PIERCING POINTS OF HOMEOMORPHISMS OF DIFFERENTIABLE MANIFOLDS

BY
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1. **Introduction.** In this paper we investigate the problem of relating homeomorphisms of euclidean n-space E^n (resp. a differentiable manifold M^n) onto itself to diffeomorphisms of E^n (resp. of M^n). Throughout this paper, "diffeomorphism" shall mean C^p-diffeomorphism, where $p > 0$. A discussion of these problems is facilitated by introducing the following equivalence relation $\sim$ on the set $H(E^n)$ of homeomorphisms of E^n onto itself. If F, $G \in H(E^n)$, we say $F \sim G$ if there exist homeomorphisms $H_0 = F$, $H_1, \cdots, H_m = G$, where each $H_i \in H(E^n)$, and nonempty open sets $U_1, U_2, \cdots, U_m$ of E^n such that $H_i | U_i = H_{i-1} | U_i$, $i = 1, 2, \cdots, m$. One asks, for example, whether a given homeomorphism $F \in H(E^n)$ is equivalent under $\sim$ to a diffeomorphism.

Fundamental in the study of this type of question is the notion of *stable* homeomorphisms of E^n onto itself. Recall that a homeomorphism $H \in H(E^n)$ is called stable if there exist homeomorphisms $H_1, \cdots, H_m$, where each $H_i \in H(E^n)$ and nonempty open sets $U_1, \cdots, U_m$ of E^n such that $H = H_1 H_2 \cdots H_m$, and $H_i | U_i = 1$, $i = 1, 2, \cdots, m$. All orientation-preserving diffeomorphisms of E^n onto itself are stable. It is readily seen that if $F \sim G$, and G is stable, then so is F. It also can be proved (cf. Theorem 5.4 of [1]) that if F and G are any two stable homeomorphisms, then $F \sim G$. It follows easily from these latter two statements that $F \sim G$ if and only if $G^{-1}F$ is a stable homeomorphism of E^n. Finally, the annulus conjecture is equivalent to the conjecture that all orientation-preserving homeomorphisms of E^n onto itself are stable. This latter conjecture is known to be true for $n = 1, 2, 3$.

In an effort to relate homeomorphisms to diffeomorphisms, we define (cf. §3) the notion of a piercing point of a homeomorphism. In §§3–7 we develop some basic properties relating to piercing points. A proof is given in §8 of a result announced by the author in [2]. Theorem 1 of §9 relates the notion of piercing point to stability. The author thanks William Huebsch for many helpful conversations.

2. **Notation.** Let the points of E^n be written $x = (x^1, \cdots, x^n)$, and provide E^n with the usual euclidean norm and metric

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$$\|x\| = \left[\sum_{i=1}^{n} (x^i)^2\right]^{1/2}, \qquad d(x,y) = \|x-y\|.$$

For c a fixed point of E^n, and $r > 0$ a constant, we denote the $(n-1)$-sphere about c, of radius r, by

$$S^{n-1}(c,r) = \{x \in E^n \,|\, d(x,c) = r\}.$$

We often delete the superscript $n-1$ when there is no danger of confusion. By a topological $(n-1)$-sphere M in E^n we mean the image in E^n of $S^{n-1}(c,r)$ under some homeomorphism h. We say that h defines M. If M is a topological $(n-1)$-sphere in E^n, we denote the bounded component of $E^n - M$ by $\dot{J}M$, and the closure of $\dot{J}M$ in E^n by JM.

A topological $(n-1)$-sphere M in E^n will be called elementary if some (hence *every*, as is readily proved) homeomorphism h defining M is extendable as a homeomorphism into E^n of an open neighborhood N of $S^{n-1}(c,r)$ relative to E^n. This is equivalent (cf. [3]) to requiring that M is locally flat. A homeomorphism h of an elementary $(n-1)$-sphere M in E^n which is extendable over a neighborhood of M as a homeomorphism will itself be termed elementary.

If M is a topological $(n-1)$-sphere in E^n, and $f: JM \to E^n$ is a homeomorphism of JM into E^n, then

$$(\alpha) \qquad f(\dot{J}M) = \dot{J}f(M).$$

A proof of (α) can be found, for example, in [4].

3. **Piercing points.** We assume in what follows that $n \geqq 2$.

Definition 1. Let $f: U \to E^n$ be a homeomorphism of U into E^n, where U is an open subset of E^n. A point $x \in U$ is called a *piercing point* of f if there exists a C^p-imbedding $(p > 0)$ $\sigma: [-1,1] \to U$, a diffeomorphism $H \in H(E^n)$, and an $(n-1)$-hyperplane P in E^n such that

(i) $\sigma(0) = x$,

(ii) $Hf\sigma([-1,1]) \cap P = Hf\sigma(0)$,

(iii) $Hf\sigma(-1)$ and $Hf\sigma(1)$ lie in opposite components of $E^n - P$.

One verifies that every point of a diffeomorphism $H \in H(E^n)$ is a piercing point of H. On the other hand, we will prove (§8) that there exist homeomorphisms $F \in H(E^n)$ having a dense set of nonpiercing points.

Proposition 1. *Let U, V be open subsets of E^n, let $f: U \to E^n$ be a homeomophism of U into E^n,* and let *$g: V \to E^n$ be a diffeomorphism of V into E^n. Then a point $x \in U \cap f^{-1}(V)$ [resp. $x \in U \cap g(V)$] is a piercing point of f if, and only if, x is a piercing point of gf [resp. $g^{-1}(x)$ is a piercing point of fg].*

Proof. In each situation considered, it can be assumed without loss of generality that g is a diffeomorphism of E^n onto itself (cf. [5, pp. 28–29]). Suppose that x is a piercing point of f. Then, corresponding to f and x, there exist σ, H, and P as in Definition 1. Then σ, Hg^{-1}, and P [resp. $g^{-1}\sigma$, H, P] can be used to verify that x is a piercing point of gf [resp. $g^{-1}(x)$ is a piercing point of fg]. Conversely, suppose that x is a piercing point of H [resp. g^{-1} is a piercing point fg]. Then σ, Hg, and P [resp. $g\sigma$, H, P] can be used to verify that x is a piercing point of f.

With the aid of Proposition 1, the notion of piercing point may be extended, in the natural way, to homeomorphisms from one nonbounded differentiable n-manifold into another. More precisely, we make the following definition.

DEFINITION 1′. Let M_1^n, M_2^n be nonbound differentiable n-manifolds having differentiable structures D_1, D_2, respectively, both of class at least $p > 0$. Let $f: U \to M_2^n$ be a homeomorphism of U into M_2^n, where U is an open subset of M_1^n. Then x is called a piercing point of f if there exist coordinate systems $(U_1, h_1) \in D_1$, $(U_2, h_2) \in D_2$ at $x, f(x)$, respectively, such that $h_1(x)$ is a piercing point (in the sense of Definition 1) of the homeomorphism

$$h_2 f h_1^{-1} : h_1(U \cap U_1 \cap f^{-1}(U_2)) \to E^n.$$

It is easily verified, using Proposition 1 and its proof, that this definition is independent of the choice of coordinate systems at x and $f(x)$.

