AN INTEGRAL REPRESENTATION OF A NORMAL FUNCTIONAL ON A VON NEUMANN ALGEBRA(1)

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1. **Introduction.** Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ on a Hilbert space H; a positive function g on $\mathscr{A}$ is included in a positive functional f on $\mathscr{A}$ (notation: $g<f$) if there is a scalar $\alpha>0$ such that $f-\alpha g$ is a positive functional on $\mathscr{A}$. A nonzero positive functional on $\mathscr{A}$ which includes only scalar multiples of itself is said to be irreducible. In this paper we shall call a positive functional f on $\mathscr{A}$ *generalized irreducible* if whenever g is a positive functional such that $g<f$ there is an element A_0 in $\mathscr{A}^+$, the cone of positive elements of $\mathscr{A}$, such that $f(A_0A)=g(A)$, for all A in $\mathscr{A}$. A positive functional f on $\mathscr{A}$ is said to be normal if for each monotonely increasing net $\{A_d \mid d\in D\}$ with least upper bound A, we have $f(A)=\text{lub}\,\{f(A_d)\mid d\in D\}$. In the present paper we obtain a representation theorem for a normal generalized irreducible functional f on a von Neumann algebra $\mathscr{A}$ of Type I. The representation has the form (A) $f(A)=\int_Z f_\zeta(A)\,d\nu(\zeta)$ where

(1) Z is the spectrum of the center $\mathscr{Z}$;

(2) $\nu=\nu_h$ ($h\in H$) is the so-called spectral measure on Z given by $f(A)=w_h(A)=\int_Z \hat{A}(\zeta)\,d\nu(\zeta)$ where $A\in\mathscr{Z}$ and the function $\hat{A}$ is the image of A in the algebra of all continuous complex-valued functions on Z under the Gelfand representation; and

(3) the f_ζ for ζ in the support Y of ν are positive functionals such that for each fixed A in $\mathscr{A}$ the map $\zeta\to f_\zeta(A)$ is continuous. We shall show that the functionals f_ζ have special properties. We also prove that normal functional f obtained from a representation of the form (A) in which the functionals f_ζ have this special property is generalized irreducible.

These theorems may be viewed as a generalization of certain results of Tomita [6]. Tomita studied positive functionals f on a C^*-algebra $\mathscr{A}$ with identity and with center $\mathscr{Z}$ with the property: if $g<f$, there is an $A_0\in\mathscr{Z}^+$ such that $g(A)=f(AA_0)$ for all $A\in\mathscr{A}$. These functionals are called centrally irreducible functionals. Some of the lemmas necessary for this generalization give results on centrally reducible functionals with little additional effort. We therefore briefly indicate in what direction this effort should be applied.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ on a Hilbert space H; let Z

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be the spectrum of $\mathscr{Z}$. For each $\zeta \in Z$ the two-sided closed ideal in $\mathscr{A}$ generated by ζ will be denoted by $[\zeta]$. The canonical homomorphism of $\mathscr{A}$ onto the C^*-algebra $\mathscr{A}/[\zeta]$ will be written as Ψ_ζ while $\Psi_\zeta(A)$ will be written as $A(\zeta)$ for each $A \in \mathscr{A}$. Glimm has proved that for each fixed A in $\mathscr{A}$, the function $\zeta \to \|A(\zeta)\|$ is continuous on Z.

2. **Centrally irreducible functionals.** A projection E in $\mathscr{A}$ is abelian if the von Neumann algebra $E\mathscr{A}E$ is commutative. If P is the central support of E (i.e., the smallest projection in $\mathscr{Z}$ such that $P \geqq E$), the map $\Phi(A) = AE$ of $\mathscr{Z}P$ onto $E\mathscr{A}E$ is an isomorphism. We define the linear function $\tau_E(A)$ on $\mathscr{A}$ as $\tau_E(A) = \Phi^{-1}(EAE)$.

Proposition 2.1. *For each ζ in the set $\{\zeta \in Z \mid \hat{P}(\zeta) = 1\}$, the functional $f(A) = \tau_E(A)^\wedge(\zeta)$ on $\mathscr{A}$ is irreducible.*

Proof. The verification that f is a nonzero positive functional is straightforward. Let g be a positive functional on $\mathscr{A}$ included in f. Since $f([\zeta]) = 0$, $g([\zeta]) = 0$. There is a functional g_1 on $\mathscr{A}(\zeta)$ such that $g_1(A(\zeta)) = g(A)$, for all A in $\mathscr{A}$. Furthermore, $g(I-E) = 0$; this means that $g(A) = g(EAE)$, for all A in $\mathscr{A}$. Therefore, we have

$$\begin{aligned} g(A) = g(EAE) &= g(\tau_E(A)E) \\ &= g_1(\tau_E(A)^\wedge(\zeta)E(\zeta)) \\ &= f(A)g_1(E(\zeta)) = f(A)g(E), \end{aligned}$$

for each A in $\mathscr{A}$. This shows that g is a scalar multiple of f and hence that f is irreducible on $\mathscr{A}$. Q.E.D.

If f is a positive functional on a C^*-algebra with identity, let $L(f)$ be the closed left-ideal $L(f) = \{A \in \mathscr{A} \mid f(A^*A) = 0\}$. The Hilbert space which is the completion of the left $\mathscr{A}$-module $\mathscr{A} - L(f)$ under the inner product $(A - L(f), B - L(f)) = f(B^*A)$ is denoted by $H(f)$. The operator $\Phi(A)$ ($A \in \mathscr{A}$) on $H(f)$ which is the extension of $\Phi(A)(B - L(f)) = AB - L(f)$ to $H(f)$ defines a continuous linear operator on $H(f)$. The homomorphism $A \to \Phi(A)$ of $\mathscr{A}$ into the $*$-algebra of bounded operators on $H(f)$ is called the canonical representation of $\mathscr{A}$ on $H(f)$ induced by f.

Lemma 2.2. *If f is a centrally reducible functional on a C^*-algebra $\mathscr{A}$ with center $\mathscr{Z}$, then the commutator $\Phi(\mathscr{A})'$ of $\Phi(\mathscr{A})$ on $H(f)$ is $\Phi(\mathscr{Z})$. Here Φ denotes the canonical representation on $H(f)$ induced by f.*

Proof. We must prove that $\Phi(\mathscr{A})' \subset \Phi(\mathscr{Z})$. Let $B \in (\Phi(\mathscr{A})')^+$; let h be a cyclic vector in $H(f)$ under $\Phi(\mathscr{A})$ such that $(\Phi(A)h, h) = f(A)$, for all A in $\mathscr{A}$. Then the functional $g(A) = (\Phi(A) \cdot Bh, h)$ on $\mathscr{A}$ is majorized by f. Consequently, there is a

$C \in \mathscr{Z}^+$ such that $(\Phi(A)Bh, h) = (\Phi(A)\Phi(C)h, h)$, for all A in $\mathscr{A}$. Thus $\Phi(C) = B$ and $(\Phi(\mathscr{A})')^+ \subset \Phi(\mathscr{Z})$. So $\Phi(\mathscr{A})' \subset \Phi(\mathscr{Z})$.

