MASSEY HIGHER PRODUCTS

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In this paper, we shall investigate some properties of a class of higher order cohomology operations of several variables. These operations, the higher products, were defined by Massey as a generalization of his triple product. There is a correspondence between the higher products and iterated Whitehead products in homotopy groups [5], [11].

It has been noted that the differentials in certain spectral sequences involving the Ext and Tor functors are related to Massey higher products [6], [9]. In particular, the differentials in a spectral sequence relating the cohomology of a space with that of its space of loops are generalized higher products.

In the first part we establish a number of properties of these higher products. These properties indicate the similarities between the higher products and the cup product. In the final section, the higher products are specialized to an operation of one variable. In certain cases, this operation may be evaluated in terms of primary Steenrod operations (Theorems 14, 19). These results are useful in the computation of the higher product structure of a space with coefficients in a field.

1. **Definitions.** Throughout this paper let (X,A) be a pair of topological spaces and let R be a commutative ring with identity. Also let $u_1,\cdots,u_k$ be positive dimensional cohomology classes, of dimensions $p_1,\cdots,p_k$ respectively, in the singular cohomology ring $H^*(X,A;R)$. Finally, let $p(i,j)=\sum_{r=i}^{j}(p_r-1)$.

Under certain conditions, we will be able to define the Massey k-fold product $\langle u_1,\cdots,u_k\rangle$ as a subset of $H^{p(1,k)+2}(X,A;R)$. We shall first describe a "higher operation" from a subset of the singular cochains to a subset of a cohomology group.

Let $C^*(X,A;R)$ be the singular cochain complex with the usual associative cup product pairing. Let $a_1,\cdots,a_k$ be cocycle representatives of $u_1,\cdots,u_k$ respectively. If $a\in C^p(X,A;R)$, then the symbol $\bar{a}$ will denote $(-1)^p a$.

Definition 1. A collection of cochains, $A=(a(i,j))$, for $1\leqq i\leqq j\leqq k$ and

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$(i,j) \neq (1,k)$ is said to be a defining system for the (cochain) product $\langle a_1, \cdots, a_k \rangle$ if

$$a(i,i) = a_i \quad \text{for} \quad i = 1, \cdots, k, \tag{1.1}$$

$$a(i,j) \in C^{p(i,j)+1}\ (X, A; R) \tag{1.2}$$

and

$$\delta a(i,j) = \sum_{r=i}^{j-1} \bar{a}(i,r)a(r+1,j). \tag{1.3}$$

The $p(1,k)+2$ dimensional cocycle, $c(A)$, defined by

$$c(A) \quad = \sum_{r=1}^{k-1} \bar{a}(1,r)a(r+1,k), \tag{1.4}$$

is called the related cocycle of the defining system A.

DEFINITION 2. The (cochain) k-fold product $\langle a_1, \cdots, a_k \rangle$ is said to be defined if there is a defining system for it. If it is defined, then $\langle a_1, \cdots, a_k \rangle$ consists of all classes $w \in H^{p(1,k)+2}(X, A; R)$ for which there exists a defining system A such that $c(A)$ represents w.

The above definition of the higher products differs from that given in [4] by the sign $(-1)^h$ where $h = \sum_{i=1}^{k-1} i p_{k-i} + (k-2)(k-1)/2$. This convention reduces the work of checking signs.

THEOREM 3. *The operation* $\langle a_1, \cdots, a_k \rangle$ *depends only on the cohomology classes of the cocycles* $a_1, \cdots, a_k$.

Proof. We need only show that $\langle a_1, \cdots, a_t, \cdots, a_k \rangle = \langle a_1, \cdots, a_t + \delta b, \cdots, a_k \rangle$ for $t = 1, \cdots, k$ and any $b \in C^{p_t - 1}(X, A; R)$. Given a defining system A for the former, we shall construct a defining system A' for the latter such that $c(A)$ is cohomologous to $c(A')$. This will imply that

$$\langle a_1, \cdots, a_t, \cdots, a_k \rangle \subset \langle a_1, \cdots, a_t + \delta b, \cdots, a_k \rangle.$$

Thee reverse inclusion will follow by symmetry.

Define a set of cochains $(a'(i,j))$ as follows:

$$a(t,t) \ = \ a_t + \delta b, \tag{1.5}$$

$$a'(i,t) \ = \ a(i,t) + a(i,t-1)b \quad \text{for } i < t, \tag{1.6}$$

$$a'(t,j) \ = \ a(t,j) - \bar{b}a(t+1,j) \quad \text{for } j > t, \tag{1.7}$$

$$a'(i,j) \ = \ a(i,j) \quad \text{if } i \neq t \quad \text{and} \quad j \neq t. \tag{1.8}$$

By a straightforward calculation, it is seen that A' is a defining system for $\langle a_1, \cdots, a_t + \delta b, \cdots, a_k \rangle$. If $1 < t < k$, then $c(A) = c(A')$. If $t = 1$, then they differ by $-\delta \bar{b} a(2,k)$. If $t = k$, they differ by $\delta a(1, k-1)b$. Q.E.D.

The preceding theorem enables us to make the following definition.

DEFINITION 4. A set of cocycles is said to be a defining system for the Massey k-fold product $\langle u_1, \cdots, u_k \rangle$ if it is one for $\langle a_1, \cdots, a_k \rangle$. The Massey k-fold product $\langle u_1, \cdots, u_k \rangle$ is said to be defined if $\langle a_1, \cdots, a_k \rangle$ is defined, in which case $\langle u_1, \cdots, u_k \rangle = \langle a_1, \cdots, a_k \rangle$ as subsets of $H^{p(1,k)+2}(X, A; R)$.

The 2-fold product $\langle u_1, u_2 \rangle$ is the subset consisting of the singular cup product $(-1)^{p_1} u_1 u_2$. The 3-fold product $\langle u_1, u_2, u_3 \rangle$ is defined if and only if the cup products $u_1 u_2 = 0$ and $u_2 u_3 = 0$. This is a secondary operation which differs from the Massey triple product by the sign $(-1)^{p_2+1}$[11].

The k-fold product is a $(k-1)$ order cohomology operation of k variables. In order for $\langle u_1, \cdots, u_k \rangle$ to be defined, it is necessary that the $(k-2)$ order operations $\langle u_1, \cdots, u_{k-1} \rangle$ and $\langle u_2, \cdots, u_k \rangle$ be defined and contain the zero element. In general, this condition is not sufficient. There must exist defining systems A' and A'' for $\langle u_1, \cdots, u_{k-1} \rangle$ and $\langle u_2, \cdots, u_k \rangle$ respectively, for which not only do the related cocycles of each cobound but also $a'(i,j) = a''(i,j)$ for $1 < i \leqq j < k$. In this case, we say that the $(k-1)$-fold products $\langle u_1, \cdots, u_{k-1} \rangle$ and $\langle u_2, \cdots, u_k \rangle$ vanish simultaneously.

The domain of the higher products may be extended somewhat. Let $\bar{X}$ be a topological space and let (X_i, A_i) be pairs of subspaces of $\bar{X}$, for $i = 1, \cdots, k$, such that $\bigcup_{r=1}^k A_r \subset \bigcap_{r=1}^k X_r$. Assume that the triads $(\bar{X}, A_i, A_j)$ are proper in the singular cohomology theory for $1 \leqq i, j \leqq k$. This condition will be satisfied if each of the spaces X_i and A_i are open in $\bar{X}$ or if $\bar{X}$ is a CW complex and each X_i and A_i are subcomplexes.

