-1))^{-1}\left|\sum_{\nu=0}^{m}\varepsilon_{n-\nu}(\alpha_n-1)s_\nu\right| \leqq (\varepsilon_p(\alpha_p-1))^{-1}\left|\sum_{\nu=0}^{p}\varepsilon_{p-\nu}(\alpha_p-1)s_\nu\right|.$$

Proof. The result is trivially true if $m=n$ or $m=0$ and we suppose that $0<m<n$. It is easily verified that for fixed m and n, $\varepsilon_{n-\nu}(\alpha_n-1)(\varepsilon_{m-\nu}(\alpha_m-1))^{-1}$ is a nondecreasing function of ν in the range $0 \leqq \nu \leqq m$.

Consequently there is a nonincreasing sequence $m_1 m_2, \cdots$ of positive integers such that

$$\begin{aligned}\left|\sum_{\nu=0}^{m}\varepsilon_{n-\nu}(\alpha_n-1)s_\nu\right| &= \left|\sum_{\nu=0}^{m}\varepsilon_{n-\nu}(\alpha_n-1)(\varepsilon_{m-\nu}(\alpha_m-1)^{-1}\ \varepsilon_{m-\nu}(\alpha_m-1)s)_\nu\right| \\ &\leqq \varepsilon_n(\alpha_n-1)(\varepsilon_m(\alpha_m-1))^{-1}\left|\sum_{\nu=0}^{m_1}\varepsilon_{m-\nu}(\alpha_m-1)s_\nu\right| \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leqq \varepsilon_n(\alpha_n-1)(\varepsilon_{m_k}(\alpha_{m_k}-1))^{-1}\left|\sum_{\nu=0}^{m_{k+1}}\varepsilon_{m_k-\nu}(\alpha_{m_k}-1)s_\nu\right|.\end{aligned}$$

(4) This lemma and its proof are given in the case of constant sequences $\{\alpha_r\}$ by Bosanquet [3, Lemma 7].

Since the sequence $\{m_k\}$ is nonincreasing there is an integer ρ such that $m_{\rho+1} = m_\rho = p$ (say) and in this case we have $0 \leqq p \leqq m$ and

$$\left| \sum_{v=0}^{m} \varepsilon_{n-v}(\alpha_n - 1)s_v \right| \leqq \varepsilon_n(\alpha_n - 1)(\varepsilon_p(\alpha_p - 1))^{-1} \left| \sum_{v=0}^{p} \varepsilon_{p-v}(\alpha_p - 1)s_v \right|$$

which is the required result.

We also note that

$$C\{\alpha_n\} = C(\{\alpha_n - \alpha\}, \alpha)\, C(\alpha). \tag{2.1}$$

Proof of Theorem 1. (i) We have to show that $s_n \to sC\{\alpha_n\}$ implies $s_n \to sC(\alpha)$, and since both $C\{\alpha_n\}$ and $C(\alpha)$ are regular matrices it is sufficient to obtain the result when $s = 0$. It is also clear, from Theorem A(ii) that we may suppose $\alpha_n \neq \alpha$ $(n = 0, 1, \cdots)$. Consequently from (2.1) it is sufficient to show that $t_n \to 0$ $C(\{\alpha_n - \alpha\}, \alpha)$ implies $t_n = o(1)$, or, writing $t_{-1} = 0, x_v = \varepsilon_{v-1}(\alpha)t_{v-1} - \varepsilon_v(\alpha)t_v$ that

$$\sum_{v=0}^{n} \varepsilon_{n-v}(\alpha_n - \alpha)x_v = o(\varepsilon_n(\alpha_n)) \tag{2.2}$$

implies

$$(t_n\varepsilon_n(\alpha) =)\ \sum_{v=0}^{n} x_v = o(\varepsilon_n(\alpha)). \tag{2.3}$$

Since $(\varepsilon_{n-v}(\alpha_n - \alpha))^{-1}$ is an increasing function of v in the range $0 \leqq v \leqq n$ we have, using Lemma 2 (case (i) with α_n replaced by $\alpha_n - \alpha - 1$)

$$\begin{aligned} \left| \sum_{v=0}^{n} x_v \right| &= \left| \sum_{v=0}^{n} (\varepsilon_{n-v}(\alpha_n - \alpha))^{-1} \varepsilon_{n-v}(\alpha_n - \alpha)x_v \right| \\ &\leqq 2 \max_{0 \leqq p \leqq n} \left| \sum_{v=0}^{p} \varepsilon_{n-v}(\alpha_n - \alpha)x_v \right| \\ &\leqq 2 \max_{0 \leqq p \leqq r} \max_{0 \leqq k \leqq p} \frac{\varepsilon_n(\alpha_n - \alpha)}{\varepsilon_k(\alpha_k - \alpha)} \left| \sum_{v=0}^{k} \varepsilon_{k-v}(\alpha_k - \alpha)x_v \right| \\ &= o(n^{\alpha_n}), \end{aligned}$$

by (2.2) and Lemma 1, so that (2.3) holds.

