FUNCTIONS SATISFYING A WEIGHTED AVERAGE PROPERTY. II (1)

BY

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Introduction. In the previous paper [1] we were interested in characterizing the class of functions, defined in a given region (open connected set) R of the n-dimensional euclidean space E_n, which satisfy the following Weighted Average Property (W.A.P.):

$$(0.1)\qquad u(P) = \frac{\int_{B(P,r)} uw\,d\rho}{\int_{B(P,r)} w\,d\rho}, \qquad P \in R,$$

$$(0.2)\qquad u(P) = \frac{\int_{S(P,r)} uw\,d\sigma}{\int_{S(P,r)} w\,d\sigma},$$

where $B(P,r)$ and $S(P,r)$ denote any ball and its surface with the point

$$P = P(x_1, x_2, \cdots, x_n),$$

for its center and radius r which lies in R; $d\rho$ and $d\sigma$ stand for the usual Lebesgue measure of B and S and w is a weight function (W.F.) defined in R.

For convenience we state here again the definitions of weight functions and functions satisfying a W.A.P.

DEFINITION. w is a W.F. in R means that

(a) w is a nonnegative, real-valued function defined in R, and

(b) w is locally summable in R, i.e., if $P \in R$ and $0 < r < d(P,T)$, T being the boundary of R, then the Lebesgue integral $\int_{B(P,r)} w\,d\rho$ over $B(P,r)$ exists and $\int_{B(P,r)} w\,d\rho > 0$.

DEFINITION. A real-valued function u is said to satisfy the W.A.P. with respect to a W.F., w in R provided uw is locally summable in R and u satisfies the property (0.1) for each ball $B(P, r)$ whose closure lies in R.

DEFINITION.

$$R^* = \{(P,r) : P \in R \text{ and } 0 < r < d(P,T)\}.$$

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Suppose now that R be a region in E_n and w be a W.F. defined in R. Let $S(w,R)$ denote the class of all functions satisfying W.A.P. with respect to w in R.

In the previous paper [1] we proved the following theorems and corollaries:

THEOREM 2*.

(i) *If the W.F. w belongs to class $C^m(R)$, where m is a nonnegative integer and $u \in S(w,R)$, then $u \in C^{m+1}(R)$.*

(ii) *If w be infinitely differentiable in R and $u \in S(w,R)$, then u is infinitely differentiable in R.*

(iii) *If w be analytic in R and $u \in S(w,R)$, then u is analytic in R.*

REMARK 1*.

(0.3) Constant functions belong to $S(w,R)$.

(0.4) $S(w,R)$ is a linear space over the reals and $\dim S(w,R) \geqq 1$.

(0.5) If w be a nonzero constant, then $S(w,R)$ is the class of all harmonic functions defined on R.

REMARK 2*. In proving Theorem 2* the following was shown:

If $w \in C^1(R)$ and $u \in S(w,R)$, then

$$(0.6) \qquad u_{x_i}(P) = \frac{\int_{B(P,r)} u_{x_i} w \, d\rho}{\int_{B(P,r)} w \, d\rho} + \frac{\int_{B(P,r)} u w_{x_i} \, d\rho - u(P) \int_{B(P,r)} w_{x_i} \, d\rho}{\int_{B(P,r)} w \, d\rho},$$

$i = 1,2,\cdots,n$, for all $(P,r) \in R^*$.

THEOREM 4*. *If the W.F. $w \in C^1(R)$ and $u \in S(w,R)$, then u satisfies the second order elliptic differential equation*

$$(0.7) \qquad w\Delta u + 2\sum_{i=1}^{n} w_{x_i} u_{x_i} = 0$$

in R, where Δu is the Laplacian of u.

REMARK 3*. In proving Theorem 4* incidentally the following was shown:

If the W.F. $w \in C^1(R)$ and $u \in S(w,R)$, then

$$(0.8) \qquad \Delta u(P) \cdot \int_{B(P,r)} w \, d\rho + 2 \sum_{i=1}^{n} u_{x_i}(P) \cdot \int_{B(P,r)} w_{x_i} \, d\rho = 0$$

for each $(P,r) \in R^*$.