THEOREM 2.3. *Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ on a Hilbert space H. A normal functional f on $\mathscr{A}$ is centrally irreducible if and only if $f = w_h$ where h is a vector in H such that the cyclic projection $E(A', h)$ corresponding to the closed subspace closure $\{A'h \mid A' \in \mathscr{A}'\}$ is abelian.*

Proof. Let f be a centrally irreducible functional on $\mathscr{A}$ and let E be the support of f. Let Φ be the canonical representation of f on $H(f)$ induced by f and let $\mathscr{A}P$ be the kernel of Φ where P is a central projection. Then there is a $k \in H(f)$ such that $f(A) = (\Phi(A)k, k)$, for all $A \in \mathscr{A}$. We have that $\Phi(E)$ is the support of w_k on $\Phi(\mathscr{A})$. Since $\Phi(E)\Phi(\mathscr{A})'\Phi(E) = \Phi(E)\Phi(\mathscr{Z})\Phi(E)$, we have that $\Phi(E)\Phi(\mathscr{A})\Phi(E) = \Phi(E)\Phi(\mathscr{Z})\Phi(E)$. Thus, $\Phi(E)$ is an abelian projection in $\Phi(\mathscr{A})$. This means that E is an abelian projection in $\mathscr{A}$ because $E \leqq I - P$. We have that $f \mid E\mathscr{A}E$ is given by $f \mid E\mathscr{A}E = w_h \mid E\mathscr{A}E$ for some $h \in E(H)$. Thus, $f = w_h$ and $E = E(A', h)$ is an abelian projection.

Conversely let $f = w_h$ where $E = E(\mathscr{A}', h)$ is abelian. Let g be a positive functional on $\mathscr{A}$ majorized by f. For each A in $\mathscr{A}$ we have $g(EAE) = g(A)$. The functional $g \mid E\mathscr{A}E$ is majorized by $f \mid E\mathscr{A}E$. There is a B in $(E\mathscr{A}E)^+ = (\mathscr{Z}E)^+$ such that $g(EAE) = f(B \cdot EAE)$ for all A in $\mathscr{A}$. If $B = CE$ where C is a member of $\mathscr{Z}^+$, then $g(A) = f(CA)$ for all A in $\mathscr{A}$. Q.E.D.

Let f be a centrally irreducible functional on a C^*-algebra $\mathscr{A}$ with identity I and center $\mathscr{Z}$; then the commutator $\Phi(\mathscr{A})'$ of the image $\Phi(\mathscr{A})$ of $\mathscr{A}$ under the canonical representation Φ of $\mathscr{A}$ on $H(f)$ induced by f is equal to $\Phi(\mathscr{Z})$. The functional w_h $(h = I - L(f))$ on the von Neumann algebra $\Phi(\mathscr{A})''$ on $H(f)$ generated by $\Phi(\mathscr{A})$ is representable as

$$w_h(B) = \int_{Z_1} \tau_E(B)^\wedge(\zeta)\, d\nu_1(\zeta) \qquad (B \in \Phi(\mathscr{A})'').$$

The measure ν_1 is the spectral measure ν_h on the spectrum Z_1 of $\Phi(\mathscr{Z})$, and the projection E is the cyclic projection $E = E(\Phi(\mathscr{A})', h)$. The functionals $f_\zeta(B) = \tau_E(B)^\wedge(\zeta)$ $(\zeta \in Z_1)$ are irreducible on $\Phi(\mathscr{A})''$ and therefore the f_ζ $(\zeta \in Z_1)$ are elements of the pure state space of $\Phi(\mathscr{A})$ [2]. The representation Φ of $\mathscr{Z}$ onto $\Phi(\mathscr{Z})$ induces a homeomorphism of Z_1 into Z; the representation for

$$w_h(\Phi(A)) = f(A) \qquad (A \in \mathscr{A})$$

can be carried over to an integral representation on the spectrum Z of $\mathscr{Z}$ of the form $\int_Z g_\zeta(A)\, d\nu(\zeta)$. Each of the functionals g_ζ ($\zeta \in$ support ν) is in the pure state space of $\mathscr{A}$; and for each fixed A in $\mathscr{A}$ $\zeta \to g_\zeta(A)$ is continuous on the support of ν. These steps will be discussed in more detail in the ensuing theorems.

3. **Generalized irreducible functionals.** Let $\mathscr{A}$ be an algebra of continuous linear operators on a Hilbert space H. The commutator of $\mathscr{A}$ is the set $\mathscr{A}'$ of all continuous linear operators A' on H such that $AA'=A'A$ for all A in $\mathscr{A}$. If h is a vector in H, the cyclic projection $E(\mathscr{A}, h)$ generated by $\mathscr{A}$ and h is that projection which corresponds to the subspace of H given by closure $\{Ah|A \text{ in } \mathscr{A}\}$. A nonzero vector h in H is called a trace element for the algebra $\mathscr{A}$ if $(ABh, h)=(BAh, h)$ for all A, B in $\mathscr{A}$.

We first clarify the notion of a generalized irreducible functional in the next proposition.