4. **Example** 1. We now construct homeomorphisms f, $g \in H(E^2)$ such that $\mathbf{0} = (0,0)$ is a piercing point of f, $f(\mathbf{0}) = \mathbf{0}$ is a piercing point of g, but $\mathbf{0}$ is *not* a piercing point of the composition gf. Now it is easily seen that a point $x \in U \subset E^2$ is a piercing point of a homeomorphism $h: U \to E^2$ if, and only if, the point $h(x)$ is a piercing point of $h^{-1}: h(U) \to E^2$. Hence if f, g are as above, we see that the homeomorphisms gf, $f = g^{-1}(gf)$, and $f^{-1} = (gf)^{-1}g$, show that all the conclusions of Proposition 1 can fail when the hypothesis is weakened to merely requiring, for example, that $g(x)$ be a piercing point of the *homeomorphism* g.

Let $x = (0,0) = \mathbf{0} \in E^2$, and set $S = S^1(\mathbf{0}, 1)$. Denote the line segments having one end point at $\mathbf{0}$ and the other end point at $(0,1)$, $(\sqrt{2}/2, \sqrt{2}/2)$, $(-\sqrt{2}/2, \sqrt{2}/2)$, $(-\sqrt{3}/2, -\frac{1}{2})$, $(-\frac{1}{2}, -\sqrt{3}/2)$, $(\frac{1}{2}, -\sqrt{3}/2)$, $(\sqrt{3}/2, -\frac{1}{2})$, by $L_0, \cdots, L_6$, respectively. Let

$$M_1 = \{y = (y^1, y^2) \in S \mid 0 \leqq y^1 \leqq \sqrt{2}/2, \sqrt{2}/2 \leqq y^2 \leqq 1\} \cup L_0 \cup L_1,$$
$$M_2 = \{y = (y^1, y^2) \in S \mid -\sqrt{2}/2 \leqq y^1 \leqq 0, \sqrt{2}/2 \leqq y^2 \leqq 1\} \cup L_0 \cup L_2,$$

and

$$M_3 = \{y = (y^1, y^2) \in S \mid -1 \leqq y^1 \leqq 1, -1 \leqq y^2 \leqq \sqrt{2}/2\} \cup L_1 \cup L_2.$$

Let

$$\{y_i\} = \{1/i(\sqrt{2}/2, \sqrt{2}/2)\}, \quad \{z_i\} = \{1/i(-\sqrt{2}/2, \sqrt{2}/2)\}, \quad i = 1, 2, \cdots.$$

We now define f on M_1 as follows. Set $f(w) = w$ for $w = (w^1, w^2) \in L_0$, and

$$f(w) = (w^1 \cos(5\pi\sqrt{2}/12w^1) + w^2 \sin(5\pi\sqrt{2}/12w^1),$$
$$- w^1 \sin(5\pi\sqrt{2}/12w^1) + w^2 \cos(5\pi\sqrt{2}/12w^1))$$

for $w \in M_1 \cap S$. We complete the definition of f on M_1 as follows.

Set $f(y_{2i-1}) = 1/i(\sqrt{3}/2, -1/2)$, and $f(y_{2i}) = 1/i(-1/2, -\sqrt{3}/2)$, $i = 1, 2, \cdots$. Then let f map the segment between y_i and y_{i+1} in the manner indicated in Figure 1 below.

Define f on M_2 by letting

$$f((w^1, w^2)) = (-f^1(-w^1, w^2), f^2(-w^1, w^2)) \text{ for } w \in L_0 \cup (M_2 \cap S) \cup \bigcup_{i=1}^{\infty} z_i.$$

Complete the definition of f on M_2 by mapping the segment between z_i and z_{i+1} in the manner indicated in Figure 1.

To define f on M_3, note that f is defined on $(M_3 - S) \cup \{\sqrt{2}/2, \sqrt{2}/2)\} \cup \{(-\sqrt{2}/2, \sqrt{2}/2)\}$. We complete the definition of f on M_3 by mapping $M_3 - (L_1 \cup L_2)$ homeomorphically onto $S - \{f(M_1 \cup M_2)\}$.

We now have defined f on $M_1 \cup M_2 \cup M_3$ consistently as a homeomorphism. Since every homeomorphism of a topological 1-sphere M in E^2 admits an extension as a homeomorphism over JM (cf. [6]), we may extend $f|M_i$ to a homeomorphism of JM_i into E^2, $i = 1, 2, 3$. Actually, it is clear that $f|M_i$ is elementary, and hence we could use the Schoenflies extension theorem (cf. [7] or [8]) to get the extension of $f|M_i$ over JM_i. This latter method of obtaining the extension is employed when modifying the example to dimensions greater than 2. Now since $fJM_i = Jf(M_i)$ (cf. (α) of §2), we obtain a homeomorphism f of JS into E^n. For convenience, we apply the Schoenflies extension theorem again to assume that $f \in H(E^2)$. Note that $\mathbf{0}$ is a piercing point of f.

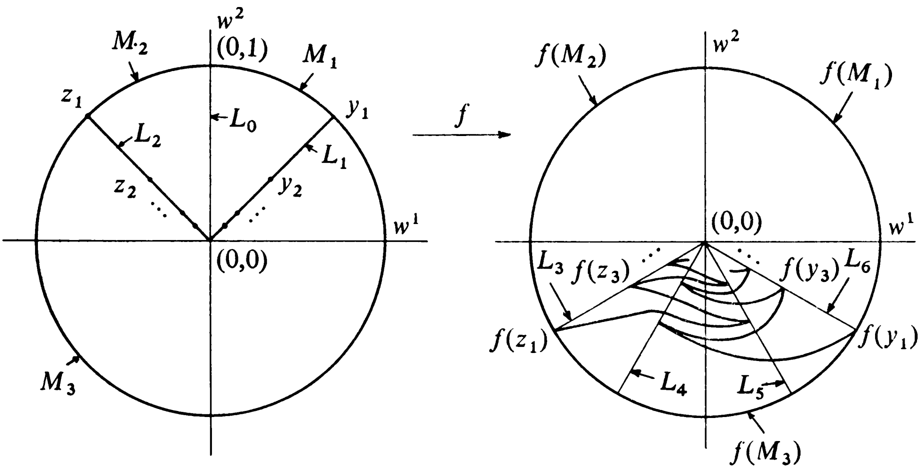

FIGURE 1

We now consider a homeomorphism $g \in H(E^2)$ with the following properties (cf. Figure 2). Let L_3, L_4, L_5, and L_6 be as in Figure 1. Denote the line segments having one end point at $\mathbf{0}$ and the other end point at $(-\frac{1}{2}, \sqrt{3}/2)$, $(-\sqrt{3}/2, \frac{1}{2})$, $(\sqrt{3}/2, \frac{1}{2})$, $(\frac{1}{2}, \sqrt{,} 3/2)$, by L_7, L_8, L_9, L_{10}, respectively. We suppose $g(L_i) = L_{i+4}$,

$i = 3,4,5,6$. We further suppose that g is the identity on the w^2-axis. A homeomorphism $g \in H(E^2)$ with these properties is easily constructed. Note that $\mathbf{0}$ is a piercing point of g.