THEOREM 5*. *Let w be a W.F. defined in R and λ be a real number such that $w \in C^2(R)$ and is a solution of the differential equation*

$$(0.9) \qquad \Delta F + \lambda F = 0$$

in R. Then a necessary and sufficient condition that $u \in S(w,R)$ is that

(i) *$u \in C^2(R)$ and*

(ii) *u satisfies the differential equation*

$$w\Delta u + 2\sum_{i=1}^{n} u_{x_i} w_{x_i} = 0 \tag{0.10}$$

in R.

REMARK 4*. In proving Theorem 5* incidentally the following was shown: If the W.F. $w \in C^2(R)$ and is a solution of (0.9), then $w(P) > 0$ for all P in R.

COROLLARY 1*. *Let w be a W.F. defined in R and λ be a real number such that $w \in C^2(R)$ and is a solution of the differential equation*

$$\Delta F + \lambda F = 0. \tag{0.11}$$

in R. Then the following are true:

(a) *If $f \in C^2(R)$ and is a solution of* (0.11), *then $f/w \in S(w,R)$.*

(b) *If $u \in S(w,R)$, then $uw \in C^2(R)$ and is a solution of* (0.11).

COROLLARY 2*. *Let w be a W.F. defined in a region R of E_n $(n>1)$ and λ a real number such that $w \in C^2(R)$, and is a solution of the equation*

$$\Delta F + \lambda F = 0$$

in R. Then $S(w,R)$ is infinite dimensional.

REMARK 5*. In E_1 the following can be proved easily:
If the W.F. $w \in C^1(R)$, then

(i) $1 \leqq \dim S(w,R) \leqq 2$

(ii) $\dim S(w,R) = 2$ if and only if $w \in C^2(R)$ and is a solution of

$$\frac{d^2w}{dx^2} + \lambda \frac{dw}{dx} = 0 \tag{0.12}$$

in R, for some real constant λ.

The importance of the weight functions which are solutions of the equation

$$\Delta w + \lambda w = 0 \tag{0.13}$$

has been demonstrated in the previous paper [1].

In this paper we propose to deduce in terms of the W.A.P. a necessary and sufficient condition for a W.F. to be a solution of (0.13). From this result we will prove our main theorem of §I, namely, "$S(w,R)$ is infinite dimensional if and only if w is a solution of (0.13)."(2)

In §II we will consider derivatives of functions belonging to $S(w,R)$ and deduce necessary and sufficient conditions for these derivatives to belong to $S(w,R)$.

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(2) Added in proof. True only in E_2. See also footnote (3).

Here again we will emphasize the role of weight functions which are solutions of (0.13), specially the exponential weight functions $\exp\{\sum_{i=1}^{n} a_i x_i\}$.

I. The differential equation $\Delta w + \lambda w = 0$.

THEOREM 1. *Let w be a W.F. belonging to class $C^1(R)$. A necessary and sufficient condition that $w \in C^2(R)$ and is a solution of*

$$\Delta w + \lambda w = 0 \tag{1.0}$$

in R, where λ is some real constant, is that
(i) $w(P) > 0$ *for all* $P(x_1, x_2, \cdots, x_n)$ *in* R *and*
(ii) $w_{x_i}/w \in S(w,R)$ *for* $i = 1,2,\cdots,n$.

Proof. *Necessity.* Let $w \in C^2(R)$ and be a solution of (1.0). Then w is analytic in R and each of the partial derivatives w_{x_i}, $i = 1,2,\cdots,n$, is also a solution of (1.0). Again by Remark 4*, $w(P) > 0$ for all $P = P(x_1, x_2, \cdots, x_n)$ in R. Hence by Corollary 1*, $w_{x_i}/w \in S(w,R)$ for $i = 1,2,\cdots,n$.