To prove (ii) it is sufficient, in view of (2.1), to show that the matrix $C(\{\alpha_n - \alpha\}, \alpha)$ is regular if and only if $(\alpha - \alpha_n)\log n = O(1)$. Now $\sum_{v=0}^{n} \varepsilon_{n-v}(\alpha_n - \alpha - 1)\varepsilon_v(\alpha) = \varepsilon_n(\alpha_n)$ and it is easily verified, using Lemma 1, that $\varepsilon_{n-v}(\alpha_n - \alpha - 1)\varepsilon_v(\gamma) \cdot (\varepsilon_n(\alpha_n))^{-1} = o(1)$ for $v = 0, 1, \cdots$. Hence $C(\{\alpha_n - \alpha\}, \alpha,)$ is regular if and only if

$$\begin{aligned} \sum_{v=0}^{n} \left| \varepsilon_{n-v}(\alpha_n - \alpha - 1) \right| \varepsilon_v(\alpha) &= 2\varepsilon_n(\alpha) - \sum_{v=0}^{n} \varepsilon_{n-v}(\alpha_n - \alpha - 1)\varepsilon_v(\alpha) \\ &= 2\varepsilon_n(\alpha) - \varepsilon_n(\alpha_n) \\ &= O(\varepsilon_n(\alpha_n)) \end{aligned}$$

i.e., using Lemma 1, if and only if $(\alpha - \alpha_n)\log n = O(1)$.

Proof of Theorem 2. A straightforward calculation shows that $C(\{\alpha_n - \alpha\},\alpha)$ is regular and (i) follows.

To prove (ii) we write $\beta_n = \alpha_n - \alpha$ and consider first the case when $\{\beta_n\}$ and $\{(1-\beta_n)/n\}$ are decreasing sequences. In this case it follows from Lemma 2 that if $0 \leqq m \leqq n$

$$(\varepsilon_n(\alpha_n))^{-1} \left| \sum_{v=0}^{m} \varepsilon_{n-v}(\beta_n - 1)\,\varepsilon_v(\alpha)s_v \right|$$

$$\leqq \max_{0\leqq p\leqq m} (\varepsilon_p(\alpha_p))^{-1} \left| \sum_{v=0}^{p} \varepsilon_{p-v}(\beta_p - 1)\,\varepsilon_v(\alpha)s_v \right| .$$

It follows [6, p. 75] that if $s_n = o(1)\,C(\{\beta_n\},\alpha)$ then

$$s_k \sup_{n\geqq k} \varepsilon_{n-k}(\beta_n - 1)\,\varepsilon_k(\alpha)\,(\varepsilon_n(\alpha_n))^{-1} = o(1) \text{ as } k \to \infty.$$

Now

$$\varepsilon_{n-k}(\beta_n - 1) = \frac{n(n-1)\cdots(n-k+1)}{(\beta_n + n - 1)(\beta_n + n - 2)\cdots(\beta_n + n - k)}\varepsilon_n(\beta_n - 1)$$

and it is easily verified that, since $\{(1-\beta_n)/n\}$ decreases, $\{(n-p)/(\beta_n + n - p - 1)\}$ is a decreasing sequence for $n \geqq k$ and $p = 0,1,\cdots,k-1$. Moreover, since $\{\beta_n\}$ decreases, $\{\varepsilon_n(\beta_n - 1)(\varepsilon_n(\alpha_n))^{-1}\}$ decreases. Consequently, for each fixed k, $\{\varepsilon_{n-k}(\beta_n - 1)\,(\varepsilon_n(\alpha_n))^{-1}\}$ decreases for $n \geqq k$ and so

$$\sup_{n\geqq k} \varepsilon_{n-k}(\beta_n - 1)\,\varepsilon_k(\alpha)\,(\varepsilon_n(\alpha_n))^{-1} = \varepsilon_k(\alpha)\,(\varepsilon_k(\alpha_k))^{-1}.$$