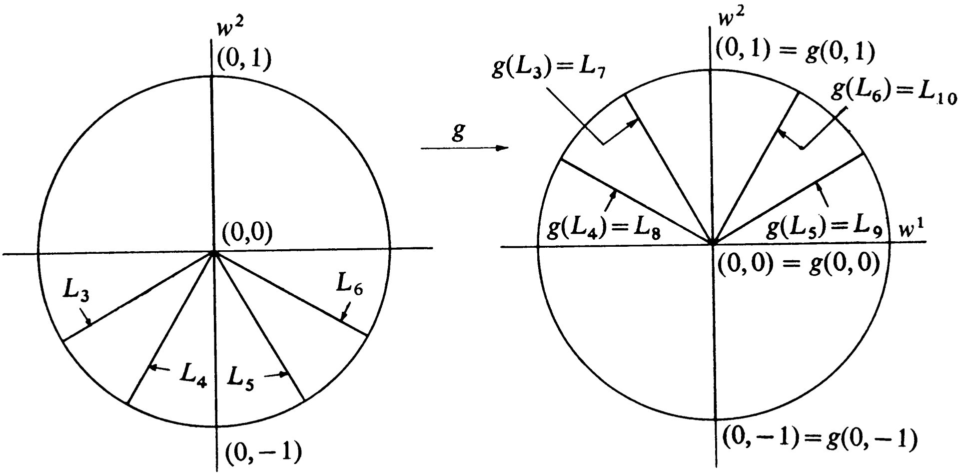

FIGURE 2

It is readily seen that $\mathbf{0}$ is not a piercing point of gf.

REMARKS. An analogous construction can be carried out for any $n \geqq 2$. Also, one verifies with the aid of Theorem 4.1 of [5] (only really needed for f), that f and g can be constructed so as to be diffeomorphisms on $E^n - \mathbf{0}$.

5. **One-sided piercing points.** Note that the homeomorphism gf of Example 1, while not having $\mathbf{0}$ as a piercing point, nevertheless leaves the segment from $(0,1)$ to $\mathbf{0}$ pointwise invariant. Hence $\mathbf{0}$ could be called a "one-sided" piercing point of gf. We make this concept precise in the following definition.

DEFINITION 2. Let $f: U \to E^n$ be a homeomorphism of U into E^n, where U is an open subset of E^n. A point $x \in U$ is called a *one-sided* piercing point of f if there exists C^p-imbedding $(p > 0)$ $\sigma: [0,1] \to U$, a diffeomorphism $H \in H(E^n)$ and an $(n-1)$-hyperplane P in E^n such that

(i) $\sigma(1) = x$,

(ii) $Hf\sigma([0,1]) \cap P = Hf\sigma(1)$.

A point $x \in U$ which is *not* a one-sided piercing point will be called a *spiral point* of f.

We see, then, that a spiral point $x \in U$ of f has the following characteristic property: given any C^p-imbedding $(p > 0)$ $\sigma: [0,1] \to U$, any diffeomorphism $H \in H(E^n)$, and any $(n-1)$-hyperplane P in E^n, there exists a sequence of points $t_i \in [0,1]$ converging to 1 and such that $Hf\sigma(t_i) \in P$. Clearly, a spiral point of f is a nonpiercing point of f, but the converse does not hold as the homeomorphism gf of Example 1 shows.

PROPOSITION 2. *Let $f: U \to E^n$ be a homeomorphism of U into E , where U is* *an open subset of* E^n. *Then the set of one-sided piercing points of* f *is dense in* U. *In fact, if* $\sigma: [0,1] \to U$ *is any* C^p*-imbedding,* $(p > 0)$, *then there exists a* $t \in (0,1)$ *such that* $\sigma(t)$ *is a one-sided piercing point of* f.

Proof. Let $\sigma: [0,1] \to U$ be a C^p-imbedding, and let P be any $(n-1)$-hyperplane in E^n such that $f\sigma(0)$ and $f\sigma(1)$ lie in opposite components of $E^n - P$. Since $f\sigma([0,1]) \cap P \neq \emptyset$, there exists a (unique) $t \in (0,1)$ such that $f\sigma[0,t] \cap P = \sigma(t)$. Hence $\sigma(t)$ is a one-sided piercing point of f.

Remark. The notion of "one-sided" piercing point extends, in the natural way (cf. Definition 1′), to differentiable manifolds. Proposition 2 holds with this notion so extended.

6. **An alternative definition.** Suppose we altered Definition 1 by requiring that H be the identity diffeomorphism of E^n onto itself. The question arises as to whether every piercing point in the old sense would also be one in this new more restrictive sense. Example 2 below answers this latter question in the negative.

Example 2. Let $S = S^1(\mathbf{0}, 1)$, and let $f: E^2 \to E^2$ be a homeomorphism of E^2 onto itself having the following properties. First $f(x) = x$ for $x \in C\dot{J}S \cup L$, where L is the radius segment of S joining $\mathbf{0} = (0,0)$ to $(0,1)$. Secondly, the image under f of every radius segment of S makes an angle of 0° with L at $\mathbf{0}$ (cf. Figure 3). Such homeomorphism clearly exists (in fact, f may be required to be a diffeomorphism on $E^2 - \mathbf{0}$). Note that $\mathbf{0}$ is *not* a piercing point of f if H is required to be the identity map of E^n.

We now construct a diffeomorphism $H \in H(E^n)$ which will show that $\mathbf{0}$ is a piercing point (in the sense of Definition 1) of f.

We can assume f is so constructed that there exists a C^∞-imbedding $\tau: (L \cup L'') \to E^2$ such that $\tau(L) = f(L')$ and $\tau \mid L'' = 1$, where L', L'' are the radius segments of S which go through $(\sqrt{2}/2, \sqrt{2}/2)$, $(0, -1)$, respectively. Let H be a C^∞-diffeomorphism of E^2 onto itself such that $H \mid (f(L') \cup L'') = \tau^{-1}$. Let $\sigma: [-1,1] \to E^2$ be the *linear* imbedding defined by $\sigma(-1) = (0, -\frac{1}{2})$, $\sigma(1) = (0, \frac{1}{2})$. Finally, let P be the x^2 axis. One verifies that σ, H, and P satisfy the conditions of Definition 1 with respect to f and $\mathbf{0}$.