Sufficiency. Suppose that $w(P) > 0$ for all $P = P(x_1, x_2, \cdots, x_n)$ in R and $w_{x_i}/w \in S(w,R)$ for $i = 1,2,\cdots,n$. $w_{x_i}/w = F^i \in S(w,R)$ implies by Theorem 2* that $F^i \in C^2(R)$ and $w \in C^3(R)$. Also by Theorem 4*, F^i is a solution of the differential equation

$$w\Delta F^i + 2\sum_{j=1}^{n} F^i_x w_{x_j} = 0, \tag{1.1}$$

$i = 1,2,\cdots,n$. Again $F^i w = w_{x_i}$ implies that

$$\Delta w + \lambda w = 0, \tag{1.2}$$

where

$$\lambda = -\sum_{i=1}^{n} \{F^i_{x_i} + (F^i)^2\}.$$

Clearly λ belongs to class $C^1(R)$ and

$$\frac{\partial \lambda}{\partial x_j} = \lambda_{x_j} = -\sum_{i=1}^{n} \{F^i_{x_i x_j} + 2F^i F^i_{x_j}\}, \qquad j = 1,2,\cdots,n.$$

Using

$$F^i_{x_j} = F^j_{x_i}, F^i_{x_j x_i} = F^j_{x_i x_i}, \qquad i,j = 1,2,\cdots,n;$$

we have

$$w\lambda_{x_j} = -\sum_{i=1}^{n} \{wF^j_{x_i x_i} + 2F^j_{x_i} w_{x_i}\} = -\left\{w\Delta F^j + 2\sum_{i=1}^{n} F^j_{x_i} W_{x_i}\right\} = 0.$$

Since

$$w \neq 0 \text{ in } R, \ \lambda_{x_j} = 0 \text{ in } R \text{ for } j = 1,2,\cdots,n.$$

Therefore λ is a constant. This completes the proof.

Dimension of $S(w,R)$.

THEOREM 2. *Let R be a region in E_2 and w be a positive W.F. belonging to class $C^1(R)$. Then $S(w,R)$ is infinite dimensional if and only if w belongs to class $C^2(R)$ and is a solution of*

$$\Delta w + \lambda w = 0 \tag{1.3}$$

in R for some real constant λ. Furthermore, if $S(w,R)$ is finite dimensional then $1 \leqq \dim S(w,R) \leqq 2$.

Proof. If $w \in C^2(R)$ and is a solution of (1.3) then $S(w,R)$ is infinite dimensional by Corollary 2*. So now suppose that w be a positive W.F. belonging to class $C^1(R)$ such that $\dim S(w,R) > 2$. Then there exists two nonconstant functions u_1 and u_2 both belonging to $S(w,R)$ such that $1, u_1, u_2$, are linearly independent over R. By Theorem 4* and Remark 3* we have

$$w\Delta u_i + 2(w_x u_{ix} + w_y u_{iy}) = 0, \ i = 1,2,. \tag{1.4}$$

in R and

$$\Delta u_i(P) + 2u_{ix}(P)\left[\frac{\int_{B(P,r)} w_x d\rho}{\int_{B(P,r)} w d\rho}\right] + 2u_{iy}(P)\left[\frac{\int_{B(P,r)} w_y d\rho}{\int_{B(P,r)} w d\rho}\right] = 0 \tag{1.5}$$

for each $(P,r) \in R^*$. Therefore we have

$$u_{ix}(P)\left[\frac{w_x(P)}{w(P)} - \frac{\int_{B(P,r)} w_x d\rho}{\int_{B(P,r)} w d\rho}\right] + u_{iy}(P)\left[\frac{w_y(P)}{w(P)} - \frac{\int_{B(P,r)} w_y d\rho}{\int_{B(P,r)} w d\rho}\right] = 0, \tag{1.6}$$

for each $(P,r) \in R^*$, $i = 1,2$.