Proposition 3.1. *Let $\mathscr{A}$ be a von Neumann algebra on a Hilbert space H. The vector functional w_h is generalized irreducible on $\mathscr{A}$ if and only if h is a trace element for $E\mathscr{A}E$ where $E=E(\mathscr{A}', h)$.*

Proof. The statement that w_h is generalized irreducible if h is a trace element on $E\mathscr{A}E$ is known [1, p. 96, Proposition 5]. We proceed with the proof of the converse. Let w_h be generalized irreducible on $\mathscr{A}$ and let E' be the projection in $\mathscr{A}'$ given by $E'=E(\mathscr{A}, h)$. If A' is an element in $(\mathscr{A}')^+$, we have that $g(A)=w_h(A'A)$ is a positive functional on $\mathscr{A}$ included in w_h. Thus, there is an element A_0 in $\mathscr{A}^+$ such that $g(A)=w_h(AA_0)$ for all A in $\mathscr{A}$. This means

$$((A_0-A')h, Ah) = 0,$$

for all A in $\mathscr{A}$. Since $\{Ah \mid A \in \mathscr{A}\}$ is dense in $E'(H)$, we have

$$E'(A_0-A')E'h = 0.$$

Therefore, for any A' in $\mathscr{A}'$, there is an element A_0 in $\mathscr{A}$ such that

$$E'A'E'h = E'A_0E'h$$

and

$$E'A'^*E'h = E'A_0^*E'h.$$

These relations show that h is a trace element for $E'\mathscr{A}'E'$. Indeed, for A', B', in $\mathscr{A}'$ we have

$$(B'A'h, h) = (B'A_0h, h) = (B'h, A_0^*h) = (B'h, A'^*h) = (A'B'h, h)$$

where A_0 is an element of $\mathscr{A}$ given by $E'A_0E'h=E'A'E'h$ and $E'A_0^*E'h=E'A'^*E'h$.

By the first part of the theorem w_h is generalized irreducible on $\mathscr{A}'$; a second application of the preceding argument shows w_h is generalized irreducible on $\mathscr{A}$.

From the preceding proposition the next decomposition follows immediately.

THEOREM 3.2. *Let $\mathscr{A}$ be a von Neumann algebra of Type* I *with center $\mathscr{Z}$ on a Hilbert space H. If f is a generalized irreducible normal functional on $\mathscr{A}$, we may write* (A) $f(A)=\int f_\zeta(A)\,d\nu(\zeta)$, *for all A in $\mathscr{A}$. Here*

(1) *ν is the spectral measure ν_h on the spectrum Z of $\mathscr{Z}$; the vector h arises from restricting f to $\mathscr{Z}$;*

(2) *f_ζ is a state for each ζ in the support Y of ν and $\zeta \to f_\zeta(A)$ is continuous on Y for each fixed A in $\mathscr{A}$;*

(3) *for each ζ in Y, $f_\zeta(A)=\hat{A}(\zeta)$ for all A in $\mathscr{Z}$;*

(4) *except on a nowhere dense set N in Y, f_ζ is a finite sum of irreducible functionals on $\mathscr{A}$; and*

(5) *except on N, f_ζ is generalized irreducible on $\mathscr{A}$.*

Proof. We prove the parts (1), (2), (3) without using the hypothesis that $\mathscr{A}$ is of Type I.

Let Φ be the canonical representation of $\mathscr{A}$ on $H(f)$ induced by f. The map Φ is normal and therefore $\Phi(\mathscr{A})$ is a von Neumann algebra on $H(f)$. Let k be the cyclic vector in $H(f)$ under $\Phi(\mathscr{A})$ such that $(\Phi(A)k, k)=f(A)$ for all A in $\mathscr{A}$. If g is a positive functional included in w_k, it is easy to see that there is a $B \in \Phi(\mathscr{A})^+$ such that $g(A)=w_k(BA)$ for all A in $\Phi(\mathscr{A})$. Thus, w_k, is generalized irreducible on the algebra $\Phi(\mathscr{A})$. Let F be the projection in $\Phi(\mathscr{A})$ given by $F=E(\Phi(\mathscr{A})', k)$. There is a projection Q in $\mathscr{Z}$ such that the kernel of Φ is $A(I-Q)$. The unique projection E in $\mathscr{A}Q$ such that $\Phi(E)=F$ is the support of f.

We claim that f is a faithful finite normal trace on $E\mathscr{A}E$. That f is faithful and finite needs no verification. To prove f is a trace on $E\mathscr{A}E$, let A and B be members of $E\mathscr{A}E$. We have

$$\begin{aligned} f(AB) &= (\Phi(AB)k, k) \\ &= (\Phi(A)\Phi(B)k, k) \\ &= (\Phi(B)\Phi(A)k, k) \\ &= (\Phi(BA)k, k) = f(BA). \end{aligned}$$

Thus, f is a trace on $E\mathscr{A}E$.

We have proved that $E\mathscr{A}E$ is a finite von Neumann algebra. If $\mathscr{A}$ is of Type I, so is $E\mathscr{A}E$. The algebra $E\mathscr{A}E$ possesses a canonical $^\#$-map $A \to A^\#$ of $E\mathscr{A}E$ onto the center $\mathscr{Z}_1$ of $E\mathscr{A}E$ [1]. We may write for all A in $E\mathscr{A}E$

$$f(A) = f(A^\#) = \int A^{\#\wedge}(\zeta_1)\,d\nu_1(\zeta_1) \tag{1}$$

where ν_1 is the spectral measure on the spectrum Z_1 of $\mathscr{Z}_1$. The measure arises from restricting f to $\mathscr{Z}_1$.

Let P be the central support of E. There is an isomorphism of $\mathscr{Z}P$ into $\mathscr{Z}_1$ given by $A \to AE$, for $A \in \mathscr{Z}P$. There is a homeomorphism ρ of the spectrum Z_1 of $\mathscr{Z}_1$ onto the spectrum

$$Y = \{\zeta \in Z \mid \hat{P}(\zeta) = 1\} \text{ of } \mathscr{Z}P \text{ such that}$$

the map $A \to AE$ of $\mathscr{Z}P$ to $\mathscr{Z}_1$ is given by $\hat{A} \to \hat{A}\cdot\rho$ when $\mathscr{Z}P$ and $\mathscr{Z}_1$ are identified by the Gelfand map with the algebras of all continuous complex-valued functions on Y and on Z_1 respectively.

We use $\rho^{-1}=\eta$ to obtain a representation on Y. For each $\zeta \in Y$, we define $f_\zeta(A)=(EAE)^{\#\wedge}(\eta(\zeta))$, for each A in $\mathscr{A}$. It is easy to see that f_ζ is a state for each $\zeta \in Y$, and that for each fixed A in $\mathscr{A}$, the function $\zeta \to f_\zeta(A)$ is continuous on Y. Furthermore, if $A \in \mathscr{Z}$,

$$\begin{aligned} f_\zeta(A) = (EAE)^{\#\wedge}(\eta(\zeta)) &= (AE)^\wedge(\eta(\zeta)) \\ &= (AP)^\wedge(\rho\cdot\eta(\zeta)) \\ &= \hat{A}(\zeta), \end{aligned}$$

since $P^\wedge(\zeta)=1$ for $\zeta \in Y$. Therefore, f_ζ satisfies condition (3).