Remark. Example 2 also shows that Proposition 1 would not be satisfied if our definition of piercing point would have been the more restrictive one. This example can be modified to yield the corresponding results for $n \geqq 2$.

7. **Property** Q. We now introduce a property which concerns the existence of piercing points of a uniform type.

Definition 3. Let $f: U \to E^n$ be a homeomorphism of U into E^n, where U is an open set in E^n. We say that *f has property Q at a point* $x_0 \in U$ if there exists a diffeomorphism L of E^n onto itself such that the following three conditions are satisfied:

(i) $L^{-1}(x_0)$ is a piercing point of fL, i.e. there exists σ, H, and P satisfying (i), (ii), and (iii) of Definition 1 relative to fL and $L^{-1}(x_0)$,

(ii) σ is linear, i.e. $\sigma([-1,1])$ is a straight line segment,

(iii) there exists an open set W in P such that for all $x \in L^{-1}f^{-1}H^{-1}(W)$, H, P, and σ_x defined by $\sigma_x(t) = \sigma(t) + x - \sigma(0)$ satisfy conditions (i), (ii), and (iii) of Definition 1 with respect to fL and x.

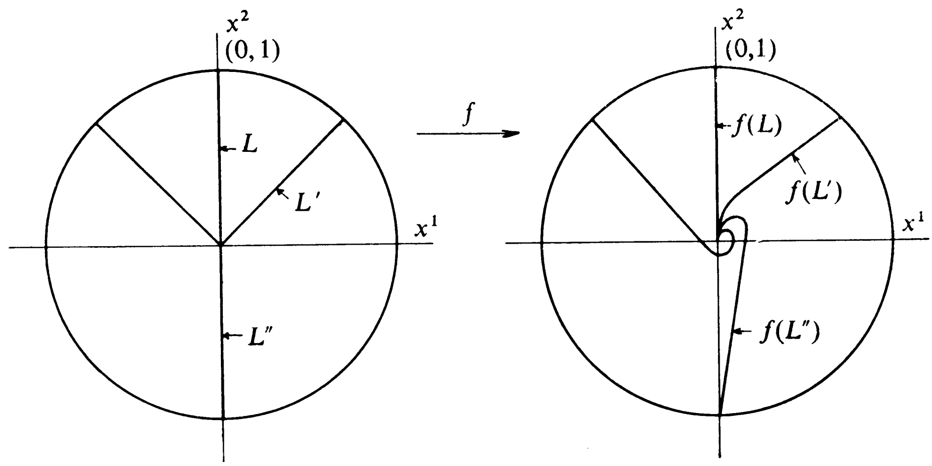

FIGURE 3

REMARKS. Again, one verifies that if $f: U \to E^n$ is a C^p-imbedding, $(p > 0)$, then f has property Q at each point of its domain. Also, it is readily seen that the definition of "property Q" can be extended in the natural way (cf. Definition 1′) to differentiable manifolds. One verifies that if M_1^n, M_2^n are nonbounded differentiable n-manifolds, and if $f: U \to M_2^n$ is a C^p-imbedding of U into M_2^n, where U is an open subset of M_1^n, then f has property Q at each point of U.

8. **A homeomorphism having a dense set of nonpiercing points.** We now construct a homeomorphism F of E^n onto itself having a dense set of nonpiercing points. For $c = (c^1, c^2, \cdots, c^n)$, $x = (x^1, x^2, \cdots, x^n)$, and $r > 0$, we define the homeomorphism $F_{c,r,i}$, $i = 2, \cdots, n$, of E^n onto itself, as follows:

$$F_{c,r,i}(x) = x, \qquad [x \in C\mathring{J}S(c,r) \cup JS(c,r/2)]$$

$$F_{c,r,i}(x) = ((x^1 - c^1)\cos\alpha(x) - (x^i - c^i)\sin\alpha(x) + c^1, x^2, \cdots, x^{i-1},$$

$$(x^1 - c^1)\sin\alpha(x) + (x^i - c^i)\cos\alpha(x) + c^i, x^{i+1}, \cdots, x^n)$$

$$[x \in JS(c,r) - \mathring{J}S(c,r/2)] \text{ where } \alpha(x) = 4\pi\left(\frac{r - \|x - c\|}{r}\right).$$

In the above formula, and throughout the following, CX denotes the complement in E^n of X. We then define the homeomorphism $F_{c,r}$ of E^n onto itself by setting $F_{c,r} = F_{c,r,2}F_{c,r,3} \cdots F_{c,r,n}$. We then choose a constant $\delta > 0$ with the following property (β).

(β) *If* $S(x_1,r_1),\cdots,S(x_m,r_m)$ *are* $(n-1)$*-spheres in* E^n *such that* $JS(x_i,r_i) \cap JS(x_j,r_j)=\varnothing$, $i\neq j$, $1\leqq i$, $j\leqq n$, *and if* $H=F_{x_n,r_n}F_{x_{n-1},r_{n-1}}\cdots F_{x_1,r_1}$ *(note that H is independent of the order of the factors), then* $d(H(x), H(y))\geqq \delta d(x,y)$ *for all* $x,y\in E^n$.

Such a δ clearly exists. We now define, inductively, a sequence of homeomorphisms $\{F_i\}$ of E^n into itself, and set $F=\lim_{i\to\infty}F_i$.

Let X be a countable dense subset in E^n of distinct points x_i, $i=1,2,\cdots$. Set $F_0=1$. Select a positive constant $r_{11}<\frac{1}{2}$ and such that $S(x_1,r_{11})\cap X=\varnothing$. Set $F_1=F_{x_1,r_{11}}=F_{11}$. Now consider $F_1(x_2)$. By our choice of $S(x_1,r_{11})$, and since $F_1 \mid S(x_1,r_{11})=1$, we have $F_1(x_2)\notin S(x_1,r_{11})$. We have two cases.