Let $u_1, \cdots, u_k$ be classes in the cohomology groups $H^{p_1}(X_1, A_1; R), \cdots, H^{p_k}(X_k, A_k; R)$ respectively. As before, under certain conditions it is possible to define the k-fold product $\langle u_1, \cdots, u_k \rangle$, this time as a subset of $H^{p(1,k)+2}(\bigcap_{r=1}^k X_r, \bigcup_{r=1}^k A_r; R)$.

2. **Properties.** The Massey higher products can be viewed as higher order analogues of the cup product. The properties we shall establish below are generalizations of well-known relations satisfied by the cup product.

The first several properties we shall list are functorial in nature. The proofs of these will be immediate from the definitions of the first section.

(2.1) *Naturality.* Let (Y, B) be a pair of spaces and R' a commutative ring with identity. Let $f: (Y, B) \to (X, A)$ be a map of pairs and $g: R \to R'$ be a map of rings. If $\langle u_1, \cdots, u_k \rangle$ is defined, then so is $\langle f^* g_\# u_1, \cdots, f^* g_\# u_k \rangle$ and $f^* g_\# \langle u_1, \cdots, u_k \rangle \subset \langle f^* g_\# u_1, \cdots, f^* g_\# u_k \rangle$. Furthermore if f^* and $g_\#$ are isomorphisms, then equality holds between the two operations.

(2.2) *Other Chain Complexes.* The higher products may be defined in an analogous way with the use of any other chain complex, $\bar{C}^*(X, A; R)$, which has an associative product. For example, for certain pairs of spaces, we could have used the simplicial theory, the Čech theory, or the Alexander-Spanier theory instead of the singular theory.

Assume that there is a chain equivalence between $\bar{C}^*(X,A;R)$ and the singular complex $C^*(X,A;R)$ such that the map from one complex to the other commutes with the cup product. Then the isomorphism induced by the chain equivalence preserves the higher products.

(2.3) *Scalar Multiplication.* Assume that the k-fold product $\langle u_1,\cdots,u_k\rangle$ is defined. Then, for any $x\in R$ and $t=1,\cdots,k$, the product $\langle u_1,\cdots,xu_t,\cdots,u_k\rangle$ is defined and $x\langle u_1,\cdots,u_k\rangle\subset\langle u_1,\cdots,xu_t,\cdots,u_k\rangle$.

(2.4) *Loop Suspension.* Let $\pi: PX\to X$ be the path-loop fibration over X. Then $E_A=\pi^{-1}(A)$ is the space of paths in X starting from the base point and ending in A. The relative loop suspension $\sigma: H^n(X,A;R)\to H^{n-1}(E_A;R)$ is defined as the composite homomorphism

$$H^n(X,A;R)\xrightarrow{\pi^*}H^n(PX,E_A;R)\underset{\sim}{\overset{\delta}{\leftarrow}}H^{n-1}(E_A;R).$$

THEOREM 5. *Assume that $\langle u_1,\cdots,u_k\rangle$ is defined as a subset of $H^{p(1,k)+2}(X,A;R)$. Then $\sigma\langle u_1,\cdots,u_k\rangle$ is the subset of $H^{p(1,k)+1}(E_A;R)$ consisting solely of the zero element.*

Proof. Since δ is an isomorphism and $\pi^*\langle u_1,\cdots,u_k\rangle\subset\langle\pi^*u_1,\cdots,\pi^*u_k\rangle$, it suffices to show that the latter higher product consists solely of the zero element. Let A' be a defining system for $\langle\pi^*u_1,\cdots,\pi^*u_k\rangle$, and let $j^\#: C^*(PX,E_A;R)\to C^*(PX;R)$ be the canonical injection. Since $j^\# a'(1,1)$ is a cocycle in the acyclic complex $C^*(PX;R)$, there is a cochain b_1 of dimension p_1-1 such that $\delta b_1=j^\# a(1,1)$. By induction on j, there are cochains $b_j\in C^{p(1,j)}(PX;R)$, for $j=1,\cdots,k-1$, such that

$$\delta b_j=j^\# a'(1,j)+\sum_{r=1}^{j-1}\bar{b}_r j^\# a'(r+1,j),$$

since the right hand side can be seen to be a cocycle. A straightforward calculation yields

$$-\delta\sum_{r=1}^{k-1}\bar{b}_r a'(r+1,k)=\sum_{r=1}^{k-1}(j^*\bar{a}'(1,r))a'(r+1,k)=c(A)'.$$

Q.E.D.

The final properties which we shall establish are more algebraic. Their proofs would be straightforward if the cup product were commutative on the cochain level. In general it is not even true that $ab\sim ba$ for singular cochains a and b. However there is a pairing of singular cochain groups, the $\cup_1$ product of Steenrod [10], from $C^p\otimes C^q$ to C^{p+q-1} such that

(2.5) $$\delta(a \cup_1 b) = -\overline{ab} + (-1)^{pq}\overline{ba} + \delta a \cup_1 b + \bar{a} \cup_1 \delta b,$$

where $a \in C^p$ and $b \in C^q$. Furthermore if $c \in C^r$ then there is a formula of G. Hirsch [3] which states

(2.6) $$(ab)\cup_1 c = a(b \cup_1 c) + (-1)^{q(r-1)}(a \cup_1 c)b.$$

THEOREM 6. *Assume* $\langle u_1,\cdots,u_k\rangle$ *is defined in* $H^{p(1,k)+2}(X,A;R)$ *and let* $v \in H^q(X,A;R)$. *Then the k-fold product* $\langle u_1,\cdots,u_tv,\cdots,u_k\rangle$ *is defined, for* $t = 1,\cdots,k$, *as a subset of* $H^{q+p(1,k)+2}(X,A;R)$. *Furthermore the following relations are satisfied.*

(2.7) $$\langle u_1,\cdots,u_k\rangle v \subset \langle u_1,\cdots,u_kv\rangle,$$

(2.8) $$v\langle u_1,\cdots,u_k\rangle \subset (-1)^q\langle vu_1,\cdots,u_k\rangle,$$

(2.9) $$\langle u_1,\cdots,u_tv,\cdots,u_k\rangle \cap (-1)^q\langle u_1,\cdots,vu_{t+1},\cdots,u_k\rangle \neq \varnothing.$$

These relations may be interpreted as equalities modulo the sum of the indeterminacies.

Proof. Let $A = (a(i,j))$ be a defining system for $\langle u_1,\cdots,u_k\rangle$ and let b be a cocycle representative of v. Set

$$a_1(i,j) = \begin{cases} a(i,j) & \text{if } j < k, \\ a(i,k)b & \text{if } j = k. \end{cases}$$

Clearly A_1 is a defining system for $\langle u_1,\cdots,u_kv\rangle$ and $c(A)b = c(A_1)$. This proves (2.7). To prove (2.8), set

$$a_2(i,j) = \begin{cases} a(i,j), & i > 1, \\ ba(1,j), & i = 1 \end{cases}$$

and note that $\bar{b}c(A) = c(A_2)$.