Thus, in the special case being considered, if $s_n = o(1)C(\{\beta_n\},\alpha)$, then $s_k\varepsilon_k(\alpha)\,(\varepsilon_k(\alpha_k))^{-1} = o(1)$, i.e., by Lemma 1, $s_k = o(k^{\beta_k})$. In particular, if $\alpha_n = \alpha + K/\log n$ for some positive K and all sufficiently large n, then $s_n = o(1)\,C(\{\beta_n\},\alpha)$ implies $s_n = o(1)$, i.e., that $C(\{\beta_n\},\alpha)$ is equivalent to convergence. Now if $(\alpha_n - \alpha)\log n = O(1)$ we have $\alpha_n \leqq \alpha + K/\log n$ for some positive K and all sufficiently large n. It follows from Theorem A (iii) that $C(\{\alpha_n - \alpha\},\alpha)$ is equivalent to convergence and in view of (2.1) this proves (ii).

To obtain (iii) we note that $C(\{\alpha_n - \alpha\},\alpha)$ is regular and that, since we may assume without loss of generality that $\alpha_n - \alpha < 1$, $\varepsilon_{n-v}(\alpha_n - \alpha - 1)\,\varepsilon_v(\alpha)$ increases with v for $0 \leqq v \leqq n$ so that, writing $C(\{\alpha_n - \alpha\},\alpha) = (a_{nv})$ we have

$$\begin{aligned}\sum_{v=0}^{\infty} \left| a_{nv+1} - a_{nv} \right| &= 2a_{nn} - a_{n0} \\ &= o(1)\end{aligned}$$

as $n \to \infty$ by Lemma 1 since $(\alpha_n - \alpha)\log n \to \infty$. It follows [1(a), p. 130] that $C(\{\alpha_n - \alpha\},\alpha)$ evaluates some divergent sequence, and (iii) follows.

3. Let $r_n(h,k)\,(h,k,$ real $n=0,1,\cdots)$ be defined by

$$(3.1)\qquad (1-x)^{-h-1}\left(\frac{\log(1-x)}{-x}\right)^k = \sum_{n=0}^{\infty} r_n(h,k)x^n \qquad (|x|<1).$$

It is known that [7, p. 193]

$$(3.2)\qquad r_n(h,k) \sim \frac{n^h}{\Gamma(h+1)}(\log n)^k \qquad (h \neq -1,-2,\cdots)$$

$$(3.3)\qquad r_n(h,k) \sim (-1)^{k-1}(|k|-1)!\,kn^h(\log n)^{k-1} \quad (h=-1,-2,\cdots).$$

Let $p_n(k)=r_n(-1,k)$, and let $L(k)$ denote the Nörlund matrix [5, p. 64] generated by the sequence $\{p_n(k)\}$. If $k \geqq 0$ we have $p_n(k) \geqq 0$ $(n=1,2,\cdots)$ and $p_0(k)=1$. Moreover from (3.1) and (3.2)

$$\sum_{v=0}^{n} p_n(k) = r(0,k) \sim (\log n)^k$$

and from (3.3)

$$p_n(k) \sim \frac{k(\log n)^{k-1}}{n}$$

so that [5, p. 64] if $k \geqq 0$, $L(k)$ is regular. The matrix $L(1)$ generates the harmonic means of M. Riesz and the matrices $L(k)$ may be regarded as iterates of $L(1)$ in the same sense that the matrices $C(\alpha)$ are iterates of $C(1)$.

We prove next

THEOREM 3. *If $\alpha \geqq 0$ and $\alpha_n - \alpha = k\log\log n/\log n$ for sufficiently large values of n, where $k \geqq 0$, then $C\{\alpha_n\} \equiv L(k)\,C(\alpha)$.*

We require some further lemmas.

LEMMA 3(5). *Suppose that $0 < p_n \leqq K$ $(n=0,1,\cdots)$, where K is independent of n, and that the sequence $\{q_n\}$ is defined by*

$$q_0p_0 = 1,\ q_np_0 + q_{n-1}p_1 + \cdots + q_0p_n = 0 \qquad (n=1,2,\cdots).$$

Suppose further that there is an integer N $(\geqq 0)$ such that

$$(3.4)\qquad q_v \leqq 0, \qquad v \geqq N,$$

$$(3.5)\qquad \sum_{r=0}^{v} q_v \geqq 0, \qquad v \geqq N.$$

Then for any sequence $\{s_v\}$

$$(3.6)\qquad \left|\sum_{v=0}^{m} p_{n-v}s_v\right| \leqq H \max_{0\leqq k\leqq m}\left|\sum_{v=0}^{k} p_{k-v}s_v\right| \qquad (0 \leqq m \leqq n)$$

where H is independent of n.