Case 1. $F_1(x_2)\in JS(x_1,r_{11})$. Then select positive constants r_{12},r_{22} such that:

$$r_{12}<\tfrac{1}{2}\,r_{11}, \tag{2.1}$$

$$\max(r_{12},r_{22})<\delta^2/2^2, \tag{2.2}$$

$$JS(F_1(x_2),r_{22})\subset \dot{J}S(x_1,r_{11}), \tag{2.3}$$

$$JS(F_1(x_1),r_{12})\cap JS(F_1(x_2),r_{22})=\varnothing, \tag{2.4}$$

$$\begin{aligned}\{S(F_1(x_1),r_{12})\cup S(F_1(x_2),r_{22})\cup S(F_1(x_1),r_{12}/2)\cup S(F_1(x_2),r_{22}/2)\}&\\ \cap\{X\cup F_1(X)\}=\varnothing.&\end{aligned} \tag{2.5}$$

Then set $F_{12}=F_{F_1(x_1),r_{12}}$, $F_{22}=F_{F_1(x_2)r_{22}}$, and

$$F_2=F_{22}F_{12}F_{11}=F_{22}F_{12}F_1. \tag{2.6}$$

Case 2. $F_1(x_2)\in CJS(x_1,r_{11})$. Then select positive constants r_{12}, r_{22} such that (2.1), (2.2), (2.4) and (2.5) hold, together with the following relation analogous to (2.3).

$$JS(F_1(x_2),r_{22})\subset CJS(x_1,r_{11}). \tag{2.3$'$}$$

Then construct F_2 as in (2.6). Note that the following relations are satisfied.

$$F_2(x_1)=F_1(x_1)=x_1,\qquad F_2(x_2)=F_1(x_2). \tag{2.7}$$

$$F_1(y)\in \dot{J}S(F_1(x_i),r_{ij})\Rightarrow F_2(y)\in \dot{J}S(F_1(x_i),r_{ij}),\qquad i=1,2;\ i\leqq j\leqq 2. \tag{2.8}$$

$$F_1(y)\in CJS(F_1(x_i),r_{ij})\Rightarrow F_2(y)\in CJS(F_1(x_i),r_{ij}),\qquad i=1,2;\ i\leqq j\leqq 2. \tag{2.9}$$

$$\begin{aligned}d(F_2(x),F_2(y))=d(F_{22}F_{12}F_1(x),F_{22}F_{12}F_1(y))&\geqq \delta d(F_1(x),F_1(y))\\ &\geqq \delta^2 d(x,y)\qquad [x,y\in E^n].\end{aligned} \tag{2.10}$$

Suppose, inductively, that positive constants r_{ij}, $i=1,2,\cdots,k-1$, $i\leqq j\leqq k-1$ have been chosen, together with homeomorphisms $F_0=1$, $F_1,\cdots,F_{k-1}$ of E^n onto itself, such that the following conditions are satisfied. First, for $1\leqq m\leqq k-1$, $F_m=F_{mm}F_{m-1m}\cdots F_{1m}F_{m-1}$, where $F_{ij}=F_{F_{j-1}(x_i),r_{ij}}$, $i=1,2,\cdots,k-1$, $i\leqq j\leqq k-1$, and, moreover:

$$(k\text{–}1.1)\qquad r_{ij} < \tfrac{1}{2}r_{ij-1} < \cdots < \tfrac{1}{2}r_{ii},$$

$$(k\text{–}1.2)\qquad \max(r_{ij}) < \delta^i/2^j,$$

$$(k\text{–}1.3)\quad JS(F_{j-1}(x_l), r_{lj}) \cap JS(F_{j-1}(x_m), r_{mj}) = \varnothing \quad [l \neq m, 1 \leqq j \leqq k-1, l,m \leqq j],$$

$$(k\text{–}1.4)\quad \Big\{ \bigcup_{i=1,\dots,k-1;\, i\leqq j\leqq k-1} S(F_{j-1}(x_i), r_{ij}) \cup S(F_{j-1}(x_i), r_{ij}/2) \Big\} \cap \{X \cup F_1(X) \cup \cdots \cup F_{k-1}(X)\} = \varnothing,$$

$$(k\text{–}1.5)\quad F_{j-1}(x_j) \in \dot{J}S(F_{j-1}(x_l), r_{lp}) \Rightarrow JS(F_{j-1}(x_j), r_{jj}) \subset \dot{J}S(F_{j-1}(x_l), r_{lp}) \quad [j = 1, \cdots, k-1;\, l \leqq p \leqq j-1],$$

$$(k\text{–}1.6)\quad F_{j-1}(x_j) \in CJS(F_{j-1}(x_l) r_{lp}) \Rightarrow JS(F_{j-1}(x_j), r_{jj}) \subset CJS(F_{j-1}(x_l), r_{lp}) \quad [j = 1, \cdots, k-1;\, 1 \leqq l \leqq p \leqq j-1].$$

Note by (k–1.4) that (k–1.5) and (k–1.6) cover the possible locations of $F_{j-1}(x_j)$. Note also that the following relations are necessarily satisfied:

$$(k\text{–}1.7)\qquad F_j(x_l) = F_{l-1}(x_l) \qquad [j = 1, \cdots, k-1,\ 1 \leqq l \leqq j],$$

$$(k\text{–}1.8)\quad F_{j-1}(y) \in \dot{J}S(F_{j-1}(x_l), r_{lp}) \Rightarrow F_q(y) \in \dot{J}S(F_{j-1}(x_l), r_{lp}) \quad [j = 1, \cdots, k-1,\ 1 \leqq l \leqq p \leqq j-1 \leqq q \leqq k-1, y \in E^n],$$

$$(k\text{–}1.9)\quad F_{j-1}(y) \in CJS(F_{j-1}(x_l) r_{lp}) \Rightarrow F_q(y) \in CJS(F_{j-1}(x_l), r_{lp}) \quad [j = 1, \cdots, k-1,\ 1 \leqq l \leqq p \leqq j-1 \leqq q \leqq k-1,\ y \in E^n],$$

$$(k\text{–}10)\qquad d(F_j(x), F_j(y)) \geqq \delta^j d(x,y) \qquad [x, y \in E^n,\ \ j = 1, \cdots, k-1],$$

$$(k\text{–}11)\quad d(F_j(x), F_{j-1}(x)) < \delta^j/2^j < 1/2^j \qquad [j = 1, \cdots, k-1].$$

Clearly, we may choose positive constants r_{ik}, $i = 1, \cdots, k$, and define $F_k = F_{kk}F_{k-1k} \cdots F_{1k}F_{k-1}$ so that the relations (k.1)—(k.11) analogous to (k–1.1)—(k–1.11) hold. Hereafter, we understand ($m.i$) to mean the relation in stage m of our construction analogous to (k–1.1) above. Set $F = \lim_{k\to\infty} F_k$. Then using ($k$.11), F is a continuous mapping of E^n onto itself. Hence to show that F is a homeomorphism, it suffices to show that F is biunique. To verify the biuniqueness of F, let x, y be distinct points of E^n. We have the following two cases.

Case 1. *There exists a sequence* $k_1 < k_2 < \cdots$ *such that, for example,* $F_{k_i}(x) \neq F_{k_i-1}(x)$.