The proof of relation (2.9) requires the use of the $\cup_1$ product. Set

$$a_3(i,j) = \begin{cases} a(i,t)b & \text{if } j = t, \\ a(i,j) & \text{if } i > t \text{ or } j < t, \\ (-1)^{m(j)}\left[a(i,j)b + \sum_{r=t}^{j-1} a(i,r)(\bar{a}(r+1,j) \cup_1 \bar{b})\right] & \text{for } i \leqq t < j, \end{cases}$$

where $m(j) = qp(t+1,j)$. To verify that A_3 is a defining system for $\langle u_1,\cdots,u_tv,\cdots,u_k\rangle$, we need only compute $\delta a_3(i,j)$ for $i \leqq t < j$. Using (2.5) and (2.6)

$$\begin{aligned}\delta a_3(i,j) &= (-1)^{m(j)}\Bigg[\sum_{r=i}^{j-1}\bar{a}(i,r)a(r+1,j)b \\ &+ \sum_{r=t}^{j-1}\sum_{s=i}^{r-1}\bar{a}(i,s)a(s+1,r)(\bar{a}(r+1,j)\cup_1\bar{b}) \\ &+ \sum_{r=t}^{j-1}\bar{a}(i,r)(-a(r+1,j)b+(-1)^{q(p(r+1,j)+1)}\,ba(r+1,j)) \\ &- \sum_{r=t}^{j-1}\sum_{s=r+1}^{j-1}\bar{a}(i,r)((a(r+1,s)\bar{a}(s+1,j))\cup_1\bar{b})\Bigg] \\ &= (-1)^{m(j)}\sum_{s=i}^{t-1}\bar{a}(i,s)\left[a(s+1,j)b+\sum_{r=t}^{j-1}a(s+1,r)(\bar{a}(r+1,j)\cup_1\bar{b})\right] \\ &+ \sum_{s=t}^{j-1}(-1)^{m(s)}\left[\bar{a}(i,s)\bar{b}-\sum_{r=t}^{s-1}\bar{a}(i,r)(a(r+1,s)\cup_1 b)\right]a(s+1,j) \\ &= \sum_{r=i}^{j-1}\bar{a}_3(i,r)a_3(r+1,j).\end{aligned}$$

Finally set

$$a_4(i,j) = \begin{cases} a_3(i,j) & \text{if } i \neq t+1 \text{ and } j \neq t, \\ a(i,t) & \text{if } j = t, \\ \bar{b}a(t+1,j) & \text{if } i = t+1. \end{cases}$$

Then A_4 is a defining system for $(-1)^q\langle u_1,\cdots,vu_{t+1},\cdots,u_k\rangle$ and $c(A_3)=c(A_4)$.

Q.E.D.

COROLLARY 7. *If $\langle u_1,\cdots,u_k\rangle$ is defined and $v_1,\cdots,v_k$ are arbitrary cohomology classes, then $\langle u_1v_1,\cdots,u_kv_k\rangle$ is defined. Furthermore if the latter higher product consists of a single class for any choice of $v_1,\cdots,v_k$ (the operation has no indeterminancy), then*

$$\langle u_1v_1,\cdots,u_kv_k\rangle = \pm\langle u_1,\cdots,u_k\rangle v_1\cdots v_k.$$

Finally we prove two analogues of the commutativity properties of the cup product.

THEOREM 8. *Assume $\langle u_1,\cdots,u_k\rangle$ is defined. Then $\langle u_k,\cdots,u_1\rangle$ is defined and*

$$\langle u_1,\cdots,u_k\rangle = (-1)^h\langle u_k,\cdots,u_1\rangle,$$

where

$$h = \sum_{1\leq r<s\leq k} p_rp + (k-1)\sum_{r=1}^{k} p_r + \frac{(k-1)(k-2)}{2}.$$

Proof. In Lemma 9 at the end of this proof we shall show by means of acyclic model theory that there exists functorial homomorphisms E_n from the n-fold tensor product $(C^*(X,A;R))^n$ to $C^*(X,A;R)$ of homogeneous degree $1-n$, for $n = 1,2,\cdots$, such that

(a) E_1 is the identity,

(b) $\delta E_n + (-1)^n E_n \delta = G_n$ with

(c)
$$G_n(b_1 \otimes \cdots \otimes b_n) = \sum_{s=1}^{n-1} (-1)^s E_{n-1}(b_1 \otimes \cdots \otimes b_s b_{s+1} \otimes \cdots \otimes b_n) + \sum_{r=1}^{n-1} (-1)^{m(r)} E_{n-r}(b_{r+1} \otimes \cdots \otimes b_n) E_r(b_1 \otimes \cdots \otimes b_r)$$

where $m(r) = (q_{r+1} + \cdots + q_n)(q_1 + \cdots + q_r - r + 1) + n(r+1)$ and $q_i = \dim b_i$.

Let A be a defining system for $\langle u_1, \cdots, u_k \rangle$. Define a system of cochains

$$a'(i,j) = (-1)^{h(i,j)} \sum_{n=1}^{j-i+1} \sum_{\alpha \in I(n)} E_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$$

where $I(n)$ is the set of all sequences

$$\alpha = \{i = m_1 < m_2 < \cdots < m_{n+1} = j+1\},$$

$$\alpha_s = \begin{cases} a(m_s, m_{s+1} - 1) & \text{if } n-s \text{ is even,} \\ \bar{a}(m_s, m_{s+1} - 1) & \text{if } n-s \text{ is odd,} \end{cases}$$

and

$$h(i,j) = \sum_{i \leqq s < t \leqq j} p_s p_t + (j-i) \sum_{s=i}^{j} p_s + \frac{(j-i)(j-i-1)}{2}.$$

Clearly $a'(i,i) = a_i$. We shall verify that

$$\delta a'(i,j) = \sum_{r=i}^{j-1} \bar{a}'(r+1,j) a'(i,r). \tag{2.10}$$

$$\begin{aligned}(-1)^{h(i,j)} \delta a'(i,j) = \sum_{n=1}^{j-i+1} \sum_{\alpha \in I(n)} \Bigg[& \sum_{r=1}^{n} (-1)^{n+1} E_n(\bar{\alpha}_1 \otimes \cdots \otimes \delta\alpha_r \otimes \cdots \otimes \alpha_n) \\ & + \sum_{s=1}^{n-1} (-1)^s E_{n-1}(\alpha_1 \otimes \cdots \otimes \alpha_s \alpha_{s+1} \otimes \cdots \otimes \alpha_n) \\ & + \sum_{t=1}^{n-1} (-1)^{m(t,\alpha)} E_{n-t}(\alpha_{t+1} \otimes \cdots \otimes \alpha_n) E_t(\alpha_1 \otimes \cdots \otimes \alpha_t) \Bigg].\end{aligned}$$

The first two summations cancel each other. The last can be seen to be the right hand side of equation (2.10) after checking signs. Thus A' is a defining system for $\langle u_k, \cdots, u_1 \rangle$.

Moreover, a similar computation yields

$$\delta \sum_{n=2}^{k} \sum_{\alpha \in I(n)} E_n(\alpha_1 \otimes \cdots \otimes \alpha_n) + c(A) = (-1)^h c(A').$$

This proves that $\langle u_1, \cdots, u_k \rangle \subset (-1)^h \langle u_k, \cdots, u_1 \rangle$. The reverse inclusion follows by symmetry.