(5) This is an extension of some known results cf. [6, p. 43 and p. 128].

Proof. Since the result is clearly true if $m = n$ we suppose that $0 \leqq m < n$. From the definition of the sequence $\{q_n\}$ we have

$$\begin{aligned} s_\nu &= \sum_{r=0}^{\nu} q_{\nu-r} \sum_{t=0}^{r} p_{r-t}s_r \\ &= \sum_{r=0}^{\nu} q_{\nu-r}t_r \qquad \text{(say),} \end{aligned}$$

for $\nu = 0, 1, \cdots$.

Consequently

$$\begin{aligned} \left| \sum_{\nu=0}^{m} p_{n-\nu}s_\nu \right| &= \left| \sum_{\nu=0}^{m} p_{n-\nu} \sum_{r=0}^{\nu} q_{\nu-r}t_r \right| \\ &= \left| \sum_{r=0}^{m} t_r \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| \qquad (3.7) \\ &\leqq \max_{0 \leqq r \leqq m} \left| t_r \right| \sum_{r=0}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| . \end{aligned}$$

If $m \leqq N$

$$\sum_{r=0}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| \leqq K \sum_{r=0}^{N} \sum_{\nu=0}^{N} |q_\nu| = K \qquad \text{(say).}$$

If $m > N$ we write

$$\sum_{r=0}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| = \sum_{r=0}^{n-N} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| + \sum_{r=m-N+1}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| .$$

Now if $0 \leqq r \leqq m - N$ and $\nu > m - r$ we have $\nu > N$ so that $q_\nu \leqq 0$ by (3.4) and so for $0 \leqq r \leqq m - N$

$$\sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \geqq \sum_{\nu=0}^{n-r} p_{n-r-\nu}q_\nu = 0.$$

Consequently if $m > N$

$$\begin{aligned} \sum_{r=0}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| &\leqq \sum_{r=0}^{m} \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu + 2 \sum_{r=m-N+1}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu \right| \\ &\leqq \sum_{r=0}^{m} \sum_{\nu=0}^{m-r} p_{n-r-\nu}q_\nu + 2KN \sum_{\nu=0}^{N} |q_\nu| \\ &= \sum_{\nu=0}^{m} p_{n-\nu} \sum_{r=0}^{\nu} q_{\nu-r} + 2K' \\ &\leqq \sum_{\nu=0}^{n} p_{n-\nu} \sum_{r=0}^{\nu} q_\nu + 2K' \qquad \text{(by (3.5) since } m > N) \\ &= 1 + 2K' = K'' \quad \text{(say).} \end{aligned}$$

If we now choose $H = \max(K', K'')$ we have

$$\sum_{r=0}^{m} \left| \sum_{\nu=0}^{m-r} p_{n-r-\nu} q_\nu \right| \leqq H$$

and (3.6) follows from this and (3.7).

Lemma 4. *If $\gamma \geqq 0$ and if $\{\alpha_n\}$ satisfies either of the two conditions of Lemma 2 then $C\{\alpha_n\} \equiv C(\{\alpha_n\}, \gamma)$.*

Proof. The case $\gamma = 0$ being trivial we suppose that $\gamma > 0$. It follows from Theorem A (iii) and known results [2, Theorem 8] that $C(\{\alpha_n\}, \gamma) \subseteq C(1)$. Consequently if $s_n = o(1)C(\{\alpha_n\}, \gamma)$ and we write $\tau_n = \max_{0 \leqq \nu \leqq n} \left| \sum_{p=0}^{\nu} s_p \right|$ we shall have $\tau_n = o(n)$. Putting $\rho_n = [n - \tau_n]$ we have

$$\sum_{\nu=0}^{n} \varepsilon_{n-\nu}(\alpha_n - 1)s_\nu = \sum_{\nu=0}^{\rho_n} + \sum_{\rho_n+1}^{n} = \sum_1 + \sum_2 \quad \text{(say)}$$