Let R_1 be the subset of R to which a point P belongs if and only if the Jacobian

$$J_{12} = \frac{D(u_1,u_2)}{D(x,y)} = \begin{vmatrix} u_{1x} & u_{1y} \\ u_{2x} & u_{2y} \end{vmatrix}$$

does not vanish at P. R_1 is a dense subset of R. For, if not, then there exists a point P in $R - R_1$ and a neighborhood $N(P)$ of the point P lying in R such that the Jacobian J_{12} vanishes identically in $N(P)$. If at least one of the minors, say u_{1x}, is not identically zero in $N(P)$, then there is a point Q and a neighborhood $N(Q)$ of the point Q lying in $N(P)$ such that $u_{1x} \neq 0$ in $N(Q)$. Hence there exists a functional relation $u_2 = \phi(u_1)$ valid in $N(Q)$. Since each of u_1 and u_2 belong to $C^2(R)$ and $u_{1x} \neq 0$ in $N(Q)$, ϕ is twice continuously differentiable and we have

$$w\Delta u_2 + 2u_{2x}w_x + 2u_{2y}w_y = w \cdot \frac{d^2\phi}{du_1^2}\{(u_{1x})^2 + (u_{1y})^2\} = 0$$

or

$$\frac{d^2\phi}{du_1^2} = 0.$$

Hence $u_2 = c_1u_1 + c_2$, in $N(Q)$, where c_1 and c_2 are real constants. This means by the maximum principle of the elliptic equation

$$w\Delta u + 2w_xu_x + 2w_yu_y = 0$$

that $u_2 = c_1u_1 + c_2$ in R, contradicting the linear independence assumption. Therefore every minor of J_{12} is identically zero in $N(P)$ implying that each of u_1 and u_2 is constant in $N(P)$ and hence in R contradicting again our hypothesis. Therefore R_1 is a dense subset of R. Now to prove our theorem consider the equations (1.6). Clearly

$$\int_{B(P,r)} w_x d\rho \Big/ \int_{B(P,r)} w d\rho = w_x(P)/w(P) \tag{1.8}$$

$$\int_{B(P,r)} w_y d\rho \Big/ \int_{B(P,r)} w d\rho = w_y(P)/w(P) \tag{1.9}$$

for each point P in R_1 and each r satisfying $0 < r < d(P,T)$. Since R_1 is dense, we conclude that the relations (1.8) and (1.9) are true for each $(P,r) \in R^*$. This means that each of the functions w_x/w and w_y/w belongs to $S(w,R)$. Hence by Theorem 1 there exists a real constant λ such that $w \in C^2(R)$ and is a solution of (1.3). Therefore $S(w,R)$ is infinite dimensional by Corollary 2*.

It is clear also that if $S(w,R)$ is finite dimensional then $1 \leqq \dim S(w,R) \leqq 2$.

THEOREM 3(3). *Let R be a region in E_n $(n>2)$ and w be a positive W.F. belonging to class $C^1(R)$. Then $S(w,R)$ is infinite dimensional if and only if w belongs to class $C^2(R)$ and is a solution of*

$$\Delta w + \lambda w = 0$$

in R, for some real constant λ. If $S(w,R)$ is finite dimensional then

$$1 \leqq \dim S(w,R) \leqq 2n - 1.$$

Proof. For simplicity we will give the proof for $n = 4$. Proof for the general case is quite similar.

(3) Added in proof. Theorem 3 is not possibly true. Mr. David Stanford of Denison University, Granville, Ohio, seems to have a counterexample. The author hopes to clarify this point in a future paper.

If $w \in C^2(R)$ and is a solution of $\Delta w + \lambda w = 0$, for some real constant λ, then by Corollary 2*, $S(w,R)$ is infinite dimensional. So now suppose that w be a positive W.F. belonging to class $C^1(R)$ such that $\dim S(w,R) > 2\cdot 4 - 1 = 7$. Then there exists nonconstant functions $u_1, u_2, \cdots, u_7$ all belonging to $S(w,R)$ such that $1, u_1, u_2, \cdots, u_7$ are linearly independent over R. As in Theorem 2, we have by Theorem 4* and Remark 3* the system of 7 equations:

$$\sum_{i=1}^{n} u_{jx_i}(P)\left[w_{x_i}(P)/w(P) - \int_{B(P,r)} w_{x_i} d\rho \bigg/ \int_{B(P,r)} w d\rho\right] = 0, \tag{1.10}$$

$j = 1, 2, \cdots, 7$; for each $(P,r) \in R^*$.