To complete the proof, we need the following lemma. Let C^* be the singular cochain complex of some pair of spaces. Let $D:(C^*)^2 \to C^*$ represent a cup product pairing (induced by some diagonal approximation $\Delta: C_* \to (C_*)^2$) and let $T:(C^*)^2 \to (C^*)^2$ be the cochain automorphism defined by $T(a \otimes b) = (-1)^{pq} b \otimes a$, where $\dim a = p$ and $\dim b = q$.

LEMMA 9. *For each $n = 1, 2, \cdots$, there is a functorial homomorphism $E_n: (C^*)^n \to C^*$ of homogeneous degree $1 - n$, satisfying the conditions*

(a) $E_1: C^* \to C^*$ is the identity,

(b) $\delta E_2 + E_2 \delta = D(1 - T) = G_2$,

(c) $\delta E_n + (-1)^n E_n \delta = G_n$ for $n > 2$,

where, for $n > 2$, G_n is any functorial homomorphism of homogeneous degree $2 - n$ with the following properties: G_n is defined whenever $E_1, \cdots, E_{n-1}$ are; $\delta G_n = (-1)^n G_n \delta$; and if E_k maps $(k-1)$-cochains of $(C^*)^k$ to 0 in C^0 for all $k = 2, \cdots, n-1$, then G_n maps $(n-2)$-cochains of $(C^*)^n$ to 0 in C^0.

Proof. We shall prove the result by constructing chain maps $E^n: C_* \to (C_*)^n$ which will be dual to E_n. Let $\Delta: C_* \to (C_*)^2$ be the diagonal approximation which induces D and let T' be the dual to T. Then $\Delta - T'\Delta$ is a functorial chain map from C_* to $(C_*)^2$ which sends 0-chains to 0. The zero homomorphism from C_* to $(C_*)^2$ is a functorial chain map which agrees with $\Delta - T'\Delta$ on the zero chains. By the acyclic model theorem [7, Chapter 4], the two maps are functorially chain homotopic. Thus there is a map $E^2: C_* \to (C_*)^2$ such that $\partial E^2 + E^2 \partial = \Delta - T\Delta$. We may choose E^2 so that it sends 0-chains to 0.

Now assume that $E^k: C_* \to (C_*)^k$ is defined, of homogeneous degree $k-1$ for all $k < n$, such that

(a′) E^k is functorial,

(b′) E^k sends 0-chains to 0, and

(c′) $(-1)^k \partial E^k + E^k \partial = G^k$, where G^k is the dual of G_k.

By hypothesis, G^n is defined, functorial and sends zero dimensional chains to 0 in the $(n-2)$-chain group of $(C_*)^n$. The map G^n is either a chain map (if n is even), or can be made into one by a sign trick. By the acyclic model theorem, G^n is chain homotopic to the zero map. That is, there is a functorial map E^n which satisfies (a′), (b′), and (c′). The dual map to E^n clearly satisfies (a), (b), and (c).

Q.E.D.

THEOREM 10. *Assume that the k-fold products $\langle u_t, \cdots, u_k, u_1, \cdots, u_{t-1} \rangle$, for*

$t = 1, \cdots, k$, *are simultaneously defined. Then there are classes* $x_t \in \langle u_t, \cdots, u_{t-1} \rangle$, *for* $t = 1, \cdots, k$, *such that*

$$\sum_{t=1}^{k} (-1)^{\pi(t)} x_t = 0 \textit{ where}$$

(2.11)

$$\pi(1) = \sum_{r=1}^{k} (k-r)p_r + (k-1)$$

and

$$\pi(t) = (p_1 + \cdots + p_{t-1} + k)(p_t + \cdots + p_k) + \sum_{r=1}^{k} (t-1-r)p_r + t(k-1) \quad \textit{for } t > 1.$$

Proof. The condition that the higher products be defined simultaneously implies that there exists cochains $(a(i,j))$ for $1 \leqq i, j \leqq k$, $(i,j) \neq (1,k)$ and $(i,j) \neq (t, t-1)$ such that $a(i,i)$ is a cocycle representative of u_i and

$$\delta a(i,j) = \sum \bar{a}(i,r)a(r+1,j)$$

where, if $i < j$, then the summation is taken over all r with $i \leqq r < j$, and if $i > j$, then the summation is taken over all r with $i \leqq r$ or $r < j$ and $a(k+1,j)$ is interpreted as $a(1,j)$. If A_t is the subset of these cochains which is a defining system for $\langle u_t, \cdots, u_{t-1} \rangle$ then $c(A_t) = \sum_{r \neq t-1} \bar{a}(t,r)a(r+1, t-1)$.

By Lemma 9, there is a set of functorial "permuting" homomorphisms $E'_n : (C^*)^n \to C^*$ of homogeneous degree $1 - n$ satisfying the following conditions:

(a) E_1 is the identity,

(b) $\delta E'_n + (-1)^n E'_n \delta = G'_n$ where

(c)
$$G_n(b_1 \otimes \cdots \otimes b_n) = (-1)^m E'_{n-1}(b_2 \otimes \cdots \otimes b_n b_1) + \sum_{s=1}^{n-1} (-1)^s E'_{n-1}(b_1 \otimes \cdots \otimes b_s b_{s+1} \otimes \cdots \otimes b_n)$$

where $m = q_1(q_2 + \cdots + q_n)$ and $q_r = \dim b_r$. Consider the cochain

$$c = \sum_{n=2}^{k} \sum_{\beta \in J(n)} (-1)^{\pi(\beta)} E'_n(\beta_1 \otimes \cdots \otimes \beta_n) \tag{2.12}$$

where $J(n)$ is the set of all sequences

$$\beta = \{1 \leqq m_1 < m_2 < \cdots < m_n \leqq k\};$$

$$\beta_n = \begin{cases} a(m_n, m_1 - 1), & m_1 > 1, \\ a(m_n, k), & m_1 = 1; \end{cases}$$

$$\beta_s = \begin{cases} a(m_s, m_{s+1} - 1) & \text{if } n - s \text{ is even, } s < n, \\ \bar{a}(m_s, m_{s+1} - 1) & \text{if } n - s \text{ is odd, } s < n \end{cases} \quad \text{and } \pi(\beta) = \pi(m_1).$$

$$\delta c = \sum_{n=2}^{k} \sum_{\beta \in J(n)} (-1)^{\pi(\beta)} \Big[\sum_{r=1}^{n} (-1)^{n+1} E'_n(\bar\beta_1 \otimes \cdots \otimes \delta\beta_r \otimes \cdots \otimes \beta_n)$$

$$+ (-1)^{m(\beta)} E'_{n-1}(\beta_2 \otimes \cdots \otimes \beta_n\beta_1)$$

$$+ \sum_{s=1}^{n-1} (-1)^s E'_{n-1}(\beta_1 \otimes \cdots \otimes \beta_s\beta_{s+1} \otimes \cdots \otimes \beta_n) \Big]$$

$$= \sum_{\beta \in J(2)} (-1)^{\pi(\beta)+m(\beta)} E_1(\beta_2\beta_1) - (-1)^{\pi(\beta)} E_1(\beta_1\beta_2)$$

$$= -\sum_{t=1}^{k} \sum_{r \neq t-1} (-1)^{\pi(t)} \bar a(t,r) a(r+1, t-1)$$

$$= -\sum_{t=1}^{k} (-1)^{\pi(t)} c(A_t). \qquad \text{Q.E.D.}$$

3. **The operation $\langle u \rangle^k$.** Let $u \in H^m(X,A;R)$ be a class such that $u^2 = 0$. Let a be a cocycle representative of u. Then the triple product $\langle u,u,u \rangle = \langle a,a,a \rangle$ is defined as a coset of $uH^{2m-1}(X,A;R)$ in $H^{3m-1}(X,A;R)$.