Then $F_{k_i-1}(x) \in \dot{J}S(F_{k_i-1}(x_{l_i}),\ r_{l_ik_i-1})$ for some $l_i \leqq k_{i-1}$. Using (k.5), (k.6) and (k.8), we see that $F(x) \in \dot{J}S(F_{k_i-1}(x_{l_i}),\ r_{l_ik_i-1})$, $i = 1,2,\cdots$. Set $\zeta = d(x,y)$, and choose $p = k_j$ when j is so large that $1/2^{p-1} < \zeta/3$. Now $F(x) \in JS(F_{p-1}(x_{l_j}), r_{l_jp-1})$, and by ($k$.10), we have

$$(\xi)\qquad d(F_{p-1}(x), F_{p-1}(y)) \geqq \delta^{p-1} d(x,y) = \delta^{p-1}\zeta.$$

Now by (k.2), (β), and our choice of p, we have

(η) $\qquad r_{l_jp-1} < \delta^{p-1}/2^{p-1} < \delta^{p-1}\zeta/3 < d(F_{p-1}(x), F_{p-1}(y))/2.$

Then $F_{p-1}(y) \in CJS(F_{p-1}(x_{l_j}), r_{l_jp-1})$, and using ($k$.9), $F(y) \notin \dot{J}S(F_{p-1}(x_{l_j}), r_{l_jp-1})$. Hence $F(x) \neq F(y)$. A similar proof holds when there exists a sequence $m_1 < m_2 < \cdots$ such that $F_{m_i}(y) \neq F_{m_{i-1}}(y)$.

Case 2. *There exist integers $M(x), N(y)$ such that $F_l(x) = F_{M(x)}(x)$, $l \geqq M(x)$, and $F_p(y) = F_{N(y)}(y)$, $p \geqq N(y)$.*

Then $F(y) = F_{N(y)}(y)$ and $F(x) = F_{M(x)}(x)$. Suppose $M(x) = N(y)$. Then since $F_{M(x)} = F_{N(y)}$ is a homeomorphism, we have $F(x) = F_{M(x)}(x) \neq F_{N(y)}(y) = F(y)$. Now if $M(x) < N(y)$, then $F(x) = F_{M(x)}(x) = F_{N(y)}(x)$, and since $F_{N(y)}$ is a homeomorphism, we have $F(x) = F_{N(y)}(x) \neq F_{N(y)}(y) = F(y)$. A similar proof holds when $N(y) < M(x)$.

This completes the proof of the biuniqueness of F, and therefore F is a homeomorphism of E^n onto itself.

CLAIM. *X consists of nonpiercing points of F.* We show first that x_1 is a nonpiercing point of F. Figure 4 illustrates the situation when $n = 2$. The situation for general n is analogous.

Clearly, x_1 is a nonpiercing point of F, in fact, x_1 is a spiral point. Now consider x_j, where $j > 1$. Construct a new sequence of homeomorphisms $\{G_{i,j}\}$, $i = 1, 2, \cdots$, where $G_{i,j}$ is obtained from F_i by deleting all factors in F_i which are of the form F_{kl}, where $k < j$ (with $G_{1,j} = G_{2,j} = \cdots = G_{j-1,j} = 1$). Setting $G_j = \lim_{i\to\infty} G_{i,j}$, one verifies, in a manner analogous to that concerning F and x_1, that G_j is a homeomorphism of E^n onto itself having $F_{j-1}(x_j)$ as a nonpiercing (spiral) point. Setting $V^j = F_{j-1}{}^{-1}(S(F_{j-1}(x_j), r_{jj}))$, we see that V^j is a neighborhood of x_j, and $F \mid V^j = G_j F_{j-1} \mid V$. Moreover, using ($j$.4) and the fact that $F_{c,r} \mid (E^n - \{S(c,r) \cup S(c, r/2)\})$ is a C^∞-diffeomorphism, there exists an open neighborhood U^j of x_j in V^j such that $F_{j-1} \mid U^j$ is a C^∞-diffeomorphism. It follows from Proposition 1 that x_j is a nonpiercing point of $G_j F_{j-1}$, and hence x_j is a nonpiercing point of F. This completes the proof of the claim.

REMARKS. Now $F \mid S(x_1, r_{11}) = 1$, and hence we see that F has property Q at every point of $S(x_1, r_{11})$. Also, F is stable, since it is equivalent under $\sim$ to the stable homeomorphism $\hat{F} \in H(E^n)$ defined by $\hat{F} \mid C\dot{J}S(x_1, r_{11}) = F \mid C\dot{J}S(x_1, r_{11})$, and $\hat{F} \mid JS(x_1, r_{11}) = 1$. Now if T is any triangulation of E^n, we see that F could have been constructed to have a dense set of nonpiercing points, and, moreover, reduce to the identity on the $(n-1)$-skeleton T^{n-1} of T. We merely take our countable dense set X, and all the spheres entering into our construction, to be disjoint from T^{n-1}. In particular, if P is an $(n-1)$-hyperplane in E^n, then F could have been required to reduce to the identity on P (and hence have property Q at each point of P).

THEOREM 1. *If M^n is any nonbounded differentiable n-manifold, then there exists a homeomorphism F of M^n onto itself having a dense set of nonpiercing points* (*cf. Definition* 1′).