Let (j_1, j_2, j_3, j_4) be a combination of four distinct integers taken from the seven positive integers $1, 2, \cdots, 7$. The system of equations (1.10) gives rise to ${}_7C_4 = 35$ Jacobians of the form

$$J_{j_1 j_2 j_3 j_4} = \frac{D(u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4})}{D(x_1, x_2, x_3, x_4)}.$$

Let R_1 be the subset of R to which a point P belongs if and only if at least one of the ${}_7C_4 = 35$ Jacobians $\{J_{j_1 j_2 j_3 j_4}\}$ does not vanish at P. R_1 is a dense subset of R. For, if not, then there exists a point P and a neighborhood $N(P)$ of the point P lying in R such that each of the 35 Jacobians $\{J_{j_1 j_2 j_3 j_4}\}$ vanishes identically in $N(P)$. This again means that every first, second, and third minor of each of the 35 Jacobians $\{J_{j_1 j_2 j_3 j_4}\}$ vanishes identically in $N(P)$. For, if possible, let one of the first minors of J_{1234}, say $D(u_1, u_2, u_3)/D(x_1, x_2, x_3)$, does not vanish identically in $N(P)$. Then there is a point Q and a neighborhood $N(Q)$ of the point Q lying in $N(P)$ such that $D(u_1, u_2, u_3)/D(x_1, x_2, x_3) \neq 0$ in $N(Q)$. Hence considering also the Jacobians $J_{1235}, J_{1236}, J_{1237}$, there exists functional relations

$$\begin{aligned} u_4 &= \phi_4(u_1, u_2, u_3),\ u_5 = \phi_5(u_1, u_2, u_3), \\ u_6 &= \phi_6(u_1, u_2, u_3),\ u_7 = \phi_7(u_1, u_2, u_3) \end{aligned} \tag{1.11}$$

each valid in $N(Q)$. Also there is a functional relation

$$\phi(u_4, u_5, u_6, u_7) = 0, \tag{1.12}$$

valid in $N(P)$.

Therefore (1.11) and (1.12) imply that there is a functional relation

$$\psi(u_1, u_2, u_3) = 0,$$

valid in $N(Q)$ contradicting the fact that $D(u_1, u_2, u_3)/D(x_1, x_2, x_3) \neq 0$ in $N(Q)$. Hence every first minor of each of the 35 Jacobians vanishes identically in $N(P)$. If possible, suppose now that one of the second minors of J_{1234}, say, $D(u_1, u_2)/D(x_1, x_2)$, is not identically zero in $N(P)$. Then there exists a point Q and a neighborhood $N(Q)$ of the point Q lying in $N(P)$ such that $D(u_1, u_2)/D(x_1, x_2)$

$\neq 0$ in $N(Q)$. Hence, considering also the Jacobian J_{1256}, there exists functional relations

$$u_3 = \phi_3(u_1,u_2),\ u_4 = \phi_4(u_1,u_2),\ u_5 = \phi_5(u_1,u_2),\ u_6 = \phi_6(u_1,u_2) \tag{1.13}$$

valid in $N(Q)$. (1.12) and (1.13) lead to functional relation $\psi(u_1,u_2)=0$ valid in $N(Q)$, contradicting the nonvanishing of $D(u_1,u_2)/D(x_1,x_2)$ in $N(Q)$. Hence every second minor of each of the 35 Jacobians $\{J_{j_1j_2j_3j_4}\}$ vanishes identically in $N(P)$. Finally suppose that one of the third minors of J_{1234}, say $\partial u_1/\partial x_1$ is not identically zero in $N(P)$. Then there exists functional relations of the form

$$u_2 = \phi_2(u_1), \quad u_3 = \phi_3(u_1), \quad u_4 = \phi_4(u_1),$$

valid in some neighborhood $N(Q)$ of a point Q lying in $N(P)$. Now applying similar arguments as in Theorem 2, we get a relationship $u_2 = c_1u_1 + c_2$ valid in R, where c_1 and c_2 are real constants which contradicts the linear independence assumption. Therefore every third minor of each of the 35 Jacobians $\{J_{j_1j_2j_3j_4}\}$ vanishes identically in $N(P)$, which means that each of the functions $u_1, u_2, u\cdots, u_7$ is constant in $N(Q)$ and hence, by the maximum principle of the elliptic equation (0.10), in R, implying again a contradiction. Therefore R_1 is a dense subset of R.