We may, however, restrict the notion of a defining system for this particular case by requiring that $a(1,2) = a(2,3)$. In other words, choose a cochain $a(2)$ so that $\delta a(2) = \bar a a$. Then $\bar a(2)a + \bar a a(2)$ is a cocycle representative of $\langle a,a,a \rangle$. Any other choice of $a(2)$ would differ by a cocycle $b \in C^{2m-1}(X,A;R)$. Thus $(\bar a(2) + \bar b)a + \bar a(a(2) + b)$ also represents $\langle a,a,a \rangle$ in this restricted sense. Since b is an odd dimensional cocycle, the two representatives of $\langle a,a,a \rangle$ differ by the coboundary $-ba + \bar a b$. Thus, with this restricted definition, the triple product has no indeterminacy. We shall call this restricted operation $\langle a \rangle^3$. Clearly $\langle a \rangle^3 \subset \langle a,a,a \rangle$.

Definition 11. A system of cochains $A^* = (a(n))$, $n = 1, \cdots, k-1$, satisfying

$$a(n) \in C^{n(m-1)+1}(X,A;R), \tag{3.1}$$

$$a(1) = a \text{ is a cocycle}, \tag{3.2}$$

$$\delta a(n) = \sum_{r=1}^{n-1} \bar a(r) a(n-r), \tag{3.3}$$

is called a defining system for $\langle a \rangle^k$. The cocycle

$$c(A^*) = \sum_{r=1}^{k-1} \bar a(r) a(k-r) \tag{3.4}$$

is called its related cocycle. Then $\langle a \rangle^k$ is said to be defined if there is a defining system for it, in which case $\langle a \rangle^k$ consists of all classes $w \in H^{k(m-1)+2}(X, A; R)$ for which there is a defining system A^* such that $c(A^*)$ represents w.

THEOREM 12. *The operation $\langle a \rangle^k$ depends only on the cohomology class of a.*

Proof. Let A^* be a defining system for $\langle a \rangle^k$ and let b be an $(m-1)$-cochain. We shall construct a defining system A'^* for $\langle a + \delta b \rangle^k$ such that the related cocycle of A^* is cohomologous to that of A'^*. Define $a'(n)$ inductively by

$$a'(1) = a + \delta b,$$
$$a'(n) = a(n) - \bar{b}a'(n-1) + a(n-1)b, \ n \geqq 2.$$

Clearly A'^* satisfies (3.1) and (3.2). By induction on n we shall show that A'^* satisfies (3.3). This is immediate if $n = 2$. Assume true for $n-1$. Then, on the one hand

$$\delta a'(n) = \sum_{r=1}^{n-1} \bar{a}(r)a(n-r) - (\delta\bar{b})a'(n-1)$$
$$- \sum_{r=2}^{n-1} b\bar{a}'(r-1)a'(n-r) + \sum_{r=2}^{n-1} \bar{a}(r-1)a(n-r)b + \bar{a}(n-1)\delta b.$$

On the other hand

$$\sum_{r=1}^{n-1} \bar{a}'(r)a'(n-r) = \sum_{r=1}^{n-1} \bar{a}(r)a(n-r) + \delta\bar{b}a'(n-1)$$
$$+ \sum_{r=2}^{n-1} [- b\bar{a}'(r-1) + \bar{a}(r-1)\bar{b}]a'(n-r)$$
$$+ \sum_{r=1}^{n-2} \bar{a}(r)[- \bar{b}a'(n-r-1) + a(n-1-r)b] + \bar{a}(n-1)\delta b.$$

By reindexing where necessary, the two equations are seen to be equal. A similar calculation shows

$$\sum_{r=1}^{k-1} \bar{a}'(r)a'(k-r) = \sum_{r=1}^{k-1} \bar{a}(r)a(k-r) + \delta[- \bar{b}a'(k-1) + a(k-1)b],$$

and so the related cocycles are cohomologous. Q.E.D.

DEFINITION 13. Let a be a cocycle representative of u. A set of cocycles is said to be a defining system for $\langle u \rangle^k$ if it is one for $\langle a \rangle^k$. If $\langle a \rangle^k$ is defined, then so is $\langle u \rangle^k$ and $\langle u \rangle^k = \langle a \rangle^k$ as subsets of $H^{k(m-1)+2}(X, A; R)$.

We establish some results concerning these operations $\langle u \rangle^k$. Let p be an odd prime and let β be the Bockstein operator associated with the exact coefficient sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$. Furthermore, let P^m be the Steenrod pth power P^m: $H^q(X; Z_p) \to H^{q+2m(p-1)}(X; Z_p)$.

THEOREM 14. *If* $u \in H^{2m+1}(X;Z_p)$, *then* $\langle u\rangle^p$ *is defined as a single class in* $H^{2mp+2}(X;Z_p)$ *and* $\langle u\rangle^p = -\beta P^m u$.

We first prove some theorems leading up to this result. Note that if $\langle u\rangle^k$ is defined, then the k-fold product $\langle u,\cdots,u\rangle$ is defined and $\langle u\rangle^k \subset \langle u,\cdots,u\rangle$. Also $\langle u\rangle^k$ is defined if and only if $\langle u\rangle^{k-1}$ is defined and contains the zero element.

THEOREM 15. *Let* Q *be the rationals. Then, if* $u \in H^{2m+1}(X;Q)$, $\langle u\rangle^k$ *is defined, and vanishes in* $H^{2mk+2}(X;Q)$ *for each* $k = 2,3,\cdots$. *Let* p *be an odd prime. Then if* $u \in H^{2m+1}(X;Z_p)$, $\langle u\rangle^k$ *is defined and vanishes in* $H^{2mk+2}(X;Z_p)$ *for each* $k = 2,3,\cdots, p-1$, *and therefore* $\langle u\rangle^p$ *is defined.*

Proof. Let a be a cocycle representative of u in $C^{2m+1}(X;R)$, where R is Q or Z_p. By means of the Steenrod $\cup_1$-product (2.5) we shall explicitly construct a defining system $(a(n))$ for $\langle u\rangle^k$.

If $R = Q$ and n is arbitrary, or if $R = Z_p$ and $n < p$, define by induction on n, $a(1) = a$, and $a(n) = (1/n)[a(n-1) \cup_1 a]$. It suffices to verify that $\delta a(n) = \sum_{r=1}^{n-1} - a(r)a(n-r)$. Note $\dim a(r)$ is odd for all r.