Let ν be the Radon measure on Z with support Y given by the formula

$$\nu(X) = \nu_1(\eta(X)) \quad \text{for each Borel set } X \subset Y.$$

Defining $f_\zeta=0$ for $\zeta \notin Y$, we have

$$f(A) = \int_Z f_\zeta(A)\, d\nu(\zeta),$$

for all A in $\mathscr{A}$.

We now show that ν is a spectral measure. There is an element h in H such that $f|\mathscr{Z}=w_h|\mathscr{Z}$. We have

$$f(A) = \int \hat{A}(\zeta)\, d\nu(\zeta) = w_h(A)$$

for all A in $\mathscr{A}$. By definition $\nu=\nu_h$.

We now employ the hypothesis that $\mathscr{A}$ is of Type I to obtain properties (4) and (5). First we construct the nowhere dense set N in Y. Let

$$E\mathscr{A}E = \prod \{(E\mathscr{A}E)\cdot P_d \mid d \in D\}$$

where the P_d are mutually orthogonal central projections in $E\mathscr{A}E$ with least upper bound E such that $(E\mathscr{A}E)\cdot P_d$ is homogeneous for each $d \in D$. Let

$$S_d = \{\zeta_1 \in Z_1 \mid \hat{P}_d(\zeta_1) = 1\} \text{ and let}$$

$$S = \bigcup \{S_d \mid d \in D\}.$$

The set S is open and dense in Z_1; we set N_1 equal to the complement of S in Z_1. The set N_1 is nowhere dense in Z_1 and thus $N=\rho(N_1)$ is nowhere dense in Y.

For each $\zeta \in Y-N$ we must prove that f_ζ satisfies properties (4) and (5). Let P_0 be the unique projection in $\{P_d \mid d \in D\}$ such that $\hat{P}_0(\zeta_0)=1$ where $\zeta_0=\eta(\zeta)$. Since the algebra $(E\mathscr{A}E)\cdot P_0$ is finite and homogeneous, there are equivalent orthogonal abelian projections $E_1, E_2, \ldots, E_n$ such that $E_1+E_2+\cdots+E_n=E$. Let U_{jk} $(1 \leqq j, k \leqq n)$ be partial isometric operators in $(E\mathscr{A}E)\cdot P_0$ such that

(1) $U_{jk}U_{lm}=\delta_{mj}U_{lk}$, where δ is the Kronecker delta;

(2) $U_{jk}^*=U_{kj}$; and

(3) $U_{jj}=E_j$,

for all $1 \leqq j, k, l, m \leqq n$. For each A in $(E\mathscr{A}E)\cdot P_0$, there are unique B_{jk} $(1 \leqq j, k \leqq n)$ in $\mathscr{Z}_1 P_0$ such that

$$A = \sum_{j,k} B_{jk}U_{jk}.$$

We prove that the map $\Psi_{\zeta_0}=\Psi$ takes $(EAE)\cdot P_0$ onto the set of all linear operators on an n-dimensional Hilbert space H_n. Let $\Psi(U_{jk})=V_{jk}$ $(1 \leqq j, k \leqq n)$. If $e_1, e_2, \ldots, e_n$ is an orthonormal basis of H_n, define $V_{jk}e_l=\delta_{jl}e_k$. Then for each A in $(EAE)\cdot P_0$ we have

$$\Psi(A) = \sum \alpha_{jk}V_{jk}$$

where $\alpha_{jk}=\hat{B}_{jk}(\zeta_0)$ if $A=\sum B_{jk}U_{jk}$, $B_{jk} \in \mathscr{Z}_1 P_0$; thus $\Psi(A)$ is defined on H_n. It is easy to see that $\Psi((E\mathscr{A}E)\cdot P_0)$ is the set of all linear functionals on H_n.

We have for each A in $\mathscr{A}$ that

$$\begin{aligned} f_\zeta(A) &= (EAE)^{\#\wedge}(\zeta_0) \\ &= (EAE\cdot P_0)^{\#\wedge}(\zeta_0) \\ &= \sum_j (E_jAE_j)^{\#\wedge}(\zeta_0) \\ &= \sum_j (\Psi(EAE)e_j, e_j). \end{aligned} \tag{1}$$

Now each functional $A \to (E_jAE_j)^{\#\wedge}(\zeta_0)$ is irreducible on $E\mathscr{A}E$. Indeed, E_j is abelian in EAE. Thus for each A in $\mathscr{A}$ there is a B in $\mathscr{Z}_1 P_0$ such that $E_jAE_j=BE_j$. So

$$\begin{aligned} (E_jAE_j)^{\#} &= (BE_j)^{\#} \\ &= (BU_{kj}U_{jk})^{\#} \\ &= (BU_{jk}U_{kj})^{\#} = (BE_k)^{\#}. \end{aligned}$$

Thus,

$$
\begin{aligned}
(E_jAE_j)^{\#\wedge}(\zeta_0) &= \left[\sum_{k=1}^{n} (BE_k)^{\#\wedge}(\zeta_0)\right]n^{-1} \\
&= \left[\left(B\sum_{k=1}^{n} E_k\right)^{\#\wedge}(\zeta_0)\right]n^{-1} \\
&= [\hat{B}(\zeta_0)]n^{-1} \\
&= [\tau_{E_j}(A)^{\wedge}(\zeta_0)]n^{-1}.
\end{aligned}
$$

Setting $f_{j\zeta}(A)=(E_jAE_j)^{\#\wedge}(\zeta_0)$ we have that $f_{j\zeta}$ is irreducible on $E\mathscr{A}E$. Consequently, $f_{j\zeta}$ is irreducible on $\mathscr{A}$.

We now show that f_ζ has property (5). Let g be the restriction of f_ζ to $E\mathscr{A}E$. It is sufficient to show g is generalized irreducible on $E\mathscr{A}E$. Let g_1 be a positive functional included in g. There is a positive functional g_2 on $L(H_n)$, the set of all linear operators on H_n, such that $g_2\cdot\Psi=g_1$ and g_2 is included in $\sum\{w_{e_k}\mid 1\leq k\leq n\}$. But $\sum w_{e_k}$ is a trace on $L(H_n)$. There is an A_0 in $(E\mathscr{A}E)^+$ such that

$$g_1(A) = g_2(\Psi(A)) = \textstyle\sum_k w_{e_k}(\Psi(A_0A)) = g(AA_0),$$

for all A in $E\mathscr{A}E$. Thus g is generalized irreducible on $E\mathscr{A}E$. Q.E.D.

The converse of the preceding theorem is contained in the next two theorems.