Now arguing similarly as in Theorem 2, we conclude that each of the functions w_{x_i}/w, $i=1,2,3,4$, belong to $S(w,R)$. Hence by Theorem 1, there exists a real constant λ such that $w \in C^2(R)$. and is a solution of (1.0) in R. Hence $S(w,R)$ is infinite dimensional by Corollary 2*. It is also clear that if $S(w,R)$ is finite dimensional then $1 \leqq \dim S(w,R) \leqq 2\cdot 4 - 1 = 7$.

II. **Derivatives of functions satisfying W.A.P.** It is known that derivatives of harmonic functions, defined in a given region R of E_n, are also harmonic in that region. But this is not, in general, true for functions satisfying W.A.P. unless they satisfy a similar W.A.P. with respect to the derivatives of the W.F. For example, consider the W.F., $w(x,y) = x + y$, defined in R, where R is the first quadrant of the plane E_2. The function $u(x,y) = x^2 - 4xy + y^2$ is a solution of

$$w\Delta u + 2(w_xu_x + w_yu_y) = 0 \tag{2.0}$$

in R. Also $u \in C^2(R)$. Since $w \in C^2(R)$ and is a solution of $\Delta w = 0$ in R, by Theorem 5*, $u \in S(w,R)$. But $u_x = 2x - 4y$ and $u_y = -4x + 2y$, do not satisfy (2.0) and hence cannot satisfy W.A.P. with respect to w in R.

On the other hand $w(x,y) = \exp(x+y)$ is a W.F. in E_2. $w \in C^2(E_2)$ and is a solution of

$$\Delta w - 2w = 0$$

in E_2. The function $u(x,y) = x^2 - 2xy + y^2 - x - y$ belongs to class $C^2(E_2)$ and is a solution of

$$w\Delta u + 2w_x u_x + 2w_y u_y = 0$$

in E_2. Hence by Theorem 5*, $u \in S(w, E_2)$. Also each of the derivatives $u_x = 2x - 2y - 1$, $u_y = -2x + 2y - 1$ belongs to class $C^2(E_2)$ and is a solution of (2.0). Therefore each of the derivatives u_x and u_y belongs to $S(w, E_2)$.

THEOREM 4. *Let R be a region in E_n and w be a W.F. belonging to class $C^1(R)$. If $u \in S(w,R)$, then a necessary and sufficient condition that the partial derivative $\partial u/\partial x_i = u_{x_i}$, $1 \leqq i \leqq n$, will belong to $S(w,R)$ is that*

$$\int_{B(P,r)} uw_{x_i} d\rho = u(P) \int_{B(P,r)} w_{x_i} d\rho \tag{2.1}$$

for each $(P,r) \in R^*$.

Proof. By Remark 2*, the partial derivative u_{x_i} satisfies the relation

$$u_{x_i}(P) \int_{B(P,r)} w d\rho = \int_{B(P,r)} u_{x_i} w d\rho + \left[\int_{B(P,r)} uw_{x_i} d\rho - u(P) \int_{B(P,r)} w_{x_i} d\rho \right] \tag{2.2}$$

for each $(P,r) \in R^*$. It is clear from (2.2) that the theorem is true.

REMARK 1. If w be a positive constant, then $S(w,R)$ is the class of all harmonic functions defined in R and $w_{x_i} = 0$, for $i = 1, 2, \cdots, n$, implies that the relation (2.1) is true for constant W.F., which simply means that derivatives of harmonic functions are also harmonic as is well known.

THEOREM 5. *Let the W.F. w belong to class $C^2(R)$ and be solution of*

$$\Delta w + \lambda w = 0 \tag{2.3}$$

in R, where λ is a real constant. If $u \in S(w,R)$ then these are equivalent:

(i) *u_{x_i}, $1 \leqq i \leqq n$, belongs to $S(w,R)$*
(ii) *uw_{x_i}/w belongs to $S(w,R)$*
(iii) *uw_{x_i} is a solution of (2.3) in R*
(iv) *$u_{x_i} w$ is a solution of (2.3) in R*
(v) *u is a solution of*

$$w_{x_i} \Delta u + 2 \sum_{j=1}^{n} u_{x_j} w_{x_i x_j} = 0 \tag{2.4}$$

in R.