By induction on n,

$$\begin{aligned}\delta a(n) &= (1/n)\delta(a(n-1) \cup_1 a)\\ &= (1/n)\left[-a(n-1)aa(n-1) - \sum_{r=1}^{n-2} (a(r)a(n-1-r)) \cup_1 a\right].\end{aligned}$$

By the formula of G. Hirsch (2.6),

$$\begin{aligned}&\sum_{r=1}^{n-2} (a(r)a(n-1-r)) \cup_1 a\\ &\quad= \sum_{r=1}^{n-2} [a(r)(a(n-1-r) \cup_1 a) + (a(r) \cup_1 a)a(n-1-r)]\\ &\quad= \sum_{r=1}^{n-2} [a(r)(n-r)a(n-r) + (r+1)a(r+1)a(n-1-r)]\\ &\quad= \sum_{r=2}^{n-2} [na(r)a(n-r)] + [(n-1)(a(n-1)a(1) + a(1)a(n-1))].\end{aligned}$$

Substituting this result into the above, we see that $(a(n))$ is a defining system for $\langle u\rangle^k$. Q.E.D.

We now show that $\langle u\rangle^p$ is defined without indeterminacy.

LEMMA 16. *Let* $u \in H^{2m+1}(X;Z_p)$ *and assume* $\langle u\rangle^k$ *is defined. Let* A^* *be a defining system for it. If* b *is a cocycle of dimension* $2ms+1$, *where* $ps > k$, *then there exists a defining system* A'^* *for* $\langle u\rangle^k$ *such that* $a'(n) = a(n)$ *if* $n < s$, $a'(s) = a(s) + b$, *and* $c(A^*) \sim c(A'^*)$.

Proof. For each pair of integers (t,q), $t<k$, define odd dimensional cochains $c_{t,q}$ by induction on q as follows:

$$\begin{aligned} c_{t,q} &= 0 \text{ if } t<0 \text{ or } q<0, \text{ or } t=q=0,\\ c_{t,0} &= a(t) \quad \text{if } t \geqq 1,\\ c_{0,1} &= b,\\ c_{t,q} &= (c_{t,q-1} \cup_1 b) \quad \text{otherwise.} \end{aligned}$$

Now we may define cochains

$$a'(n) = \sum_{q=0}^{[n/s]} \frac{1}{q!} c_{n-qs,q} \quad \text{for} \quad n = 1,\cdots,k-1,$$

since $[n/s] < p$. If $n < s$, then $a'(n) = a(n)$. Also, $a'(s) = a(s) + b$. To verify (3.3) we need the following formula

$$\delta c_{t,q} = -\sum_{r=0}^{q}\sum_{j=0}^{t}\binom{q}{r} c_{j,r}c_{t-j,q-r}, \tag{3.5}$$

which we establish by induction on q. For $q=0$, the formula reduces to

$$\delta a(t) = -\sum_{j=1}^{t-1} a(j)a(t-j).$$

Assume now the formula holds for q, then

$$\begin{aligned} \delta c_{t,q+1} &= \delta(c_{t,q} \cup_1 b)\\ &= -c_{t,q}b - bc_{t,q} - \sum_{r=0}^{q}\sum_{j=0}^{t}\binom{q}{r}(c_{j,r}c_{t-j,\,q-r}) \cup_1 b\\ &= -c_{t,q}b - bc_{t,q} - \sum_{r=0}^{q}\sum_{j=0}^{t}\binom{q}{r}[c_{j,r}(c_{t-j,q-r} \cup_1 b) + (c_{j,r} \cup_1 b)c_{t-j,q-r}]. \end{aligned}$$

(If $(j,r) \neq (0,0)$, then $c_{j,r} \cup_1 b = c_{j,r+1}$ and $c_{0,0} = 0$)

$$\begin{aligned} &= -\sum_{r=0}^{q}\sum_{j=0}^{t}\binom{q}{r}[c_{j,r}c_{t-j,q-r+1} + c_{j,r+1}c_{t-j,q-r}]\\ &= -\sum_{r=0}^{q+1}\sum_{j=0}^{t}\left[\binom{q+1}{r} c_{j,r}c_{t-j,q-r+1}\right]. \end{aligned}$$

This completes the induction step for the formula.

Now for all n, $1 \leqq n \leqq k$,

$$\sum_{i=1}^{n-1} a'(i)a'(n-i) = \sum_{i=1}^{n-1} \sum_{r=0}^{[i/s]} \frac{1}{r!} c_{i-rs,r} \sum_{\rho=0}^{[(n-i)/s]} \frac{1}{\rho!} c_{n-i-\rho s,\rho}$$

$$= \sum_{i=1}^{n-1} \sum_{r=0}^{[i/s]} \sum_{\rho=0}^{[(n-i)/s]} \frac{1}{r!\rho!} c_{i-rs,r} c_{n-i-\rho s,\rho}.$$

(Reverse the order of the outer two sums)

$$= \sum_{r=0}^{[n/s]} \sum_{i=rs}^{n-1} \sum_{\rho=0}^{[(n-i)/s]} \frac{1}{r!\rho!} c_{i-rs,r} c_{n-i-\rho s,\rho}.$$

(Reverse the inner two sums and set $j = i - rs$)

$$= \sum_{r=0}^{[n/s]} \sum_{\rho=0}^{[n/s]-r} \sum_{j=0}^{n-(r+\rho)s} \frac{1}{r!\rho!} c_{j,r} c_{n-(r+\rho)s-j,\rho}.$$

(Set $q = r + \rho$ and sum over q)

$$= \sum_{q=0}^{[n/s]} \sum_{r=0}^{q} \sum_{j=0}^{n-qs} \frac{1}{r!(q-r)!} c_{j,r} c_{n-qs-j,q-r}.$$

$\left(\text{Since } \binom{q}{r} = \frac{q!}{r!(q-r)!}, \text{ and by formula 3.5}\right)$

$$= -\sum_{j=1}^{n-1} a(j)a(n-j) - \sum_{q=1}^{[n/s]} \frac{1}{q!} \delta c_{n-qs,q}.$$

If $n < k$, then the last line is equal to $\delta a'(n)$. This proves simultaneously that A'^* is a defining system for $\langle u \rangle^k$ and that the related cocycles of A^* and A'^* are cohomologous. Q.E.D.

THEOREM 17. *For an odd prime p, let $u \in H^{2m+1}(X; Z_p)$. Then $\langle u \rangle^p$ is defined as a single class of $H^{2mp+2}(X; Z_p)$.*

Proof. Let A^* and A'^* be two defining systems for $\langle a \rangle^p$, where a is a cocycle representative of u. We shall show that their related cocycles are cohomologous.

For this, it suffices to construct a set of defining systems A_q^* of $\langle a \rangle^p$, for $q = 1, \cdots, p-1$, so that

(i) $a_1(n) = a(n)$,

(ii) $a_q(n) = a'(n)$ for $n \leqq q$, and thus $a_{p-1}(n) = a'(n)$ for $n = 1, \cdots, p-1$,

(iii) the related cocycles of A_q^* and A_{q+1}^* are cohomologous.

Assume inductively on q that $(a_q(n))$ is defined. Then $\delta a_q(q+1) = \delta a'(q+1)$ so the two differ by a cocycle. Thus, by Lemma 16, there is a defining system A^*_{q+1} satisfying (i), (ii), (iii). Q.E.D.