where, since $\varepsilon_{n-\nu}(\alpha_n - 1)$ is an increasing function of ν for $0 \leqq \nu \leqq n$

$$\begin{aligned} \left| \sum_1 \right| &\leqq 2\varepsilon_{n-\rho_n}(\alpha_n - 1)\tau_n \\ &= o(n^{\alpha_n}) \end{aligned}$$

by Lemma 1. By Lemma 2 we have since $(\varepsilon_\nu(\gamma))^{-1}$ decreases as ν increases

$$\begin{aligned} \left| \sum_2 \right| &= \left| \sum_{\nu=\rho_n+1}^{n} \varepsilon_{n-\nu}(\alpha_n - 1)\,\varepsilon_\nu(\gamma)(\varepsilon_\nu(\gamma))^{-1} s_\nu \right| \\ &\leqq (\varepsilon_{\rho_n+1}(\gamma))^{-1} \max_{\rho_n+1 \leqq p \leqq n} \left| \sum_{\nu=0}^{p} \varepsilon_{n-\nu}(\alpha_n - 1)\,\varepsilon_\nu(\gamma) s_\nu \right| \\ &\leqq (\varepsilon_{\rho_n+1}(\gamma)) \max_{\rho_n+1 \leqq p \leqq n} \max_{0 \leqq k \leqq p} \frac{\varepsilon_n(\alpha_n - 1)}{\varepsilon_k(\alpha_k - 1)} \left| \sum_{\nu=0}^{k} \varepsilon_{k-\nu}(\alpha_{k-1})\,\varepsilon_\nu(\gamma) s_\nu \right| \\ &= o(n^{\alpha_n}) \end{aligned}$$

by Lemma 1. Consequently $s_n = o(1)\,C\{\alpha_n\}$.

Conversely if $s_n = o(1)\,C\{\alpha_n\}$ then, by Lemma 2,

$$\begin{aligned} \left| \sum_{\nu=0}^{n} \varepsilon_{n-\nu}(\alpha_n - 1)\,\varepsilon_\nu(\gamma)s_\nu \right| &\leqq 2\varepsilon_n(\gamma) \max_{0 \leqq p \leqq n} \left| \sum_{\nu=0}^{p} \varepsilon_{n-\nu}(\alpha_n - 1)s_\nu \right| \\ &\leqq 2\varepsilon_n(\gamma) \max_{0 \leqq p \leqq n} \max_{0 \leqq k \leqq p} \frac{\varepsilon_n(\alpha_n - 1)}{\varepsilon_k(\alpha_k - 1)} \\ &\quad \cdot \left| \sum_{\nu=0}^{k} \varepsilon_{k-\nu}(\alpha_{k-1})s_\nu \right| \\ &= o(n^{\alpha_n+\gamma}), \end{aligned}$$

and the result follows.

Proof of Theorem 4. It is clearly sufficient after (2.1) and Lemma 4 to show that $L(k) \equiv C\{\beta_n\}$ where $\beta_n = \alpha_n - \alpha = k \log\log n/\log n$ for $n \geqq 4$ and the initial values of β_n are chosen so that $\{(1-\beta_n)/n\}$ is decreasing. (Alteration of a finite number of the values of β_n clearly has no effect on the summability properties of $C\{\beta_n\}$.)

We first show that

$$C\{\beta_n\} \subseteq L(k). \tag{3.8}$$

Since both methods in (3.8) are regular it is sufficient to show that

$$\sum_{v=0}^{n} \varepsilon_{n-v}(\beta_n - 1)s_v = o((\log n)^k) \tag{3.9}$$

implies

$$\sum_{v=0}^{n} p_{n-v}(k)s_v = o((\log n)^k). \tag{3.10}$$

We abbreviate $p_n(k)$ to p_n for the rest of this proof. We then have, by Lemma 2,

$$\begin{aligned} \left| \sum_{v=0}^{n} p_{n-v}s_v \right| &= \left| \sum_{v=0}^{n} \rho_{n-v}\varepsilon_{n-v}(\beta_n - 1)s_v \right| \\ &= o(n^{\beta_n})\left\{ \sum_{v=0}^{n-1} \left| \rho_{n-v} - \rho_{n-v-1} \right| + \rho_0 \right\} \end{aligned}$$

where $\rho_s = p_s(\varepsilon_s(\beta_n - 1))^{-1}$. Now

$$\frac{\rho_s}{\rho_{s+1}} = \frac{p_s}{p_{s+1}}\,\frac{\beta_n + s}{s+1},$$

so that, by (3.3) and a routine calculation

$$\rho_s - \rho_{s+1} = \beta_n \rho_{s+1} O\left(\frac{1}{s}\right) + \rho_{s+1} O\left(\frac{1}{s\log s}\right).$$