Proof. By hypothesis and from Theorem 2*, each of u and w is analytic in R. Also, by Theorem 1, $w(P) > 0$ for each point $P = P(x_1, x_2, \cdots, x_n)$ in R and each of the functions w_{x_i}/w, $i = 1, 2, \cdots, n$, belongs to $S(w,R)$.

Now suppose that (i) is true. Using Remark 3* we can write

$$u_{x_i}(P) \int_{B(P,r)} w d\rho = \int_{B(P,r)} \{u_{x_i} + (uw_{x_i})/w\} w d\rho - u(P) \int_{B(P,r)} \{w_{x_i}/w\} w d\rho,$$

or

$$\{u_{x_i}(P) + u(P)w_{x_i}(P)/w(P)\} \int_{B(P,r)} w d\rho = \int_{B(P,r)} \{u_{x_i} + (uw_{x_i})/w\} w d\rho$$

for each $(P,r) \in R^*$ which implies that $u_{x_i} + uw_{x_i}/w \in S(w,R)$. Since $S(w,R)$ is a linear space, $uw_{x_i}/w \in S(w,R)$. Hence (i) implies (ii). Next suppose that (ii) is true. Then by Corollary 1*, uw_{x_i} is a solution of (2.3). Therefore (ii) implies (iii). Now suppose that (iii) is true. By hypothesis and from Corollary 1*, uw is a solution of (2.3) and hence $(uw)_{x_i} = u_{x_i}w + uw_{x_i}$ is also a solution of (2.3). We conclude from the linearity of (2.3) that $u_{x_i}w$ is a solution of (2.3). Thus (iii) implies (iv).

Next suppose that (iv) is true. We have

$$\Delta(u_{x_i}w) + \lambda(u_{x_i}w) = 0. \tag{2.5}$$

Also by hypothesis $\Delta w + \lambda w = 0$ and

$$w\Delta u + 2\sum_{j=1}^{n} u_{x_j}w_{x_j} = 0. \tag{2.6}$$

Differentiation of (2.6) with respect to x_i yields

$$\left\{w_{x_i}\Delta u + 2\sum_{j=1}^{n} u_{x_j}w_{x_ix_j}\right\} + \{\Delta(u_{x_i}w) + \lambda(u_{x_i}w)\} = 0$$

or

$$w_{x_i}\Delta u + 2\sum_{j=1}^{n} u_{x_j}w_{x_ix_j} = 0.$$

Therefore (iv) implies (v).

Finally suppose that (v) is true. Now differentiation of (2.6) with respect to x_i gives

$$\left\{w_{x_i}\Delta u + 2\sum_{j=1}^{n} u_{x_j}w_{x_ix_j}\right\} + \left\{w\Delta u_{x_i} + 2\sum_{j=1}^{n} u_{x_ix_j}w_{x_j}\right\} = 0.$$

Therefore we have $w\Delta u_{x_i} + 2\sum_{j=1}^{n} u_{x_ix_j}w_{x_j} = 0$, which by Theorem 5* means that $u_i \in S(w,R)$. This completes the cycle.

Exponential Weight Functions, Definition. Weight functions of the form

$$w(x_1, x_2, \cdots, x_n) = k \exp(a_1x_1 + a_2x_2 + \cdots + a_nx_n),$$

$(x_1, x_2, \cdots, x_n) \in E_n$, where a_i's are real constants and k is a positive real constant, are called exponential weight functions.

Properties of exponential weight functions. If w be an exponential W.F., then

(2.7) w is an analytic W.F. in any subregion R of E_n,

(2.8) w is a solution of $\Delta w + \lambda w = 0$ in E_n, where $\lambda = -\sum_{i=1}^{n} a_i^2$.

(2.9) $w_{x_i}/w = a_i$, $i = 1, 2, \cdots, n$.