THEOREM 3.3. *Let $\mathscr{A}$ be a von Neumann algebra of Type* I *with center $\mathscr{Z}$ on a Hilbert space H and let $\nu=\nu_h$ be a spectral measure on the spectrum Z of $\mathscr{Z}$. Let $E_1, E_2, \ldots, E_n$ be equivalent orthogonal abelian projections in $\mathscr{A}$ with central support P such that $Ph=h$. Let $\tau_j(A)=\tau_{E_j}(A)$ $(1\leq j\leq n)$ for $A\in\mathscr{A}$ and define $f_{j\zeta}(A)=\lambda_j\tau_j(A)^{\wedge}(\zeta)$ $(1\leq j\leq n)$ where λ_j is a strictly positive scalar and $\zeta\in Z$. If we set*

$$f(A) = \sum_{j=1}^{n}\int_Z f_{j\zeta}(A)\,d\nu(\zeta)$$

for all A in $\mathscr{A}$, there are elements B_0 and B_0' in $\mathscr{A}^+$ such that

(1) $B_0B_0'=B_0'B_0=E=E_1+E_2+\cdots+E_n$; and

(2) if g is a positive functional on $\mathscr{A}$ included in f, there is an A_0 in $\mathscr{A}^+$ such that $g(A)=f(B_0'A_0B_0A)=f(AB_0A_0B_0')$ for all A in $\mathscr{A}$. If $\lambda_1=\lambda_2=\cdots=\lambda_n=1$, we may take $B_0=B_0'=E$.

Proof. Let $B_0=\sum\{\lambda_j^{1/2}E_j\mid 1\leq j\leq n\}$ and let

$$B_0' = \sum\{\lambda_j^{-1/2}\mid E_j\, 1\leq j\leq n\}.$$

Set $f_1(A)=f(B_0'AB_0')=\sum\{(\tau_j(A)h, h) \mid 1\leqq j\leqq n\}$ for all A in $\mathscr{A}$. Since $(\tau_j(A)h, h)$ is a normal functional on $E\mathscr{A}E$, $f_1(A)$ is a normal functional on $E\mathscr{A}E$. The support of f_1 is E and, consequently, f_1 is a normal functional on $\mathscr{A}$. We show that f_1 is a trace when restricted to $E\mathscr{A}E$. For each ζ in the spectrum Z of $\mathscr{Z}$ and for each A, B in $E\mathscr{A}E$ it is sufficient to show that

$$\sum\{\tau_j(AB)^\wedge(\zeta) \mid 1 \leqq j \leqq n\} = \sum\{\tau_j(BA)^\wedge(\zeta) \mid 1 \leqq j \leqq n\}.$$

We may assume $\zeta\in\{\zeta\in Z \mid \hat{P}(\zeta)=1\}$. There is an orthonormal basis $e_1, e_2, \ldots, e_n$ for an n-dimensional Hilbert space H_n, such that $\Psi_\zeta(E\mathscr{A}E)$ is faithfully representable on H_n as the algebra of all linear operators and $(\Psi_\zeta(A)e_j, e_j)=\tau_j(A)^\wedge(\zeta)$, for all $A\in E\mathscr{A}E$ and $1\leqq j\leqq n$. Since $\sum\{w_{e_j} \mid 1\leqq j\leqq n\}$ is a trace on H_n, we have $\sum\{\tau_j(BA)^\wedge(\zeta) \mid 1\leqq j\leqq n\}=\sum\{\tau_j(BA)^\wedge(\zeta) \mid 1\leqq j\leqq n\}$. Thus, f_1 is a normal finite trace on $E\mathscr{A}E$.

Let g be a positive functional included in f. We have that the functional $g_1(A)=g(B_0'AB_0')$ is included in f_1. There is an element A_0 in $(E\mathscr{A}E)^+$ such that $g_1(A)=f_1(A_0A)$ for all A in $E\mathscr{A}E$. Thus,

$$\begin{aligned} g(A) &= g_1(B_0AB_0) = f_1(A_0B_0AB_0) = f_1(B_0AB_0A_0) \\ &= f(B_0'A_0B_0A) = f(AB_0A_0B_0') \end{aligned}$$

for all A in $\mathscr{A}$. Q.E.D.

The next theorem is obtained by reducing it to the previous one by repeated use of the following lemma.

Lemma 3.4. *Let $\mathscr{Z}$ be a commutative von Neumann algebra and let $Q_1, Q_2, \ldots, Q_n$ be projections in $\mathscr{Z}$. There are orthogonal projections $P_1, P_2, \ldots, P_m$ in $\mathscr{Z}$ such that*

(1) $\sum\{P_j \mid 1\leqq j\leqq m\}=\operatorname{lub}\{\dot{Q}_j \mid 1\leqq j\leqq n\}$;

(2) *for each j $(1\leqq j\leqq n)$ there is a finite subset π_j of $\{1, 2, \ldots, m\}$ such that $k\in\pi_j$ implies $P_k\leqq Q_j$ and $k\notin\pi_j$ implies $P_k\cdot Q_j=0$.*

Theorem 3.5. *Let $\mathscr{A}$ be a von Neumann algebra of Type* I *with center $\mathscr{Z}$ on a Hilbert space H. Let $\nu=\nu_h$ be a spectral measure on the spectrum Z of $\mathscr{Z}$. For each ζ in the support Y of ν and for each j $(1\leqq j\leqq n)$ let us assume that*

(1) *$f_{j\zeta}(A)$ is a pure state on $\mathscr{A}$;*

(2) *$f_{j\zeta}(A)=\hat{A}(\zeta)$, for all A in $\mathscr{Z}$; and*

(3) *$\zeta\to f_{j\zeta}(A)$ is continuous on Y for each fixed A in $\mathscr{A}$. Let us set*

$$f(A) = \sum_{j=1}^{n} \int f_{j\zeta}(A)\, d\nu(\zeta),$$

for each A in $\mathscr{A}$. Assume f is normal. Consider the hypotheses:

(a) *for* $1 \leqq j < k \leqq n$, $\zeta \in Y$

$$f_{j\zeta} + f_{k\zeta}$$

is generalized irreducible on $\mathscr{A}$;

(b) *hypothesis* (a) *holds, and for* $1 \leqq j < k < l \leqq n$, $\zeta \in Y$

$$f_{j\zeta} + f_{k\zeta} + f_{l\zeta}$$

is generalized irreducible on $\mathscr{A}$.