Using Lemma 1 and (3.3) we have

$$\begin{aligned} \beta_n \sum_{s=1}^{n} \rho_{s+1} O\left(\frac{1}{s}\right) &= \sum_{s=1}^{n} O\left(\frac{(\log s)^{k-1}}{s^{\beta_n+1}}\right) \\ &= O(1); \end{aligned}$$

while, if we write

$$t_s = \sum_{r=s}^{\infty} \frac{(\log r)^{k-2}}{r^2} \quad \left(= O\left(\frac{(\log s)^{k-2}}{s}\right) \right)$$

and observe that $(\varepsilon_s(\beta_n - 1))^{-1} - (\varepsilon_{s+1}(\beta_n - 1))^{-1} = (\varepsilon_s(\beta_n))^{-1}$ we have

$$
\begin{aligned}
\sum_{s=2}^{n} \rho_{s+1} O\left(\frac{1}{s \log s}\right) &= O\left(\sum_{s=2}^{n} (\varepsilon_s(\beta_n - 1))^{-1} \frac{(\log s)^{k-2}}{s^2}\right) \\
&= O\left(\sum_{s=2}^{n} t_s(\varepsilon_s(\beta_n))^{-1} + t_n(\varepsilon_n(\beta_n - 1))^{-1}\right) \\
&= O\left(\sum_{s=1}^{n} \frac{(\log s)^{k-2}}{s^{\beta_{n+1}}}\right) + O\left(\frac{(\log n)^{k-2}}{\beta_n n^{\beta_n}}\right) \\
&= O(1).
\end{aligned}
$$

It follows that

$$\sum_{s=1}^{n} |\rho_s - \rho_{s+1}| = O(1).$$

Using this and (3.11) we obtain (3.10). To obtain the inclusion reverse to (3.8) we suppose that $s_n = o(1) L(k)$. If $q_n = q_n(k) = r_n(-1, -k)$ then $p_0 q_0 = 1$, $p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 = 0$ $(n \geqq 11)$. Moreover by (3.2)

$$q_n \sim -k \frac{(\log n)^{k-1}}{n} \text{ and } \sum_{\nu=0}^{n} q_\nu = r_n(0, -k) \sim (\log n)^{-k}.$$

Consequently $\{p_n\}$ and $\{q_n\}$ satisfy the conditions of Lemma 3 and so we have

$$
\begin{aligned}
\left| \sum_{\nu=0}^{n} \varepsilon_{n-\nu}(\beta_{n-1}) s_\nu \right| &= \left| \sum_{\nu=0}^{n} \tau_{n-\nu} p_{n-\nu} s_\nu \right| \\
&\leqq \max_{0 \leqq k \leqq n} \left| \sum_{\nu=0}^{k} p_{k-\nu} s_\nu \right| \left\{ \sum_{\nu=0}^{n-1} |\tau_{n-\nu} - \tau_{n-\nu-1}| + \tau_0 \right\} \\
&= o(\log n)^k \left\{ \sum_{\nu=1}^{n} |\tau_s - \tau_{s+1}| + \tau_0 \right\}
\end{aligned}
$$

where $\tau_s = \varepsilon_s(\beta_n - 1)/p$. The required result now follows by arguments similar to those used above.

A counterpart to Theorem 3 is:

THEOREM 4. *If* $\alpha > 0$ *and* $\alpha - \alpha_n = k \log\log n / \log n$ *for sufficiently large values of* n, *where* $k \geqq 0$, *then* $L(k)\, C\{\alpha_n\} \equiv C(\alpha)$.

The proof is similar to that of Theorem 3 and we omit the details.

We remark in conclusion that the form of Theorem 3 and 4 suggests that if we write $\beta_n = |\alpha - \alpha_n|$ and H for the Nörlund method generated by the sequence $\{(n+1)^{\beta_{n+1}} - n^{\beta_n}\}$ then $C\{\alpha_n\} \equiv HC(\alpha)$ or $HC\{\alpha_n\} = C(\alpha)$ according as $\{\alpha_n\}$ is decreasing or increasing.

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