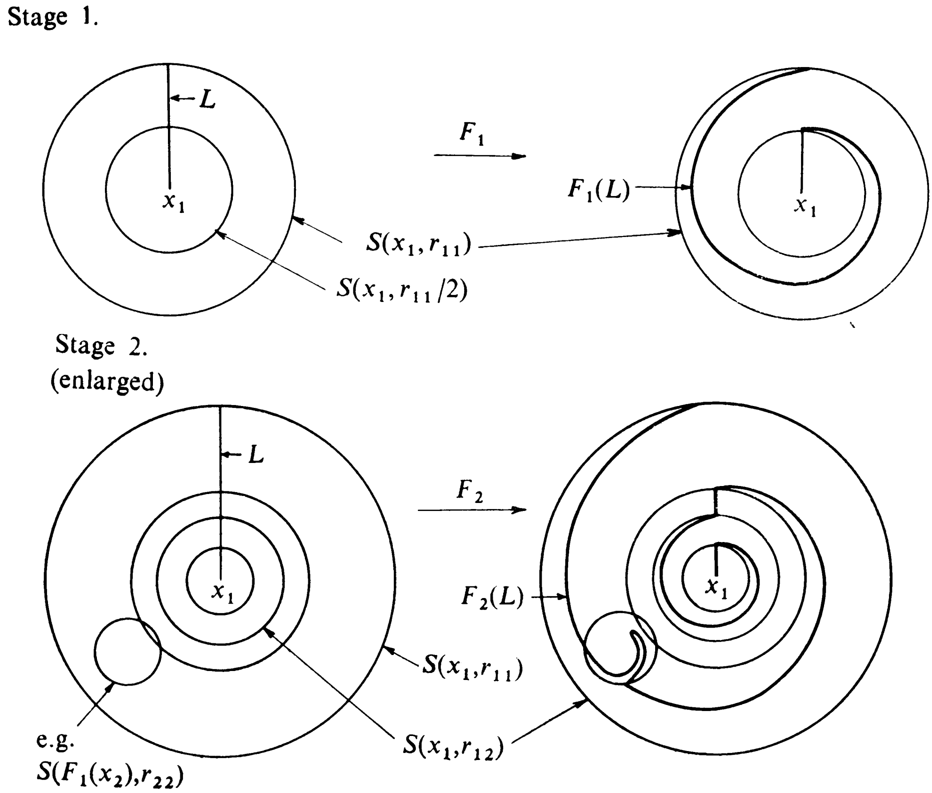

FIGURE 4

Proof. We take a countable covering (U_i, h_i) of M^n by coordinate systems. Let $f_1: h_1(U_1) \to h_1(U_1)$ be a homeomorphism of $h_1(U_1)$ onto itself having a dense set of nonpiercing points, and such that $d(x, f_1(x)) \to 0$ as x approaches the boundary of $h_1(U_1)$. Then let $F_1: M^n \to M^n$ be the homeomorphism of M^n onto itself defined by $F_1 | U_1 = h_1^{-1} f_1 h_1$, and $F_1 | (M^n - U_1) = 1$. Note that there is a dense subset of U_1 consisting of nonpiercing points of F_1. Let $f_2: h_2(U_2 - \bar{U}_1) \to h_2(U_2 - \bar{U}_1)$ be a homeomorphism of $h_2(U_2 - \bar{U}_1)$ onto itself having a dense set of nonpiercing points, and such that $d(x, f_2(x)) \to 0$ as x approaches the boundary of $h_2(U_2 - \bar{U}_1)$. Then let $F_2: M^n \to M^n$ be the homeomorphism of M^n onto itself defined by $F_2 | U_1 = F_1 | U_1$, $F_2 | (U_2 - \bar{U}_1) = h_2^{-1} f_2 h_2$, and $F_2 | (M^n - (U_1 \cup U_2)) = 1$. Inductively, we construct homeomorphisms $F_i: M^n \to M^n$ of M^n onto itself such that $F_i | (U_1 \cup \cdots \cup U_{i-1}) = F_{i-1} | (U_1 \cup \cdots \cup U_{i-1})$, $F_i | (M^n - (U_1 \cup \cdots \cup U_i)) = 1$, and there exists a dense subset of $U_1 \cup \cdots \cup U_i$ consisting of nonpiercing points of F_i. Then set $F = \lim_{i \to \infty} F_i$. It is readily seen that F is a homeomorphism of M^n onto itself having a dense set (in M^n) of nonpiercing points.

9. **Applications.** We now prove a theorem which relates the notion of piercing point to stability of homeomorphisms.

THEOREM 2. *For any integer n, if every homeomorphism* $H \in H(E^k)$, $2 \leqq k \leqq n$, *is such that* $H \sim G$, *where G has property Q at some point, then all orientation preserving homeomorphisms of* E^n *onto itself are stable.*

We first prove the following lemma.

LEMMA 1. *Suppose, for some n, that all orientation-preserving homeomorphisms of* E^{n-1} *onto itself are stable. If* $F \in H(E^n)$ *is orientation-preserving, and if there exist* $(n-1)$*-hyperplanes* P, P' *in* E^n, *and an open set U in* P', *such that* $f(U) \subset P$, *then F is stable.*

Proof of Lemma 1. Choose a point $x \in U$, and let $JS^{n-2}(x,r) = B$ be a closed $(n-1)$-ball in $P' \cap U$. Let M be an elementary topological $(n-1)$-sphere in E^n such that $M \cap P' = B$. Since we have assumed that all orientation-preserving homeomorphisms of E^{n-1} onto itself are stable, and since $F(B) \subset P$, we can modify $F|B$ (cf. Theorem 5.4 of [1]) to obtain a homeomorphism $\tilde{F}: B \to E^n$ such that $\tilde{F}|S^{n-2}(x,r) = F|S^{n-2}(x,r)$, $\tilde{F}(B) \subset P$, and $\tilde{F}|\dot{J}S^{n-2}(x,s)$ is a C^p-imbedding $(p > 0)$ for some $s < r$. Note that $\tilde{F}(B) = F(B)$ (cf. (α) of §2). We then obtain a homeomorphism $\hat{F}: M \to E^n$ by setting

$$\hat{F}(x) = F(x), \qquad [x \in M - \dot{J}S^{n-2}(x,r)],$$
$$\hat{F}(x) = \tilde{F}(x), \qquad [x \in JS^{n-2}(x,r)].$$

Since $\hat{F}(M) = F(M)$, $\hat{F}$ is elementary. Hence, using the Schoenflies extension theorem, $\hat{F}$ admits extension to a homeomorphism of E^n onto itself, which we still denote by $\hat{F}$. Moreover, using the tubular neighborhood theorem, $\hat{F}|JS^{n-2}(x,s)$ may be extended over an open neighborhood of $\dot{J}S^{n-2}(x,s)$ *in* E^n as a diffeomorphism. Hence, we can assume (cf. [9]) that $\hat{F}|V$ is a diffeomorphism for some open neighborhood V of x in E^n. Therefore $\hat{F}$ is stable. Since

$$\hat{F}|(M-B) = F|(M-B),$$

we can assume (cf. [9]), moreover, that $\hat{F} \sim F$. Hence F is stable.

REMARK. One verifies that the hypotheses of Lemma 1 can be weakened by allowing P and P' to be diffeomorphs of E^{n-1} in E^n.

Proof of Theorem 2. Using induction, the above remark, and the fact that all orientation-preserving homeomorphisms of E^n onto itself are stable for $n = 1,2,3$, it suffices to prove the following proposition (γ).

(γ) *If the homeomorphism* $G \in H(E^n)$ *has property Q at some point, then* $G \sim F$, *where* $F(U) \subset P_1$ *for an open set U in* P_2, *and* P_1, P_2 *are diffeomorphs of* E^{n-1} *in* E^n.

To verify (γ), suppose G has property Q at a point $x_0 \in E^n$. Hence there exist σ, H, P, W, and σ_x in Definition 3 relative to G and x_0. To simplify our discussion, we note that for the purposes of verifying (γ), we can assume without loss of generality that L is the identity diffeomorphism of E . Indeed, GL has property Q

at $L^{-1}(x_0)$ using σ, identity, H, P, W, and it is clear that L can be taken as orientation-preserving, which implies that $GL \sim G$.