Then if hypothesis (a) *is true, there are elements B_0, B_0' in $\mathscr{A}^+$ with $B_0B_0' = B_0'B_0 = E$, where E is the support of f, such that if g is a positive functional on $\mathscr{A}$ included in f, there is an element A_0 in $\mathscr{A}^+$ such that $g(A) = f(B_0'A_0B_0A) = f(AB_0A_0B_0')$, for all A in $\mathscr{A}$. If hypothesis* (b) *holds, f is generalized reducible on $\mathscr{A}$.*

Proof. Let

$$f_j(A) = \int f_{j\zeta}(A)\, d\nu(\zeta) \qquad (1 \leqq j \leqq n),$$

for all A in $\mathscr{A}$. Since f_j $(1 \leqq j \leqq n)$ is included in f, f_j is a normal functional on $\mathscr{A}$. Further, by Tomita's theorem [4, Theorem 1, §40] f_j $(1 \leqq j \leqq n)$ is centrally irreducible on $\mathscr{A}$. The support E_j of f_j $(1 \leqq j \leqq n)$ is an abelian projection.

Let $Q_1', Q_2', \ldots, Q_n'$ be the central supports of $E_1, E_2, \ldots, E_n$ respectively. Let $P_1, P_2, \ldots, P_m$ be the sequence of projections in $\mathscr{Z}$ which satisfy properties (1) and (2) of the lemma with respect to $Q_1', Q_2', \ldots, Q_n'$. For each j consider the projections $P_jE_1, P_jE_2, \ldots, P_jE_n$. There is no loss in generality in assuming $P_jE_1, P_jE_2, \ldots, P_jE_r$ are nonzero and that $P_jE_{r+1} = P_jE_{r+2} = \cdots = P_jE_n = 0$. Since $P_jE_1, P_jE_2, \ldots, P_jE_r$ are nonzero, we have $P_j \leqq Q_k'$ for $k = 1, 2, \ldots, r$. Thus, $P_jE_1, P_jE_2, \ldots, P_jE_r$ are abelian projections with central support P_j. This implies that $P_jE_1, P_jE_2, \ldots, P_jE_r$ are equivalent.

Let us define the functional

$$\begin{aligned} f_{p_j}(A) = f(P_jA) &= \sum_k \int \hat{P}_j(\zeta) f_{k\zeta}(A)\, d\nu(\zeta) \qquad (1 \leqq k \leqq r) \\ &= \sum_k \int_{Z_j} f_{k\zeta}(A)\, d\nu(\zeta) \qquad (1 \leqq k \leqq r) \end{aligned}$$

where Z_j is the open and closed subset of Z given by $Z_j = \{\zeta \in Z \mid \hat{P}_j(\zeta) = 1\}$. We have that the functionals $f_{1\zeta}, f_{2\zeta}, \ldots, f_{r\zeta}$ $(\zeta \in Z_j)$ enjoy property (a) (respectively, property (b)) whenever $f_{1\zeta}, f_{2\zeta}, \ldots, f_{r\zeta}$ $(\zeta \in Y)$ enjoy property (a) (respectively, property (b)). If each f_{p_j} has the property set forth in the conclusion of the theorem for $\mathscr{A} \cdot P_j$ depending on whether the $f_{j\zeta}$ $(1 \leqq j \leqq r)$ satisfy (a) or (b), then it is easy to see that $f=\sum\{f_{p_j} \mid 1\leqq j\leqq m\}$ will have the same property for $\mathscr{A}$. So we may assume that the supports $E_1, E_2, \ldots, E_n$ of $f_1, f_2, \ldots, f_n$ are equivalent abelian projections.

Let us assume that $\Psi_\zeta(\mathscr{A})$ $(\zeta\in Y)$ is represented as an algebra of continuous linear operators on a Hilbert space $H(\zeta)$ such that $\mathscr{A}(\zeta)\supset C(H(\zeta))$, where $C(H(\zeta))$ is the algebra of completely continuous operators on $H(\zeta)$. Now we have for each $\zeta\in Y$ that $f_{j\zeta}([\zeta])=(0)$ $(1\leqq j\leqq n)$ by hypothesis (2). Furthermore, for each $\zeta\in Y$ and $1\leqq j\leqq n$, we have

$$0 = f_j(I-E_j) = \int f_{j\zeta}(I-E_j)\, d\nu(\zeta).$$

Thus, for $1\leqq j\leqq n$, $f_{j\zeta}(I-E_j)=0$ for all $\zeta\in Y-N$, where N is a set of ν-measure zero in Y. Now the set N contains no open sets since ν is a Radon measure whose support is Y. Thus the set $\{\zeta\in Y \mid f_{j\zeta}(I-E_j)=0\}$ $(1\leqq j\leqq n)$ is dense in Y. Since $\zeta\to f_{j\zeta}(I-E_j)$ $(1\leqq j\leqq n)$ is continuous on Y, $f_{j\zeta}(E_j)=f_{j\zeta}(I)$ $(1\leqq j\leqq n)$ for all $\zeta\in Y$. Since for fixed $\zeta\in Y$ $\Psi_\zeta(E_j)$ $(1\leqq j\leqq n)$ is an abelian projection in $\mathscr{A}(\zeta)$ and since $\mathscr{A}(\zeta)$ is irreducible on $H(\zeta)$, we have that either $E_j(\zeta)$ $(1\leqq j\leqq n)$ is a 1-dimensional projection on $H(\zeta)$ or $E_j(\zeta)$ $(1\leqq j\leqq n)$ is equal to 0. If $E_j(\zeta)=0$ $(1\leqq j\leqq n)$, $E_j\in[\zeta]$ $(1\leqq j\leqq n)$ and $1=f_{j\zeta}(E_j)=0$ $(1\leqq j\leqq n)$. Thus $E_j(\zeta)$ $(1\leqq j\leqq n)$ is a 1-dimensional projection in $H(\zeta)$. We may therefore write $f_{j\zeta}=w_{x_j}\cdot\Psi_\zeta$ $(1\leqq j\leqq n)$ where x_j $(1\leqq j\leqq n)$ is a unit vector of $H(\zeta)$.