By the preceding results, we may interpret the operation

$$\langle\ \rangle^p : H^{2m+1}(X;Z_p) \to H^{2mp+2}(X;Z_p)$$

as a primary operation of type $(Z_p, 2m+1; Z_p, 2mp+2)$. Also, by obvious scalar properties, if $g \in Z_p$, $\langle gu\rangle^p = g^p\langle u\rangle^p = g\langle u\rangle^p$.

Now let ι be the fundamental class of $H^{2m+1}(Z_p, 2m+1; Z_p)$. It is well known that $H^*(Z_p, 2m+1; Z_p)$ is the tensor product of polynomial rings and exterior algebras each generated by $P^I(\iota)$ for admissible sequences I [1]. Therefore $\langle\iota\rangle^p = \Sigma c_i P^{I(i,1)}(\iota)\cdots P^{I(i,j_i)}(\iota)$ with $j_i \geqq 1$ and $c_i \neq 0$ for all i.

Let g be a primitive generator for the multiplicative group of Z_p. Then

$$0 = \langle g\iota\rangle^p - g\langle\iota\rangle^p = \Sigma c_i(g^{j_i} - g)P^{I(i,1)}(\iota)\cdots P^{I(i,j_i)}(\iota).$$

This only happens if the coefficients vanish, i.e. $g^{j_i} - g = 0$ in Z_p. Since g is primitive, this is true if and only if $j_i \equiv 1 \pmod{p-1}$ for each i. Since dim $\iota^p > 2mp+2$, no (nontrivial) p-fold product can exist in $H^{2mp+2}(Z_p, 2m+1; Z_p)$. Therefore $j_i = 1$ for all i; i.e. $\langle\iota\rangle^p = \Sigma c_i P^{I_i}(\iota)$.

Now if $\sigma: H^*(Z_p, 2m+1; Z_p) \to H^*(Z_p, 2m; Z_p)$ is the loop suspension, then by Theorem 5, $0 = \sigma\langle\iota\rangle^p = \Sigma c_i P^{I_i}(\sigma\iota)$ and thus $P^{I_i}(\sigma\iota) = 0$ for each i. The only Steenrod operation P^I of degree $2mp+2-2m-1$ with this property is easily seen to be βP^m (see [1]). We sum up these results in the following lemma for Theorem 14.

LEMMA 18. *For some constant* $c \in Z_p$, $\langle u\rangle^p = c\beta P^m u$ *for all* $u \in H^{2m+1}(X;Z_p)$.

The next theorem will completely characterize the operation $\langle u\rangle^{p^k}$ for $k \geqq 1$ where u is a one dimensional class mod p.

THEOREM 19. *Let* $\iota \in H^1(Z_{p^k}, 1; Z_p)$ *be the* mod p *reduction of the fundamental class* ι_k *of* $H^1(Z_{p^k}, 1; Z_{p^k})$. *Then* $\langle\iota\rangle^{p^k}$ *is defined as the single class* $-\beta_k\iota_k \in H^2(Z_{p^k}, 1; Z_p)$, *where* β_k *is the Bockstein coboundary operator associated with the exact sequence of coefficient groups.*

$$0 \to Z_p \to Z_{p^{k+1}} \to Z_{p^k} \to 0.$$

Proof. Let C^* $(= C^*(Z_{p^k}; Z_p))$ be the cochain complex of the standard resolution of Z_{p^k} with coefficients in Z_p. Then C^* is cochain equivalent to the complex of singular cochains of a $K(Z_{p^k}, 1)$ space [2]. We use only the following properties of this complex:

(a) The cochains $a \in C^n$ are set maps from the n-fold Cartesian product $(Z_{p^k})^n$ to Z_p, with the sole condition that $a(x_1, \cdots, x_n) = 0$ whenever $x_i = 0$ for some $i = 1, \cdots, n$.

(b) If $a \in C^1$, then $\delta a \in C^2$ is the map defined by $(\delta a)(x, y) = a(x) + a(y) - a(x+y)$.

(c) If a, $b \in C^1$, then the product $ab \in C^2$ is defined by $(ab)(x, y) = a(x)b(y)$.

We shall now explicitly construct a defining system of 1-cochains $(a(n))$ for $\langle \iota \rangle^{p^k}$. Let a be the mod p reduction of the "identity" map, i.e. $a(x)$ is the mod p reduction of x. Then a represents ι.

For each $n = 1, \cdots, p^k - 1$, define $a(n)$ by $a(1) = a$ and by specifying that

$$a(n)(x) = \binom{\hat{x}}{n} \pmod p.$$

This binomial coefficient is to be interpreted as follows: Let $\hat{x}$ be the integer $0 \leqq \hat{x} < p^k$ which represents x. Then

$$\binom{\hat{x}}{n}$$

is an integer, defined to be zero if $\hat{x} < n$. $a(n)(x)$ will be the mod-p reduction of this integer.

To verify the coboundary formula, we use a classical formula about binomial coefficients:

$$\binom{i+j}{n} = \sum_{r=0}^{n} \binom{i}{r} \binom{j}{n-r}. \tag{3.6}$$

(Both are the coefficient of t^n in $(1+t)^{i+j} = (1+t)^i(1+t)^j$.)

It suffices to verify (3.3), i.e.

$$\binom{\hat{x}}{n} + \binom{\hat{y}}{n} - \binom{(x+y)^\wedge}{n} = -\sum_{r=1}^{n-1} \binom{\hat{x}}{r} \binom{\hat{y}}{n-r} \qquad \pmod p.$$

Now $\hat{x} + \hat{y} = (x+y)^\wedge + \varepsilon p^k$ where $\varepsilon = \varepsilon(x, y)$ is 0 or 1. So by (3.6), the following formula holds:

$$\sum_{r=0}^{n} \binom{\hat{x}}{r} \binom{\hat{y}}{n-r} = \sum_{r=0}^{n} \binom{(x+y)^\wedge}{r} \binom{\varepsilon p^k}{n-r}. \tag{3.7}$$

Since $n < p^k$, the right hand side of (3.7) is

$$\binom{(x+y)^\wedge}{n} \bmod p.$$

Thus $(a(n))$ is defining system for $\langle \iota \rangle^p$. Furthermore

$$c(A^*)(x, y) = -\sum_{r=0}^{p^k} \binom{\hat{x}}{r} \binom{\hat{y}}{p^k - r} = -\varepsilon(x, y).$$

To compute $\beta_k \iota_k$, take a fundamental cocycle $a' \in C^1(Z_{p^k}; Z_{p^k})$ and pull it back to $a'' \in C^1(Z_{p^k}; Z_{p^{k+1}})$. Then $(1/p^k)\delta a''$ is a cocycle in $C^2(Z_{p^k}; Z_p)$ which represents

$\beta_k \iota_k$. We take a' to be defined by $a'(x) = x$ and $a''(x) = \hat{x}$ in $Z_{p^{k+1}}$, i.e. $\hat{x}(\mathrm{mod}\, p^{k+1})$. Then

$$(\delta a'')(x, y) = \hat{x} + \hat{y} - (x + y)\hat{}\,(\mathrm{mod}\, p^{k+1}) = \varepsilon(x, y)p^k. \qquad \text{Q.E.D.}$$

We need a few technical lemmas in order to complete the proof of Theorem 14. For a space X, coefficient ring R and integer $k > 2$, let B_k denote the following condition:

B_k — Whenever $v_1, \cdots, v_r$, for $r < k$, are odd dimensional classes of $H^*(X;R)$ such that $\langle v_1, \cdots, v_r \rangle$ is defined, then $\langle v_1, \cdots, v_r \rangle$ consists solely of the zero element. It is clear that B_k implies B_{k-1}. Also, if v and w are odd dimensional classes, then B_k implies that $vw = 0$.