The differential equation $\Delta u + 2\sum_{i=1}^{n} a_i u_{x_i} = 0$.

THEOREM 6. *For an exponential W.F.* $w(x_1, x_2, \cdots, x_n) = k \exp(\sum_{i=1}^{n} a_i x_i)$, *a necessary and sufficient condition that* $u \in S(w, R)$ *is that* $u \in C^2(R)$ *and is a solution of*

$$\Delta u + 2 \sum_{i=1}^{n} a_i u_{x_i} = 0, \tag{2.10}$$

in R.

Proof. The theorem follows at once from Theorem 5*.

THEOREM 7. *Let* $w(x_1, x_2, \cdots, x_n) = k \exp(\sum_{i=1}^{n} a_i x_i)$ *be an exponential W.F. If* $u \in S(w, R)$, *then*

(i) u *is analytic in* R *and*

(ii) *if* m *be a positive integer and* $D^m u$ *be any mth order partial derivative of* u, *then* $D^m u \in S(w, R)$.

Proof. Part (i) is an immediate consequence of Theorem 2*. Again $S(w, R)$ is a linear space and $u \in S(w, R)$ implies that each of the functions $uw_{x_i}/w = a_i u$, $i = 1, 2, \cdots, n$, also belongs to $S(w, R)$. Hence by Theorem 5 each of the derivatives u_{x_i}, $i = 1, 2, \cdots, n$, belongs to $S(w, R)$. Applying mathematical induction, we see that the theorem is true.

Characterisation of exponential W.F. Property (2.9) and part (ii) of Theorem 7 characterises exponential weight function in the sense of the following theorem:

THEOREM 8. *Let* w *be a W.F. belonging to class* $C^2(R)$ *and is a solution of* $\Delta w + \lambda w = 0$ *for some real constant. Then the following are equivalent:*

(i) *If* $u \in S(w, R)$, *then each of the derivatives* u_{x_i}, $i = 1, 2, \cdots, n$, *also belongs to* $S(w, R)$.

(ii) w *is an exponential W.F.*

Proof. By Theorem 1, $w(P) = w(x_1, x_2, \cdots, x_n) > 0$ for all $P(x_1, x_2, \cdots, x_n)$ in R and each of the functions $w_{x_i}/w \in S(w, R)$, $i = 1, 2, \cdots, n$. Suppose that (i) is true. Then each of the partial derivatives $(\partial/\partial x_i)(w_{x_i}/w)$, $i = 1, 2, \cdots, n$, also belongs to $S(w, R)$. Therefore by Theorem 5 each of the functions $(w_{x_i}/w)^2$ belongs to $S(w, R)$. Let $P_0 \in R$. Since $S(w, R)$ is a linear space, each of the functions $F_i = \{w_{x_i}/w - w_{x_i}(P_0)/w(p_0)\}^2$, $i = 1, 2, \cdots, n$, also belongs to $S(w, R)$. Since F_i has a minimum (zero) at P_0 in R, it follows from the minimum-principle of the differential equation

$$w\Delta u + 2 \sum_{i=1}^{n} w_{x_i} u_{x_i} = 0$$

that $w_{x_i} = a_i w$, $i = 1, 2, \cdots, n$, where a_i's are real constants. Hence $w(x_1, x_2, \cdots, x_n)$

$= k \exp(\sum_{i=1}^{n} a_i x_i)$, for each $(x_1, x_2, \cdots, x_n) \in R$, where k is a positive real constant. Therefore (i) implies (ii). Now suppose that (ii) is true. Let $w(x_1, x_2, \cdots, x_n) = k \exp(\sum_{i=1}^{n} a_i x_i)$ and $u \in S(w, R)$. Then by Theorem 7 each of the derivatives u_{x_i}, $i = 1, 2, \cdots, n$, belongs to $S(w, R)$. Hence (ii) implies (i). This completes the proof.

Bibliography

1. Anil K. Bose, *Functions satisfying a weighted average property*, Trans. Amer. Math. Soc. **118** (1965), 472–487.

2. Edouard Goursat, *A course in mathematical analysis*, Vol. I, Dover, New York, 1904.

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