Now if V is a sufficiently small neighborhood of $HG(x_0)$ *in* W, the straight line segments $\sigma_x([-1,1])$, as x varies throughout V, are mutually disjoint. To see this, note first that Definition 1 and condition (iii) of Definition 3 (with $L = 1$) imply that $\sigma_x([-1,1]) \cap G^{-1}H^{-1}(W) = x$ for all $x \in W$. Since the segments $\sigma_x([-1,1])$ are all translates of one another, we see that if $V \subset W$ is small enough, the $\sigma_x([-1,1]) \cap \sigma_y([-1,1]) \neq \varnothing$ for x, $y \in V$ implies that $x \in \sigma_y([-1,1])$, and hence $x = y$. Actually, it can be proved that the segments $\sigma_x([-1,1])$ are mutually disjoint for all $x \in W$, but we won't need this fact.

Let $B = JS^{n-2}(HG(x_0), r)$ be an $(n-1)$-ball in P such that $B \subset V$. Let P^* be be the $(n-1)$-hyperplane in E^n going through $x_0 = \sigma(0)$ and such that the segment $\sigma([-1,1])$ is normal to P^* at $\sigma(0)$. We also suppose r chosen so small that the $(n-1)$-hyperplanes P', P'' which are parallel to P^* and go through $\sigma(\frac{1}{2}), \sigma(-\frac{1}{2})$, respectively, have the following property: for each $x \in G^{-1}H^{-1}(B)$, the segments $\sigma_x([-1,1])$ intersect P', P'' in (continuously varying) points $\sigma_x(t'_x), \sigma_x(t''_x)$, respectively, where $-1 < t''_x < 0 < t'_x < 1$. Note that $t''_{x_0} = -\frac{1}{2}, t'_{x_0} = \frac{1}{2}$. Hence the segments $\sigma_x([t''_x, t'_x])$, as x varies throughout $G^{-1}H^{-1}(B)$, "fiber" the neighborhood

$$N = \{\bigcup \sigma_x([t''_x, t'_x]) \mid x \in G^{-1}H^{-1}(B)\}.$$

Let $B' = JS^{n-1}(HG(x_0), r/2)$, and let $B^* = \{\bigcup \sigma_x([t''_x, t'_x]) \mid x \in G^{-1}H^{-1}(B')\}$. It is clear that

$$\mathrm{Bd}B^* = \{\bigcup \sigma_x([t''_x, t'_x]) \mid x \in G^{-1}H^{-1}(S^{n-2}(HG(x_0), r/2))\} \cup \{B^* \cap (P' \cup P'')\}$$

is an elementary topological $(n-1)$-sphere in E^n. Setting $M = \mathrm{Bd}B^*$, we consider homeomorphism $\Phi\colon M \to E^n$ which maps the segment $\sigma_x([t''_x, t'_x])$ homeomorphically onto the segment $\sigma_x([t''_x, 0])$, and reduces to the identity $B^* \cap P''$. Then Φ is an elementary homeomorphism such that $\Phi(M) = M'$, where

$$M' = \{\bigcup \sigma_x([t''_x, 0]) \mid x \in G^{-1}H^{-1}(S^{n-2}(HG(x_0), r/2))\} \cup \{B^* \cap P''\}.$$

Set $F = G\Phi$. Then F is an elementary homeomorphism of M into E^n, and hence may be extended to a homeomorphism of E^n onto itself, which we still denote by F. Since $F \mid (B^* \cap P'') = G \mid (B^* \cap P'')$, we can assume (cf. [9]) that $F \sim G$. Moreover, setting $U = \{\bigcup \sigma_x(t_x) \mid x \in G^{-1}H^{-1}(\mathring{J}S^{n-2}(HG(x_0), r/2))\}$, we see that U is an open set *in* P', and $F(U) = G\Phi(U) = G(G^{-1}H^{-1}(\mathring{J}S^{n-2}(HG(x_0), r/2))) = H^{-1}(\mathring{J}S^{n-2}(HG(x_0), r/2)) \subset H^{-1}(P)$. Then setting $P_1 = H^{-1}(P)$, $P_2 = P'$, we see that Proposition (γ) is verified, which completes the proof of Theorem 2.

We now show that any homeomorphism $F \in H(E^n)$ is equivalent under $\sim$ to a homeomorphism having piercing points.

Since all orientation-preserving homeomorphisms of E^n onto itself are stable for $n = 1, 2, 3$, it follows that if $F \in H(E^n)$, $n = 1, 2, 3$, then $F \sim H$ where H has an n-cell of piercing points. The following theorem extends this latter result to a similar (but weaker) result in the higher dimensions.

THEOREM 3. *Let $F \in H(E^n)$, where $n \geqq 4$. Then $F \sim H$ where H has a k-cell of piercing points, $k \leqq 2n/3 - 1$.*

Proof. Let K be any k-cell in E^n. Then $F(K)$ is a flat k-cell in E^n. Moreover, $F(K)$ is stably flat by a theorem of P. Roy (cf. [10]), and $F^{-1} | F(K)$: $F(K) \rightarrow K$ admits an extension to a stable homeomorphism G of E^n onto itself. Since G is stable, we can assume that $G | U = 1$, where U is some nonempty open set in E^n. Set $H = GF$. Then $H | K = 1$, and hence K consists entirely of piercing points of H. Since $H | F^{-1}(U) = F | F^{-1}(U)$, we see that $F \sim H$ and the theorem is proved.

REMARK. Theorem 3 can be strengthened by requiring that $H | (E^n - V) = F | (E^n - V)$, where V is any nonempty open set in E^n. Hence Theorem 3 can be extended to an analogous result about differentiable manifolds.

We conclude by noting a relationship between our notion of piercing point and a notion of "piercing" which has been discussed in the literature. A topological $(n-1)$-sphere M in E^n is said to be pierced by a straight line segment yz at a point $x_0 \in M$ if $yz \cap M = x_0$, and if y and z lie in opposite components of $E^n - M$. An example was given by Fort (cf. [11]) of a wild sphere which can be pierced at each point by a straight line segment. On the other hand, our example of §8 shows that there exist *elementary* spheres M in E^n which can *not* be pierced, even by diffeomorphs of straight line segments, at a dense set of points of M. To verify this latter statement, observe first that if $G \in H(E^n)$, and if $M = G(S^{n-1}(c,r))$ can be pierced at a point x_0 by a diffeomorph of a straight line segment, then x_0 is a piercing point of G^{-1}. We now note that the homeomorphism $F \in H(E\ )$ constructed in §8 is such that $F(x)$ is a nonpiercing (spiral) point of F^{-1}, for all $x \in X$. Hence if we choose X so that $X \cap S(c, r)$ is dense in $S(c, r)$, then the elementary sphere $M = F(S(c, r))$ can *not* be pierced at the points of the dense subset $F(X \cap S(c, r))$ of M.

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