Lemma 20. *Assume that condition B_k holds. If $u_1, \cdots, u_k$ are odd dimensional classes of $H^*(X;R)$ such that $\langle u_1, \cdots, u_k \rangle$ is defined, then $\langle u_1, \cdots, u_k \rangle$ has no indeterminacy, i.e. it consists of a single class.*

Proof. Let $a_1, \cdots, a_k$ be cocycle representatives of the classes, $u_1 \cdots, u_k$ and let $A = (a(i,j))$ and $A' = (a'(i,j))$ be two defining systems for $\langle a_1, \cdots, a_k \rangle$. We need to show that their related cocycles are cohomologous. First, note that $\dim a(i,j)$ is odd for all i,j.

To prove the lemma, it suffices to construct defining systems A_q for $q = 1, \cdots, k-1$ of $\langle a_1, \cdots, a_k \rangle$ so that

(i) $A_1 = A$,

(ii) $a_q(i,j) = a'(i,j)$ for $j - i \leqq q - 1$ and thus $A_{k-1} = A'$,

(iii) $c(A_q) \sim c(A_{q+1})$ for $q = 1, \cdots, k-2$.

We define these systems by induction on q. Assume that A_q is defined. Then $b_s = a'(s, q + s) - a_q(s, q + s)$ is a cocycle for each $s = 1, \cdots, k - q$.

To construct A_{q+1}, it will suffice to construct defining systems $A_{q,s}$ of $\langle a_1, \cdots, a_k \rangle$ for $s = 0, 1, \cdots, k - q$ so that $a_{q,s}(i,j) = a_q(i,j)$ if $j - i \leqq q - 1$,

$$a_{q,s}(i, i + q) = \begin{cases} a_q(i, i + q) & \text{if } i > s, \\ a_q(i, i + q) + b_i & \text{if } i \leqq s, \end{cases}$$

and $c(A_{q,s}) \sim c(A_{q,s+1})$. Thus $A_q = A_{q,0}$ and we may define A_{q+1} to be $A_{q,k-q}$.

We define these systems by induction on s. Assume $A_{q,s}$ is defined. We will construct cochains $b(i,j)$ for $1 \leqq i \leqq j \leqq k$ so that we may define

$$a_{q,s+1}(i,j) = a_{q,s}(i,j) + b(i,j). \tag{3.8}$$

If $i > s + 1$ or $j < s + q + 1$, then set $b(i,j) = 0$. If $i \leqq s + 1$ and $j \geqq s + q + 1$, then we will construct the cochains $b(i,j)$ by induction on $j - i = q, \cdots, k - 1$.

First set $b(s + 1, s + q + 1) = b_{s+1}$. Assume now that $b(i,j)$ is defined for all (i,j) with $j - i < t$ so that

$$(3.9) \qquad \delta b(i,j) = -\sum_{r=i}^{s} a_{q,s}(i,r)b(r+1,j) - \sum_{r=s+q+1}^{j-1} b(i,r)a_{q,s}(r+1,j).$$

Then the b's and the a's form defining systems for higher products with odd dimensional classes. In fact it is not hard to see that the right hand side of (3.9) is defined when $j = i + t$ and is a cocycle representative of the higher product $\langle a_i, \cdots, a_s, b_{s+1}, a_{s+q+2}, \cdots, a_j \rangle$. By condition B_k, this cocycle must cobound, i.e. there is a cochain $b(i, i+t)$ which satisfies (3.9).

It follows immediately from (3.8) and (3.9) that

$$\delta b(i,j) = -\sum_{r=i}^{j-1} a_{q,s+1}(i,r)a_{q,s+1}(r+1,j) + \sum_{r=i}^{j-1} a_{q,s}(i,r)a_{q,s}(r+1,j)$$

for all i,j with $1 \leqq i \leqq j < k$. This shows both that $A_{q,s+1}$ is a defining system for $\langle a_1, \cdots, a_k \rangle$ and that the related cocycles of $A_{q,s}$ and $A_{q,s+1}$ are cohomologous. Q.E.D.

LEMMA 21. *For some space X, ring R and integer $k \geqq 2$, assume the following two conditions hold:*

(3.10) *If u and v are odd dimensional classes in $H^*(X;R)$, then $uv = 0$.*

(3.11) *If $u_1, \cdots, u_r$, for $r \leqq k$, are arbitrary odd dimensional classes of $H^*(X;R)$, then $\langle u_1, \cdots, u_r \rangle$ is defined.*

Then for arbitrary odd dimensional classes $u_1, \cdots, u_r$, $r \leqq k$, $\langle u_1, \cdots, u_r \rangle$ has no indeterminacy.

Proof. Clearly, we need only show that condition B_s holds for $s = 3, 4, \cdots, k$. We shall prove this by induction on s. For $s = 3$, it is trivial. Assume that condition B_{s-1} holds. Since all s-fold products are defined, all $(s-1)$-fold products must contain the zero elements. By Lemma 20, the $(s-1)$-fold products consist solely of the zero element. Thus condition B_s holds. Q.E.D.

Proof of Theorem 14. Let ι be the generator of $H^1(Z_p, 1; Z_p)$ and let $\beta\iota$ be the generator of $H^2(Z_p, 1; Z_p)$. Every odd dimensional class of $H^*(Z_p, 1; Z_p)$ can be written in the form ιv for some even dimensional class v. Since $\iota^2 = 0$, (3.10) is satisfied. If $k \leqq p$, $\langle \iota \rangle^k$ is defined, and thus also the k-fold product $\langle \iota, \cdots, \iota \rangle$. Also if $v_1, \cdots, v_k$, are even dimensional classes, then by Corollary 7 $\langle \iota v_1, \cdots, \iota v_k \rangle$ is defined. Thus (3.11) is satisfied. By Lemma 21 all p-fold products in $H^*(Z_p, 1; Z_p)$ have no indeterminacy.

Thus, by Corollary 7 and Theorem 19,

$$\begin{aligned} \langle \iota(\beta\iota)^m \rangle^p &= \langle \iota(\beta\iota)^m, \cdots, \iota(\beta\iota)^m \rangle \\ &= \langle \iota, \cdots, \iota \rangle (\beta\iota)^{pm} \\ &= \langle \iota \rangle^p (\beta\iota)^{pm} \\ &= -(\beta\iota)^{pm+1}. \end{aligned}$$

By the Cartan formula,

$$\beta P^{m}(\iota(\beta\iota)^{m}) = \beta(\iota(\beta\iota)^{pm}) = (\beta\iota)^{pm+1}.$$

Since this element is nonzero in $H^{2mp+2}(Z_p, 1; Z_p)$, the constant c in Lemma 18 is -1. Q.E.D.

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