Consider $f_{j\zeta}+f_{k\zeta}$ for $j\neq k$. Then we have $f_{j\zeta}+f_{k\zeta}=(w_{x_j}+w_{x_k})\cdot\Psi_\zeta$. Because $f_{j\zeta}<f_{j\zeta}+f_{k\zeta}$, there is an element B in $\mathscr{A}^+$ with the property

$$\begin{aligned} f_{j\zeta}(A) &= f_{j\zeta}(BA)+f_{k\zeta}(BA)\\ &= f_{j\zeta}(AB)+f_{k\zeta}(AB)\end{aligned}$$

for all A in $\mathscr{A}$ (hypothesis (a)). Then

$$\begin{aligned} w_{x_j}[A(\zeta)] &= w_{x_j}[B(\zeta)A(\zeta)]+w_{x_k}[B(\zeta)A(\zeta)]\\ &= w_{x_j}[A(\zeta)B(\zeta)]+w_{x_k}[A(\zeta)B(\zeta)]\end{aligned}$$

for all A in $\mathscr{A}$. Since $\mathscr{A}(\zeta)\supset C[H(\zeta)]$, we have that

(1) $B(\zeta)^{1/2}x_j=\lambda_j x_j$ and $B(\zeta)^{1/2}x_k=\lambda_k x_j$, where λ_j and λ_k are scalars. Indeed the set $\mathscr{B}=\{A_1(\zeta)\in\mathscr{A}(\zeta) \mid (w_{x_j}+w_{x_k})[A_1(\zeta)A(\zeta)]=(w_{x_j}+w_{x_k})[A(\zeta)A_1(\zeta)]$, for all $A(\zeta)$ in $\mathscr{A}(\zeta)\}$ is a uniformly closed $*$-subalgebra of $\mathscr{A}(\zeta)$ with identity. Thus $B(\zeta)\in\mathscr{B}$ and $B(\zeta)\geqq 0$ implies $B(\zeta)^{1/2}\in\mathscr{B}$. This means $(A(\zeta)x_j, x_j)=(A(\zeta)B(\zeta)^{1/2}x_j, B(\zeta)^{1/2}x_j)$ $+(A(\zeta)B(\zeta)^{1/2}x_k, B(\zeta)^{1/2}x_k)$, for all $A(\zeta)\in\mathscr{A}(\zeta)$.

From this expression (1) follows. Thus,

$$B(\zeta)x_j = \lambda_j^2 x_j \quad\text{and}\quad B(\zeta)x_j = \lambda_j\lambda_k x_j.$$

We have

$$(A(\zeta)x_j, x_j) = \lambda_j^2(A(\zeta)x_j, x_j)+\lambda_j\lambda_k(A(\zeta)x_j, x_k),$$

for all A in $\mathscr{A}$. So

$$x_j = \bar{\lambda}_j^2 x_j+\bar{\lambda}_j\bar{\lambda}_k x_k. \tag{2}$$

Because $B(\zeta)^{1/2}\geqq 0$, $\lambda_j(x_j, x_j)=(B(\zeta)^{1/2}x_j, x_j)\geqq 0$ and so $\lambda_j\geqq 0$. We also have that $\lambda_j>0$ since $(x_j, x_j)=f_{j\zeta}(E_j)\neq 0$ (expression (2)). Therefore (from expression (2)),

$$(1-\lambda_j^2)x_j = \lambda_j\bar{\lambda}_k x_k.$$

Either x_j is a scalar multiple of x_k or $\lambda_k=0$. If $\lambda_k=0$,

$$\begin{aligned}\lambda_j(x_j, x_k) &= (B(\zeta)^{1/2}x_j, x_k)\\ &= (x_j, B(\zeta)^{1/2}x_k)\\ &= \bar{\lambda}_k(x_j, x_j)\\ &= 0,\end{aligned}$$

by expression (1). This means that x_j and x_k are orthogonal since $\lambda_j>0$. Since $E_j(\zeta)$ and $E_k(\zeta)$ are both 1-dimensional projections with $E_j(\zeta)x_j=x_j\neq 0$ and $E_k(\zeta)x_k=x_k\neq 0$, we have $E_j(\zeta)$ is orthogonal to $E_k(\zeta)$ if $\lambda_k=0$ and $E_j(\zeta)=E_k(\zeta)$ if $\lambda_k\neq 0$.

Now by the preliminary remarks the map $\zeta\to\|(E_j\cdot E_k)(\zeta)\|=\|E_j(\zeta)E_k(\zeta)\|$ is continuous on Y. Since $E_j(\zeta)=E_k(\zeta)$ or $E_j(\zeta)\cdot E_k(\zeta)=0$ for each $\zeta\in Y$, the function $\|(E_j\cdot E_k)(\zeta)\|$ assumes at most two values, namely 0 and 1. Hence the set

$$X_{jk} = \{\zeta\in Y \mid E_j(\zeta) = E_k(\zeta)\}$$

is open and closed in Y and hence in Z since Y is open and closed in Z. Let P be the projection in $\mathscr{Z}$ with the property

$$\{\zeta\in Z \mid P^\wedge(\zeta) = 1\} = Y$$

and let Q_{jk} be the projection in $\mathscr{Z}P$ with the property

$$X_{jk} = \{\zeta\in Z \mid Q_{jk}^\wedge(\zeta) = 1\}.$$

Such a projection Q_{jk} exists because $X_{jk}\subset Y$.

Suppose we have found projections Q_{jk} with the properties just specified for all $1\leqq j<k\leqq n$. Consider the set of projections,

$$S = \{Q_{jk}\mid 1 \leqq j < k \leqq n\}\cup\{P-Q_{jk}\mid 1 \leqq j < k \leqq n\}$$

in $\mathscr{Z}P$. There are projections $Q_1, Q_2,\ldots, Q_r$ in $\mathscr{Z}P$ which satisfy (1) and (2) of the lemma with respect to the set S.

For arbitrary Q_l in $\{Q_1, Q_2, \ldots, Q_r\}$ consider the set $Q_lE_1, Q_lE_2, \ldots, Q_lE_n$. Since $E_1 \sim E_2 \sim \cdots \sim E_n$, we have $Q_lE_1 \sim Q_lE_2 \sim \cdots \sim Q_lE_n$. For each $j \neq k$, Q_lE_j is orthogonal to Q_lE_k or is equal to Q_lE_k. Indeed, either $Q_l \leqq Q_{jk}$ or $Q_l \leqq P - Q_{jk}$. In the former case $\|(E_j - E_k)(\zeta)\| = 0$ for all ζ in $Z_l = \{\zeta \in Y \mid Q_l^\frown(\zeta) = 1\}$. Thus, $\|((E_j - E_k)\cdot Q_l)(\zeta)\| = 0$ for all ζ in Z. Since

$$\bigcap \{[\zeta] \mid \zeta \in Z\} = 0$$

we have $Q_lE_j = Q_lE_k$. In the latter case, on the other hand,

$$\|(Q_l \cdot (E_j \cdot E_k))(\zeta)\| = 0$$

for all $\zeta \in Z$, and therefore Q_lE_j and Q_lE_k are orthogonal.

We have

$$f_{Q_l}(A) = f(Q_lA) = \sum_{j=1}^{n} \int_{Z_l} f_{j\zeta}(A)\, d\nu(\zeta),$$

for all A in $\mathscr{A}$. We divide the set $\{1, 2, \ldots, n\}$ into disjoint parts $\pi_1, \pi_2, \ldots, \pi_m$ such that (1) $\bigcup_k \pi_k = \{1, 2, \ldots, n\}$; (2) j and j' in π_k implies $Q_lE_j = Q_lE_{j'}$; and (3) $j \in \pi_k$ and $j' \in \pi_{k'}$ $(k \neq k')$ implies Q_lE_j is orthogonal to $Q_lE_{j'}$. We have

$$f_{Q_l}(A) = \sum_{j=1}^{m} \int \lambda_j g_{j\zeta}(A)\, d\nu(\zeta), \tag{3}$$

for all A in $\mathscr{A}$. Here, $g_{j\zeta}(A) = f_{k\zeta}(A)$ where $k \in \pi_j$ $(1 \leqq j \leqq m)$ and λ_j is a scalar equal to the cardinality of π_j $(1 \leqq j \leqq m)$. Since $f_{k\zeta}(A) = \tau_{Q_lE_k}(A)^\frown(\zeta) = \tau_{Q_lE_{k'}}(A)^\frown(\zeta) = f_{k'\zeta}(A)$ for $k, k' \in \pi_j$ (because $Q_lE_k = Q_lE_{k'}$), the functionals $g_{j\zeta}$ are uniquely defined.

If each f_{Q_l} $(l = 1, 2, \ldots, r)$ possesses the properties set forth in either conclusion for (a) and (b) with regard to $\mathscr{A}Q_l$, then $f = \sum_l f_{Q_l}$ possesses the same properties with regard to $\mathscr{A}$. Thus, we may assume that

$$f(A) = \sum_{j=1}^{n} \int \lambda_j \tau_{E_j}(A)^\frown(\zeta)\, d\nu(\zeta),$$

for all A in $\mathscr{A}$, where $E_1, E_2, \ldots, E_n$ are orthogonal equivalent abelian projections. Thus, we have reduced to Theorem 3.3.

Let $E = E_1 + E_2 + \cdots + E_n$; let B_0 be the element in $\mathscr{A}^+$ given by

$$B_0 = \lambda_1^{1/2}E_1 + \lambda_2^{1/2}E_2 + \cdots + \lambda_n^{1/2}E_n$$

and let

$$B_0' = \lambda_1^{-1/2}E_1 + \lambda_2^{-1/2}E_2 + \cdots + \lambda_n^{-1/2}E_n.$$

If g is a positive functional on $\mathscr{A}$ included in f, by Theorem 3.3 there is an A_0 in $\mathscr{A}^+$ such that

$$g(A) = f(B_0'A_0B_0A) = f(AB_0A_0B_0'),$$

for all A in $\mathscr{A}$. This completes the proof of part (a).

Let us assume that (b) applies. Using hypothesis (a), we can reduce to the form (3)

$$f_{Q_\iota}(A) = \sum_{j=1}^{m} \int \lambda_j g_{j\zeta}(A)\, d\nu(\zeta).$$

We shall show either $m=1$ or $\lambda_1=\lambda_2=\cdots=\lambda_m=1$. In both of these situations Theorem 3.3 can be applied to give the desired conclusion.

We argue by contradiction. Assume $\lambda_1>1$ and $m\neq 1$. In the notation of the proof

$$2f_{1\zeta}(A)+f_{2\zeta}(A) = (2w_{x_1}+w_{x_2})[A(\zeta)]$$

is generalized irreducible. Here x_1 and x_2 are orthogonal unit vectors in $H(\zeta)$. This is impossible as the next proposition will demonstrate.

Proposition 3.6. *Let $\mathscr{A}$ be a B^*-algebra with identity operating on a Hilbert space H. Assume $\mathscr{A}\supset C(H)$, where $C(H)$ is the algebra of completely continuous operators on H. If x_1 and x_2 are orthogonal unit vectors in H, then there is a positive functional g on $\mathscr{A}$ included in $2w_{x_1}(AB)+w_{x_2}(AB)$, for all A in $\mathscr{A}$, implies $B\neq B^*$.*

Proof. Let $g=w_y$ where $y=x_1+x_2$. If $A\in L(H)^+$, we have

$$(A^{1/2}(x_1-x_2),\, A^{1/2}(x_1-x_2)) \geqq 0,$$

and so

$$(A^{1/2}x_1,\, A^{1/2}x_1)+(A^{1/2}x_2,\, A^{1/2}x_2) \geqq (A^{1/2}x_1,\, A^{1/2}x_2)+(A^{1/2}x_2,\, A^{1/2}x_1).$$

Therefore, we have

$$(Ay, y) \leqq 2(A^{1/2}x_1,\, A^{1/2}x_1)+2(A^{1/2}x_2,\, A^{1/2}x_2) \leqq 2(2w_{x_1}(A)+w_{x_2}(A)).$$

Thus, $w_y<2w_{x_1}+w_{x_2}$ on $L(H)$ and thus on $\mathscr{A}$. Let $B\in L(H)$ have the property

$$w_y(A) = 2w_{x_1}(AB)+w_{x_2}(AB),$$

for all A in $\mathscr{A}$. Let $A_1x_1=0$, $A_1x_2=x_1$ and $A_1(K)=(0)$ where K is the orthogonal complement of the subspace of H generated by x_1 and x_2; let $A_2x_1=x_2$, $A_2x_2=0$ and $A_2(K)=(0)$. We have $A_1, A_2\in C(H)\subset A$. Then

$$\begin{aligned} 1 = w_y(A_1^*) &= 2w_{x_1}(A_1^*B)+w_{x_2}(A_1^*B) \\ &= 0+(Bx_2, x_1), \end{aligned}$$

and

$$1 = w_y(A_2^*) = 2w_{x_1}(A_2^*B)+w_{x_2}(A_2^*B)$$
$$= 2(Bx_1, x_2)+0.$$

So

$$(Bx_1, x_2) = \tfrac{1}{2} \quad \text{and} \quad (Bx_2, x_1) = 1.$$

We have

$$(x_1, Bx_2) = \overline{(Bx_2, x_1)} = 1.$$

Thus $B \neq B^*$. Q.E.D.

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